Divide and Conquer Strategy for Problem Solving - Recursive Functions

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References:
1. Ch. 4, Downey.
2. Introduction to Recursion Online Notes Ch. 70 by B. Kjell : http://chortle.ccsu.edu/java5/Notes/chap70/ch70_1.html
3.

Divide and Conquer

- Basic Idea of Divide and Conquer:
  - If the problem is easy, solve it directly
  - If the problem cannot be solved as is, decompose it into smaller parts,. Solve the smaller parts

Some Examples

- Finding the exit from within a hotel
- Finding your car in a parking lot
- Looking up a name in a phone book

Mental Exercise

- You are at the end of a very long hotel lobby with a long series of doors, with one door next to you. You are looking for the door that leads to the fire exit.
- What is your first step?

First Step

What do you do if the first step does not work?

FindExit Strategy

- Try the door next to you for the exit
- If it does not lead to an exit, advance to the next door. And repeat the FindExit strategy

Elements of the Solution

- Try a direct solution: check the nearby door for the exit
- If it does not work, use the same strategy on the smaller problem after advancing to the next door

Recursion in Words of Wisdom

- Philosopher Lao-tzu:
  - The journey of a thousand miles begins with a single step

Breaking a Stone into Dust

- BreakStone: You want to ground a stone into dust (very small stones)
- What is your first step?

First Step

- Use a hammer and strike the stone

Next Step

- If a stone pieces that result is small enough, we are done with that part
- For pieces that are too large, repeat the BreakStone process

Common Elements

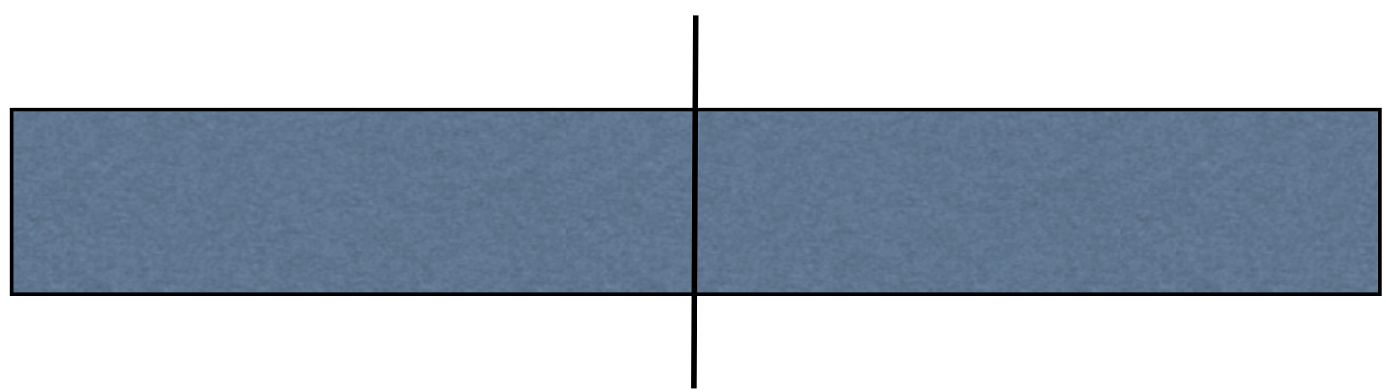
- If the problem is small enough to be solved directly, do it
- If not, find a smaller problem and use its solution to create the solution to the larger problem

Looking up a Phone Number

- You have a phone book with names in alphabetical order
- Given a name, what is your first step?

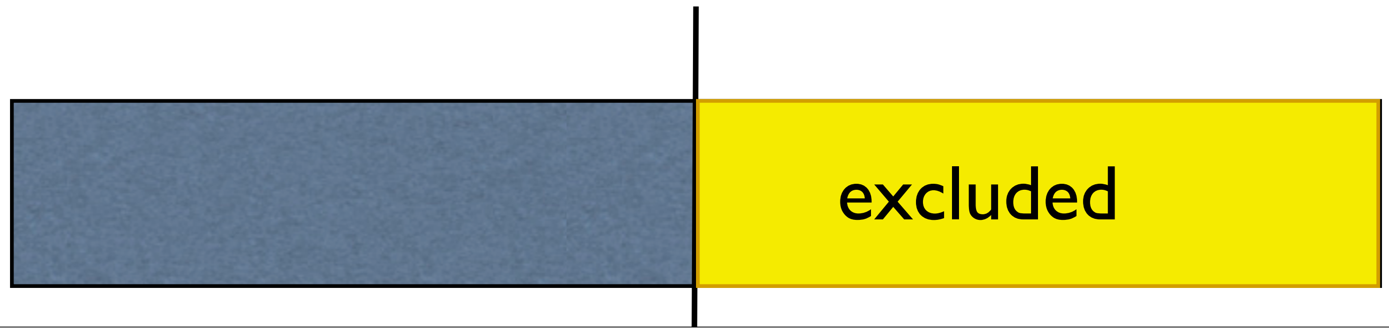
First Step

- Open the phone book in the middle (or on a random page)
- If name within the range of names on that page, find it, and we are done

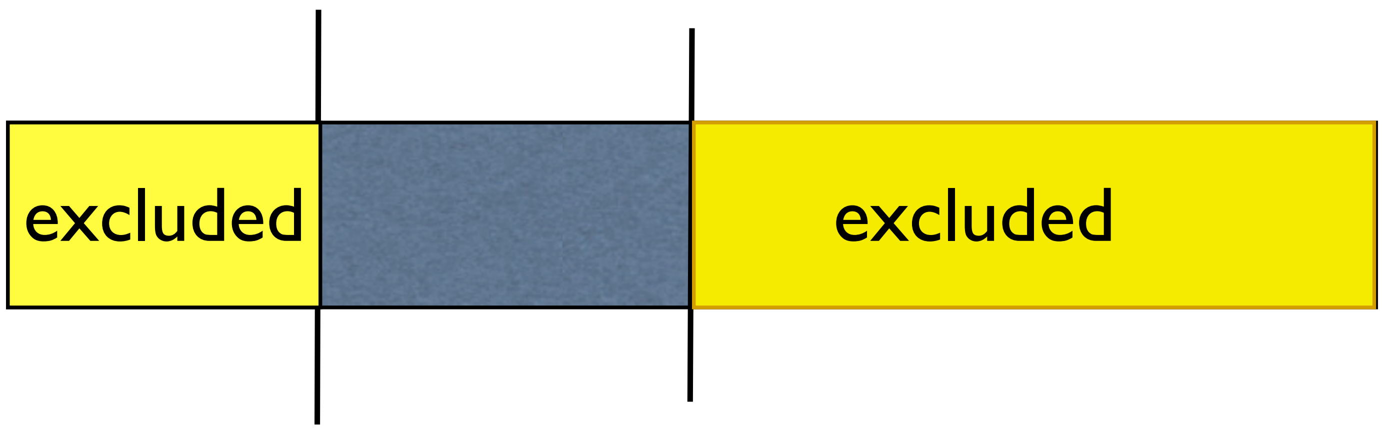


Smaller Problem Step

- If name not in the page, you can exclude either the left part or the right part
- Search in the remaining part



After another step

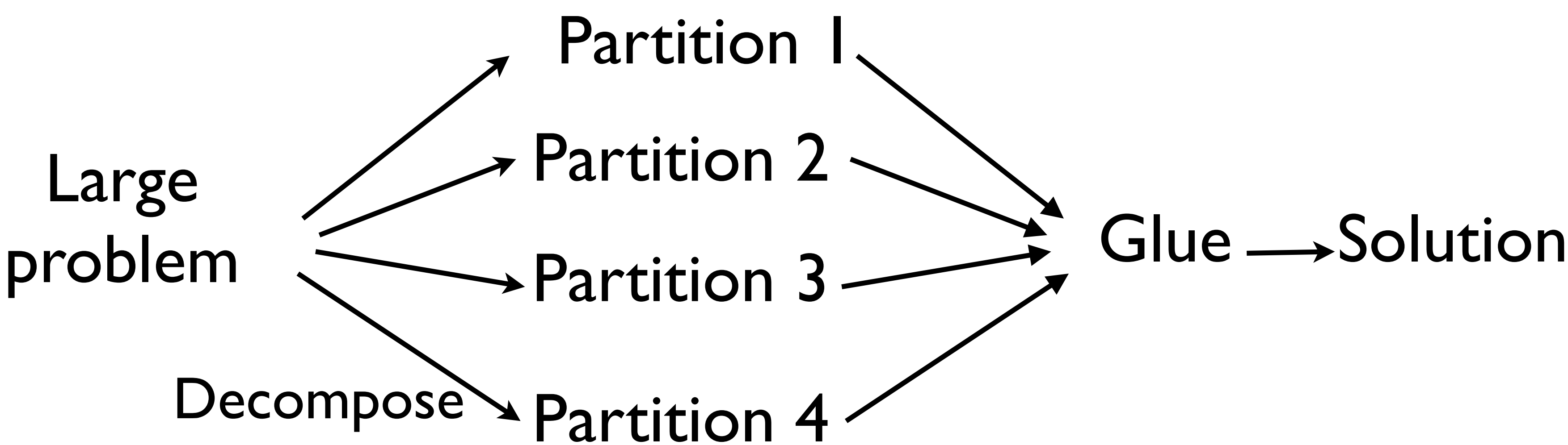


Other Problems

- Recursively-defined functions
  - Factorial: n!
  - Fibonacci numbers
  - Ackermann Function
- Tower of Hanoi
- Fractals
- Tree and data searches

Glue in Divide and Conquer

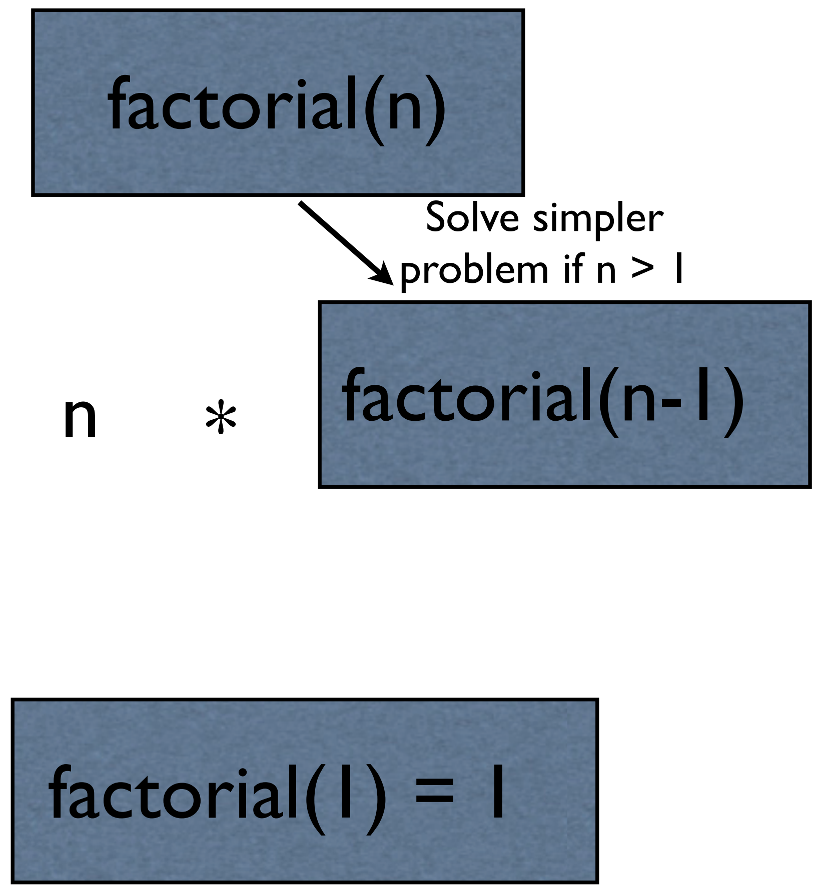
- Often, the parts must be “glued”into a solution



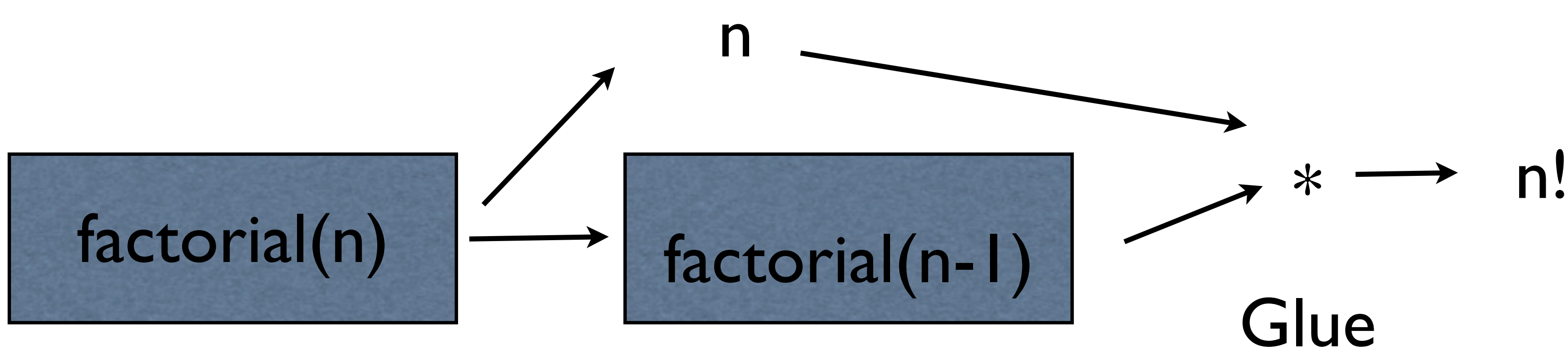
Factorial Problem

- Example: Finding factorial of n >= 1
- n! = n(n-1)(n-2)…1
- Divide and Conquer Strategy:
  - if n = 1: n! = 1 (direct solution),

    else: n! = n * (n-1)!



Divide and Conquer



Recursion in Functions

- When a function makes use of itself, as in a divide-and-conquer strategy, it is called recursion
- Recursion requires:
  - Base case or direct solution step. (e.g., factorial(1))
- Recursive step(s):
  - A function calling itself on a smaller problem. E.g., n*factorial(n-1)
- Eventually, all recursive steps must reduce to the base case

Java Code

- Definition: n! is defined as 1 if n = 0 (direct solution),

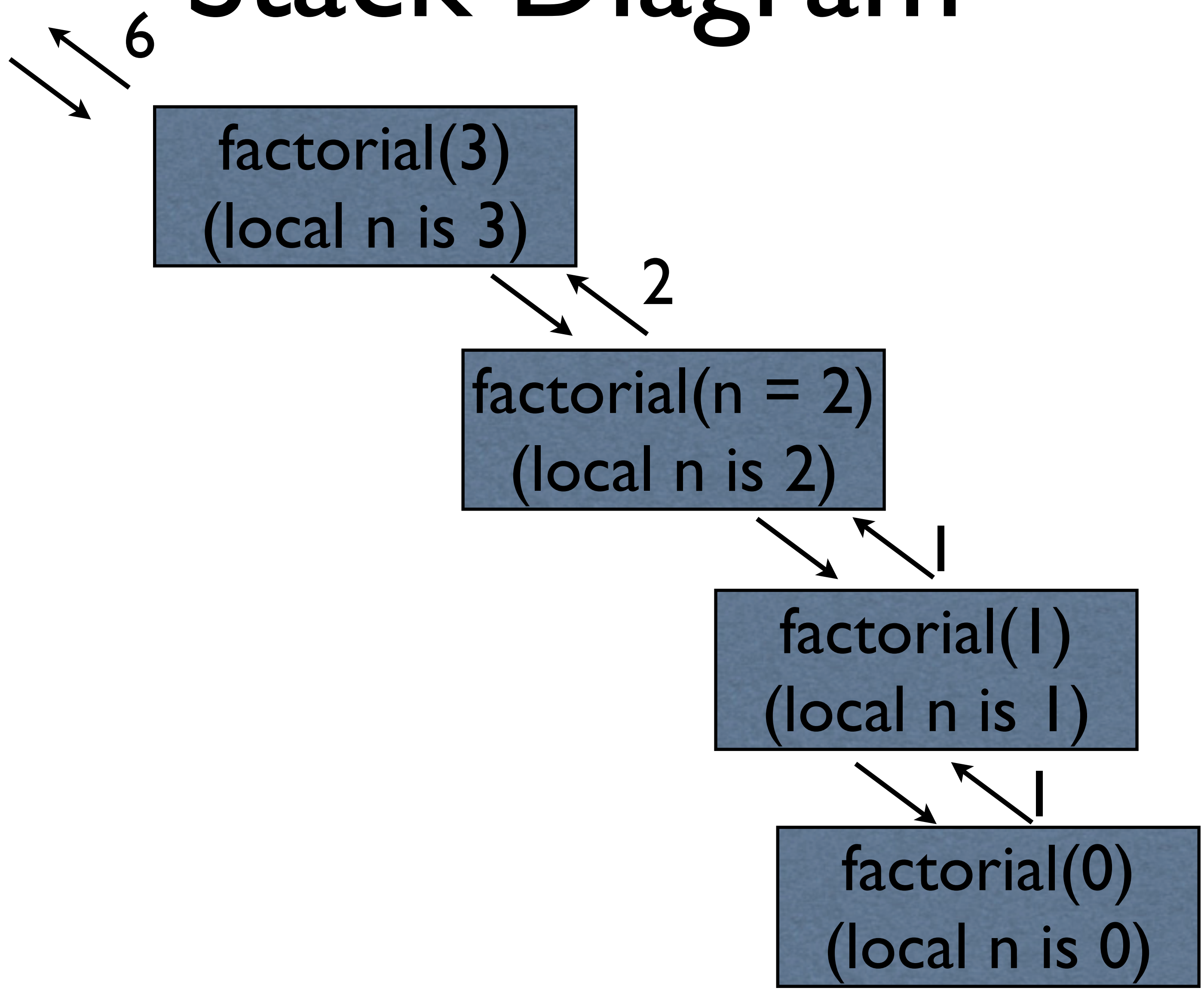
  Otherwise, n! = n * (n-1)! (divide-and-conquer)

Direct solution for base case

Divide and Conquer for larger case

```
public static int factorial(int n) {
   if (n == 0) return 1;
   else return n * factorial(n-1);
   // post-condition: returns n!
}
```

Stack Diagram



Understanding Recursive Programs

- Why does it work?

```
public static int factorial(int n) {
   if (n == 0) return 1;
   else return n * factorial(n-1);
   // post-condition: returns n!
}
```

Proof by induction:
(1) Solution works for n = 0
(2) If it works for n-1, it works for n
(3) 1. and 2. imply, it works for n = 1
(4) 2. and 3. imply it works for n = 2 and in fact any larger n

Handling Errors

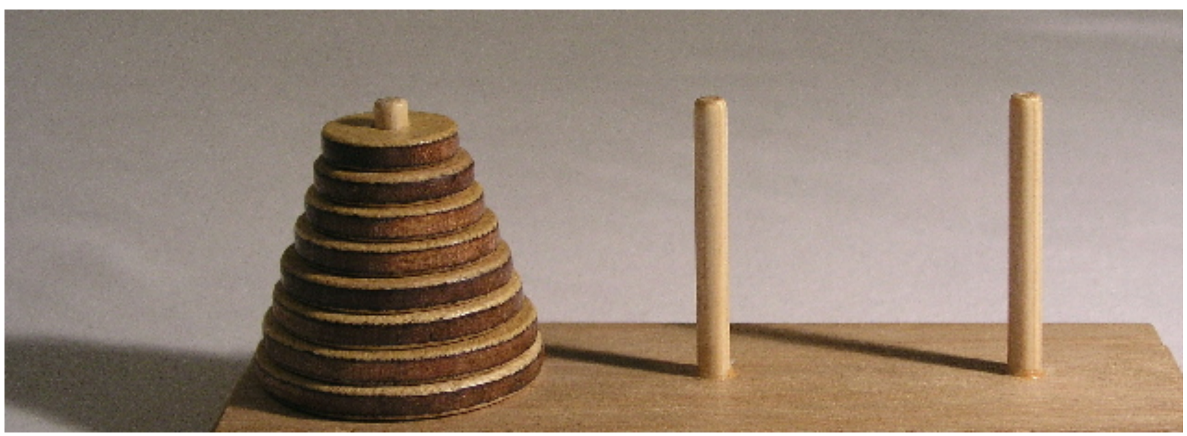
- What if n is < 1 in the factorial program?
  - factorial(-1) will call factorial (-2), which will call factorial(-3), etc.
  - Recursion will never reach the base case
- Document pre-conditions and post-conditions

Use assert or error checks to indicate something that is assumed or expected to be True

```
public static int factorial(int n) {
    assert(n >= 0); // pre-condition
    if (n == 0) return 1;
    else return n * factorial(n-1);
    // post-condition: returns n!
}
```

Tower of Hanoi

- Initial state: n disks in decreasing order of size on one peg
- Goal: move all the disks to the 2nd peg.
- Move one disk at a time
- Constraint: a disk can never be on top of a smaller disk

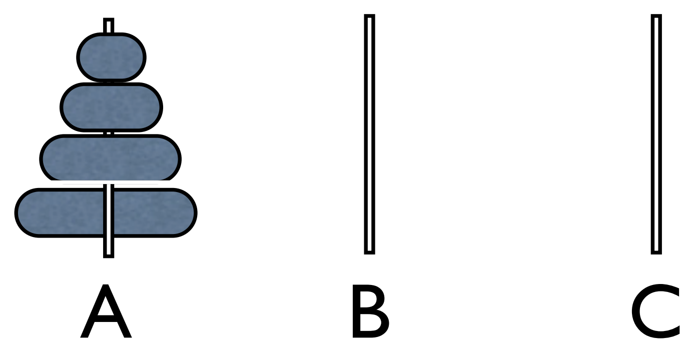


Divide-and-Conquer Strategy

- If n is 1, the solution is trivial. Just move the disk to the desired peg
- For n > 1, let's assume we know how to solve the problem for n-1 disks.
- Can we use that to construct a solution for n disks?

Solution Strategy

- Base case: if n is 1, the solution is trivial. Just move the disk
- Otherwise:
    - Move (n-1) disks from peg A to peg C *using Hanoi for n-1 disks*
- Move the left-over disk from page A to peg B
- Move (n-1) disks from peg C to peg B *using Hanoi for (n-1) disks*



Java Code

```java
public static void move(Object pegA, Object pegB) {
    System.out.println( "move disk from " + pegA + " to " + pegB);
}

public  static void hanoi(int n, Object pegA, Object pegB, Object pegC) {
    //precondition: n >= 1. disks are on pegA
    assert(n >= 1);
    if (n == 1) {
        move(pegA, pegB);
    } else {
        hanoi(n-1, pegA, pegC, pegB);
        move(pegA, pegB);
        hanoi(n-1, pegC, pegB, pegA);
    }
    // post-condition: top n disks moved from pegA to pegB
}
```

```java
hanoi(3, "peg 1", "peg 2", "peg 3");
```

Show a run in Eclipse

Tower of Hanoi Analysis

```
>>> recursion.hanoi(2, 'peg 1', 'peg 2', 'peg 3')
move disk from  peg 1 to peg 3
move disk from peg 1 to peg 2
move disk from  peg 3 to peg 2
>>>
>>> recursion.hanoi(2, 'peg 1', 'peg 3', 'peg 2')
move disk from  peg 1 to peg 2
move disk from peg 1 to peg 3
move disk from  peg 2 to peg 3
>>>
>>> recursion.hanoi(2, 'peg 3', 'peg 2', 'peg 1')
move disk from  peg 3 to peg 1
move disk from peg 3 to peg 2
move disk from  peg 1 to peg 2
```

```
>>> recursion.hanoi(3, 'peg 1', 'peg 2', 'peg 3')
move disk from  peg 1 to peg 2
move disk from peg 1 to peg 3
move disk from  peg 2 to peg 3
move disk from peg 1 to peg 2
move disk from  peg 3 to peg 1
move disk from peg 3 to peg 2
move disk from  peg 1 to peg 2
>>>
```

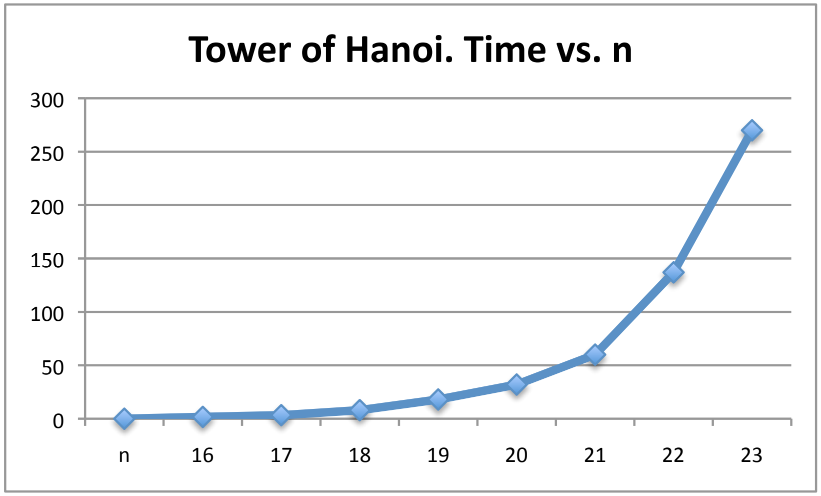
Tower of Hanoi is pretty slow for larger values of n.

Q: How many disk moves (approximately) for a given n?

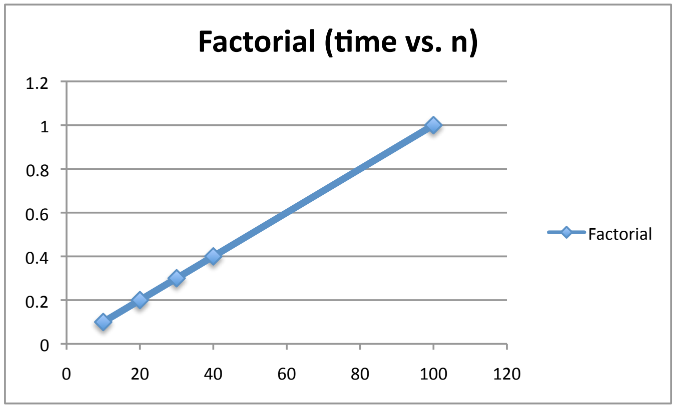
Note: The above is on Python version. Java is analogous

OPTIONAL

Performance



Tower of Hanoi really slows down for large n



Factorial's time is roughly linear with n.
We say that the time for factorial is O(n), called Big-Oh(n), or linear in n

Big-Oh

- A way to measure how execution time or memory use will grow with input size
- Formally, f(n) is O(g(n)) iff for sufficiently large values of n, f(n) is within constant times of g(n). That is,
  - f(n) < c.g(n) for all n > N and some constant c.

Big-Oh examples

- $3n + 2$ is $O(n)$ because $3n+1 < 4n$ for large n
- $1000n + 100000$ is also $O(n)$
- $10n^2 + 3$ is $O(n^2)$
- $2^n + n^3$ is $O(2^n)$

Basic Points

- Ignore the small stuff
  - $n + 10$: ignore the 10
  - $n^2 + n$: ignore the n
- Simplify
  - Replace 10 by 1. Both are O(1)
  - 2n can be replaced by n. Both are O(n)

Factorial Time Analysis

- Factorial of 0: constant time.
  - $T(0) = 1$ (treat constants as 1)
- Time required to compute factorial of n:
  - $T(n) = T(n-1) + 1$ (treat constants as 1)

$$
\begin{aligned}
T(4) &= T(3) + 1 \\
&= T(2) + 1 + 1 \\
&= T(1) + 1 + 1 + 1 \\
&= T(0) + 1 + 1 + 1 + 1 = 5
\end{aligned}
$$

In general, $T(n)$ is $n+1$ or $O(n)$

Hanoi: Number of Moves

- Let $T(n)$ = number of disk moves for n disks
- $T(1) = 1$
- $T(2) = 2*T(1) + 1 = 3$
- $T(3) = 2*T(2) + 1 = 7$
- $T(4) = 2*T(3) + 1 = 15$
- See a pattern?
- $T(n) = 2^n - 1$ or $O(2^n)$
- Hanoi for 64 disks would take a very, very long time!
- This is an example of an exponential-time program.

Advantage of Big-Oh Analysis

- Big-Oh gives you trends versus problem size
- Big-Oh analysis holds even if computers become 10 times faster

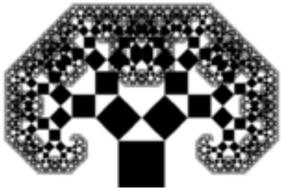
Common Growth Rates

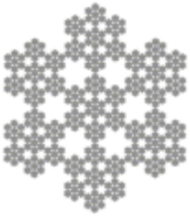
- $O(1)$: constant time. For example, array lookup, given an index
- $O(n)$: linear time. For example, scan an array of length n for a value
- $O(\log n)$: Between constant and linear time.
- $O(2^n)$: Exponential time. Very bad

We will see lots of examples later

Fractals

- Fractals are recursive drawings. They occur a lot in nature and there is a field called fractal geometry. Can use recursion to draw them





But, how to do drawings in Java?

Drawing in Java

- Java has several graphics packages: awt, swing, etc.
- We will use ACM Graphics package for Java, as it is designed for educational use
- Download acm.jar and*.java from CTools in 11-lecture-code folder
- Documentation:
  - http://jtf.acm.org/tutorial/Introduction.html

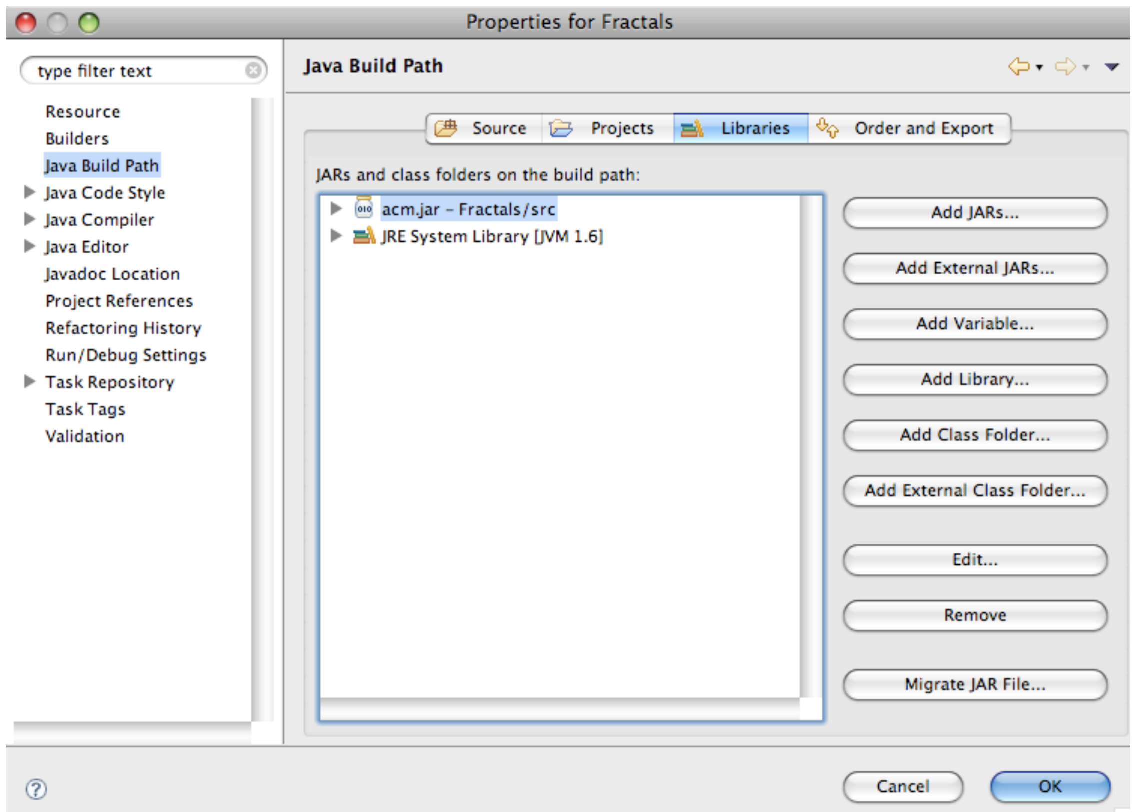
Using acm.jar

Command line (use semi-colons on Windows):

javac -cp .:/path/to/acm.jar *.java

java -cp .:/path/to/acm.jar MainClass

- In Eclipse, go to Project -> Properties -> Java Build Path
- Add acm.jar to the build path



ACM Graphics Package

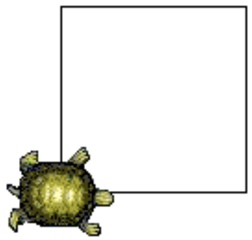
- Create shapes, e.g., GLabel, GLine, GTurtle, etc. in a GraphicsProgram
- Add them to the canvas using the add() routine
- getWidth() and getHeight() return the height of a canvas.
- Coordinate system: Top-left corner is (0,0).

Mini-exercise

- Compile and run one of the Hello programs that use the ACM jar file. Submit a screen snapshot
- Generate the stack of squares and submit a screen snapshot

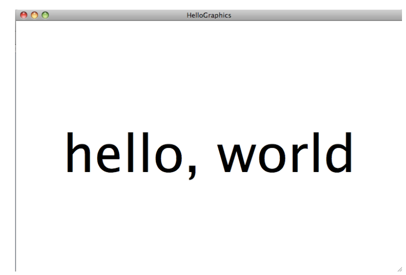
Turtle Programming: Drawing a Square

```
public static void drawSquare(GTurtle t, double len) {
    t.penDown();
    for (int i = 0; i < 4; i++) {
        t.forward(len);
        t.left(90);
    }
}

   public void run() {
    // Place turtle in the center of the canvas
    GTurtle turtle = new GTurtle(getWidth()/2.0, getHeight()/2.0);
    add(turtle);
    drawSquare(turtle, 100.0);
   }
```



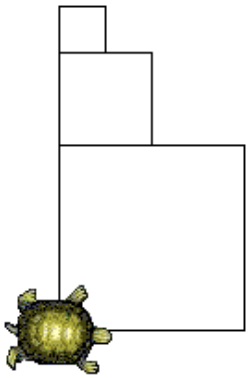
Hello World Program

```java
public class HelloGraphics extends GraphicsProgram {

    public void run() {
        GLabel label = new GLabel("hello, world");
        label.setFont("SansSerif-100");
        double x = (getWidth() - label.getWidth()) / 2;
        double y = (getHeight() + label.getAscent()) / 2;
        add(label, x, y);
    }

/* Standard Java entry point. Call MainClass.start(args)
   to get graphics program going  */

    public static void main(String[] args) {
        new HelloGraphics().start(args);
    }
}
```



Stack of Squares using Recursion

```
public void drawStack(GTurtle t, double len, int squarecount) {
    // precondition: turtle at "origin"
    if (squarecount == 0) return;
    drawSquare(t, len);   // draw big square, ending at the start location.
    t.left(90); t.forward(len); t.right(90);  // go to the top-left
    drawStack(t, len/2.0, squarecount-1);
    t.right(90); t.forward(len); t.left(90); // return to origin
    // post-condition: draw the stack and return to origin
}

public void run() {
    GTurtle turtle = new GTurtle(getWidth()/2.0, getHeight()/2.0);
    add(turtle);
    drawStack(turtle, 100.0, 3);
     }
```



Another Way

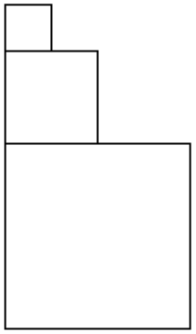
- Use shape drawing functions rather than turtles. Basic primitive
  - draw a shape of a given size at (x, y)
  - Shapes include lines, squares, circles, rectangles, etc.
  - Shapes can have attributes, such as line thickness, color, fill, etc.

Drawing a square

```
// File: SquareStackWithShapes.java

// draws a square at (x, y) of length len. Origin top-left corner.
public void drawSquare(double x, double y, double len) {
    GRect r = new GRect(x, y, len, len);
    add(r);
}

public void run() {
    drawSquare(getWidth()/2, getHeight()/2, 100.0);
}
```

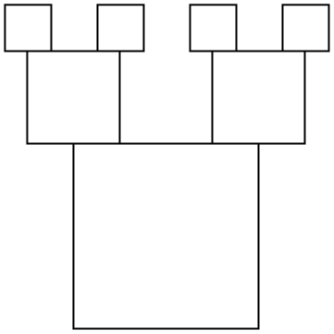
Drawing a Stack

```java
// draw a stack squarecount deep at (x, y), with squares becoming
// half the size as you go up the stack.
public void drawStack(double x, double y, double len, int squarecount) {
    if (squarecount == 0) return;
    drawSquare(x, y, len);   // draw big square, ending at the start location.
    drawStack(x, y-len/2.0, len/2.0, squarecount-1);
}
```

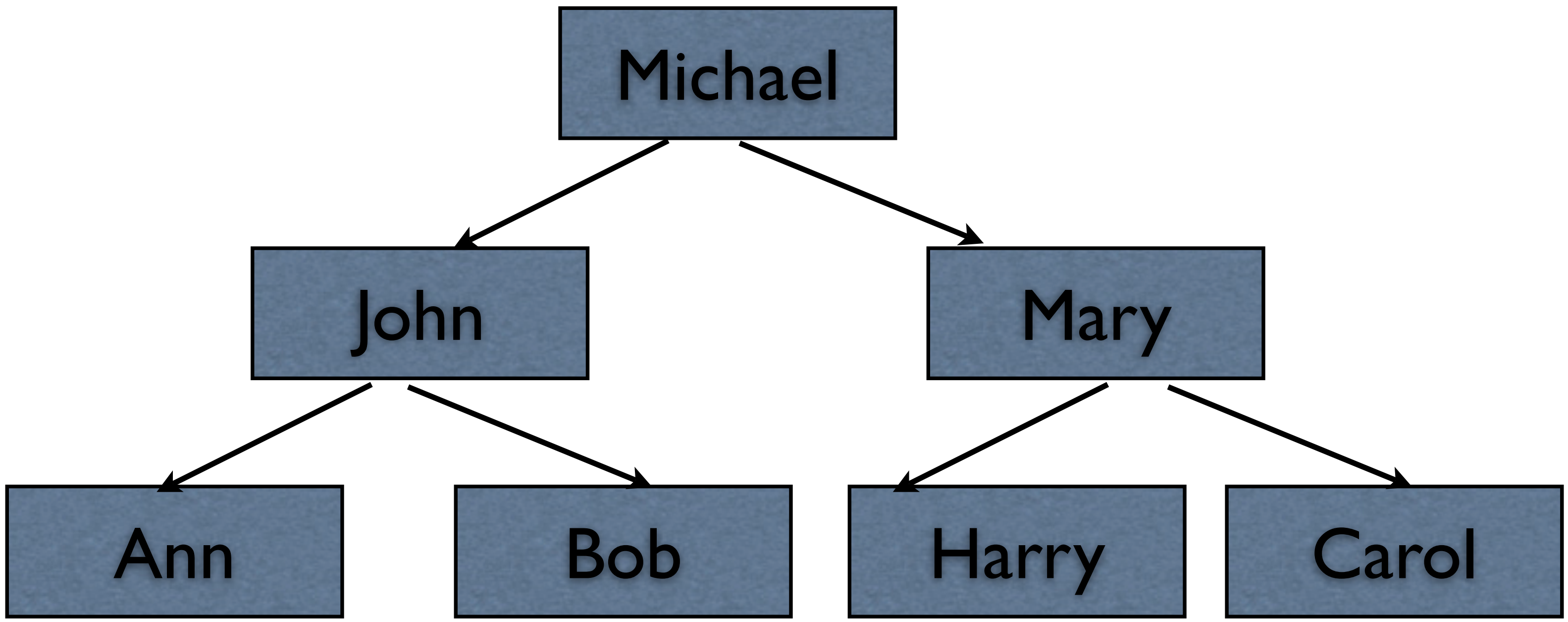


Draw the big square. Note: origin top-left corner.
Draw the remaining stack with origin 1/2 length up.

```java
public void drawTreeOfSquares(double x, double y, double len, int squarecount) {
    if (squarecount == 0) return;
    drawSquare(x, y, len);
    drawTreeOfSquares(x-len*0.25, y-len*0.5, len*0.5,  squarecount-1);
    drawTreeOfSquares(x+len*0.75, y-len*0.5, len*0.5, squarecount-1);
}
```



Hierarchical Data (trees)



Example: child-dad/mom relationship

Count nodes in a Tree

If tree empty, return 0

Else, return

one

+ count of left subtree

+ count of right subtree

Summary

- Divide and Conquer is a common problem solving strategy
- It often maps to recursive algorithms
- Big-Oh notation a way to estimate how time required to solve a problem will grow as the problem size increases