Random Processes:

Common Examples,

Autocovariance,

Autocorrelation, and

Stationarity

Random (stochastic) processes

A <u>random process</u> is a family of random variables indexed by the integers, $x[n]$, $n \in \mathbb{Z}$. In practice we observe a finite-duration realization $x[0], \dots, x[N-1]$. We'll use the same notation for a RP and a realization of that RP.

Examples

1. <u>White noise</u>: A white noise process $w[n]$ satisfies three properties:

   (a) values at different times are uncorrelated

   (b) $E\{w[n]\} = 0 \quad \forall n \in \mathbb{Z}$

   (c) $\mathrm{Var}\{w[n]\} = \sigma_w^2 \quad \forall n \in \mathbb{Z}$

An important special case is Gaussian white noise,

$$w[n] \sim N(0, \sigma_w^2).$$

Why do you think "white" is used to describe such a process?

2. <u>Moving average process</u> : a linear combination of terms in a white noise process,

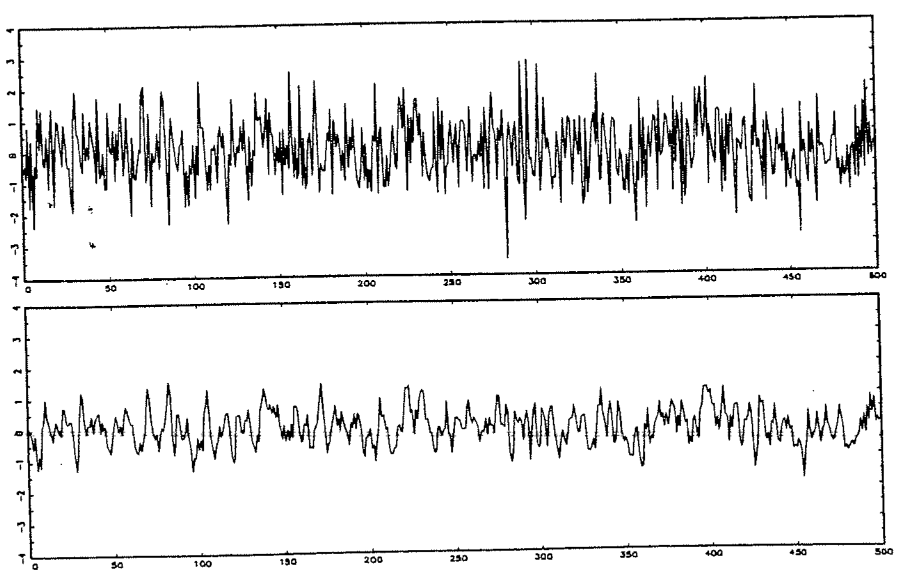
$$x[n] = \sum_{k=0}^{q} b[k]\, w[n-k]$$

such as

$$x[n] = \frac{1}{3} w[n] + \frac{1}{3} w[n-1] + \frac{1}{3} w[n-2] .$$

In general, an MA process is obtained by passing white noise through a finite impulse response (FIR) filter

$$w[n] \longrightarrow \boxed{b[k]} \longrightarrow x[n] .$$



white noise

filtered

3. <u>Autoregressive process</u>

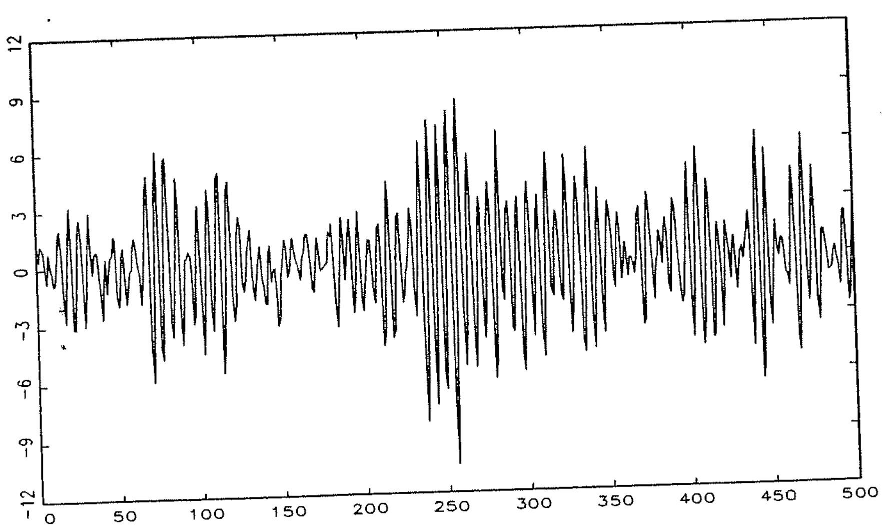
$$x[n] = \sum_{k=1}^{p} a[k]\, x[n-k] + w[n]$$

where $w[n]$ is white noise. An AR process is obtained by passing white noise through an IIR filter

$$w[n] \longrightarrow \boxed{\; a[k] \;} \longrightarrow x[n].$$

Simple example:

$$x[n] = x[n-1] - .9\, x[n-2] + w[n].$$



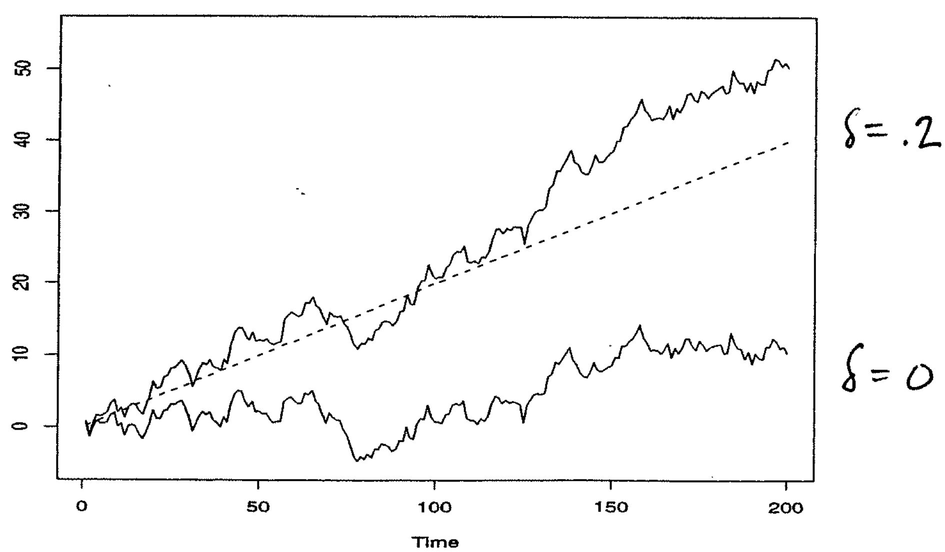
4. <u>Random walk (with drift)</u>

$$x[n] = \delta + x[n-1] + w[n]$$

where δ is fixed and called the "drift."

If $x[-1] = 0$, then for $n \geq 0$ we have

$$x[n] =$$



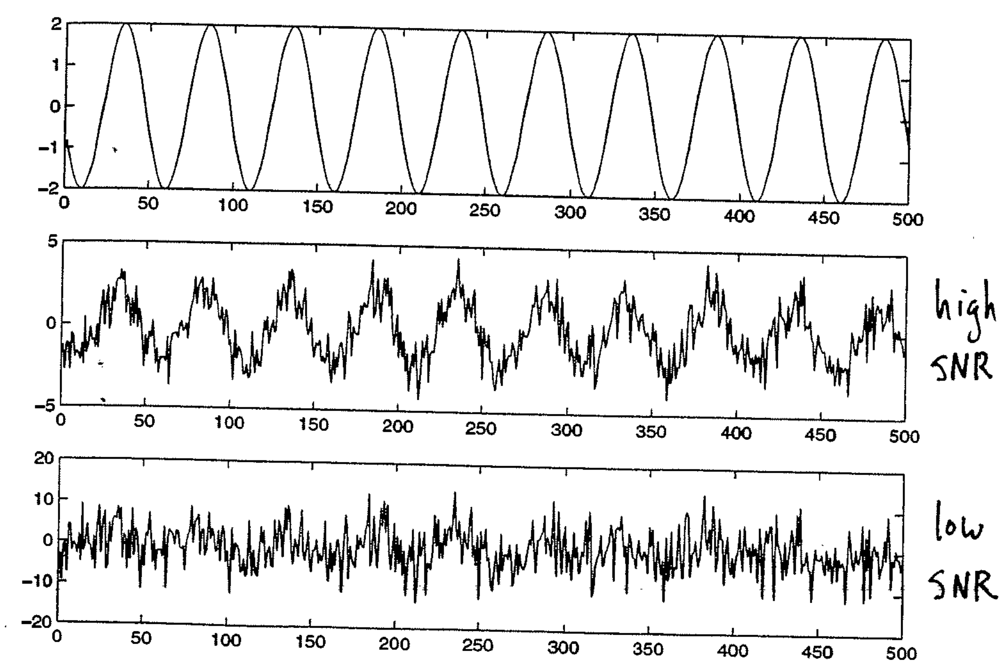
5. Signal in noise

$$x[n] = s[n] + w[n]$$

where $s[n]$ is deterministic and $w[n]$ is noise.

Example: $s[n] = A \cdot \cos(2\pi f n + \phi)$

The ratio of A to σ_w (or some function thereof) is called to "signal-to-noise ratio" (SNR).



Mean / Covariance / Correlation

The <u>mean</u> of a random process $x[n]$ is the deterministic function

$$\mu_x[n] = E\{x[n]\}.$$

The <u>autocovariance function</u> of $x[n]$ is

$$\gamma_x[m,n] = E\{(x[m]-\mu[m])(x[n]-\mu[n])\}.$$

The <u>autocorrelation function</u> (ACF) of $x[n]$ is

$$\rho_x[m,n] = \frac{\gamma[m,n]}{\sqrt{\gamma[m,m]\cdot\gamma[n,n]}}.$$

By the Cauchy-Schwarz inequality,

$|\gamma[m,n]|^2 \le \gamma[m,m]\cdot\gamma[n,n]$ and

therefore $-1 \le \rho[m,n] \le 1$.

<u>Notational remark</u>: The subscript x may be dropped when the RP is clear from context.

<u>Exercise</u> Consider $x[n] = \frac{1}{3}\left(w[n] + w[n-1] + w[n-2]\right)$ where $w[n] \sim wn(0,1)$. Determine $\mu[n]$, $\gamma[m,n]$.

<u>Solution</u> Clearly $\mu[n] \equiv 0$ and

$$\gamma[m,n] = E\{x[m] \cdot x[n]\}$$

$$= \frac{1}{9} E\Big\{\big(w[m] + w[m-1] + w[m-2]\big)\cdot\big(w[n]+w[n-1]+w[n-2]\big)\Big\}$$

Set $k = m-n$ and consider these cases:

$\boxed{k=0}$ $\quad \gamma[m,m] = \frac{1}{9}(1+1+1) = \frac{3}{9}$

$\boxed{k=1}$ $\quad \gamma[m,m-1] = \frac{1}{9}(1+1) = \frac{2}{9}$

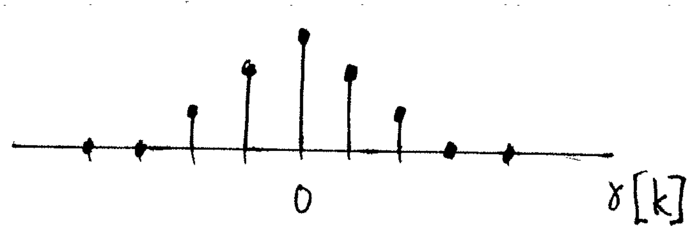
$\boxed{k=2}$ $\quad \gamma[m,m-2] = \frac{1}{9}(1) = \frac{1}{9}$

$\boxed{k\geq 3}$ $\quad \gamma[m,m-k] = 0.$

By symmetry, similar arguments apply to $k<0$.

In summary,

$$\gamma[m,n] = \begin{cases} 3/9 & m=n \\ 2/9 & m=n\pm 1 \\ 1/9 & m=n\pm 2 \\ 0 & |m-n|\geq 3 \end{cases}$$



Cross-covariance / correlation

Let $x[n]$ and $y[n]$ be two random processes.

The <u>cross-covariance</u> between $x[n]$ and $y[n]$ is

$$\gamma_{xy}[m,n] = E\left\{ \left(x[m] - \mu_x[m]\right)\left(y[n] - \mu_y[n]\right) \right\}.$$

The <u>cross-correlation function</u> (CCF) is

$$\rho_{xy}[m,n] = \frac{\gamma_{xy}[m,n]}{\sqrt{\gamma_x[m,m] \cdot \gamma_y[n,n]}}$$

These definitions can easily be extended to multiple (>2) processes $x_1[n], \ldots, x_T[n]$.

Stationarity

Conceptually, a random process is stationary if the distributional characteristics of a finite segment of that RP do not depend on when the segment was observed.

Examples:

Stationary: Cosmic background radiation

Nonstationary: Internet traffic

Formally, we say $x[n]$ is strictly stationary if, for any $n_1, \ldots, n_T$ and any k and any $c_1, \ldots, c_T$,

$$P\{x[n_1] \le c_1, \ \ldots, \ x[n_T] \le c_T\}$$

$$= P\{x[n_1+k] \le c_1, \ldots, x[n_T+k] \le c_T\}.$$

Implications:

$$\mu[n] = \mu$$

$$\gamma[m,n] = \gamma[m+k, n+k] \quad \text{for all } k.$$

Stationarity is a key assumption that will allow us to estimate properties of a RP from a single realization.

For example, suppose we observe $x[0], \ldots, x[N-1]$. What is the variance of $x[0]$? If $x[n]$ is not stationary, we're out of luck. If it is stationary, then we can use

$$\widehat{Var}\{x[0]\} = \frac{1}{N} \sum_{k=0}^{N-1} (x[k] - \hat{\mu})^2$$

where

$$\hat{\mu} = \frac{1}{N} \sum_{k=0}^{N-1} x[k].$$

Moreover, the variances of these estimates decrease as $N \to \infty$.

Technically, no real-world signal is stationary because no real-world signal has infinite duration. However, as long as the distributional characteristics don't change too rapidly, the approximation is still useful.

Strict stationarity is usuall difficult to verify from a single realization of $x[n]$ and furthermore it is stronger than we'll need.

Instead, we'll assume a weaker condition that only imposes restrictions on the first and second (but not higher order) moments of $x[n]$:

We say $x[n]$ is <u>weakly stationary</u> if

(a) $\mu[n]$ is constant

(b) $\gamma[m,n] = \gamma[m+k, n+k]$ for all k

Sometimes $x[n]$ is also called <u>wide-sense stationary</u> or simply <u>stationary</u> instead of weakly stationary.

<u>Example</u>: The moving average process

$$x[n] = \frac{1}{3}\left(w[n] + w[n-1] + w[n-2]\right)$$

is stationary.

<u>Note</u>: Strongly stationary $\Rightarrow$ weakly stationary.
The converse is generally not true except for Gaussian processes.

Joint Stationarity

We say $x[n]$ and $y[n]$ are <u>jointly stationary</u> if they are both stationary and if

$$\gamma_{xy}[m,n] = \gamma_{xy}[m+k, n+k] \quad \text{for all } k.$$

Lag Notation

When stationarity holds it is customary to write

$$\gamma_x[k] = \gamma_x[m, m+k]$$

$$\rho_x[k] = \rho_x[m, m+k] = \frac{\gamma_x[k]}{\gamma_x[0]}$$

$$\gamma_{xy}[k] = \gamma_{xy}[m, m+k]$$

$$\rho_{xy}[k] = \rho_{xy}[m, m+k] = \frac{\gamma_{xy}[k]}{\sqrt{\gamma_x[0] \cdot \gamma_y[0]}}$$

Here k is called the <u>lag</u>

<u>Remark</u>: Kay assumes stationarity from the onset and calls

$$r_{xx}[k] = E\{x[m]\,x[m+k]\}$$

the auto-correlation function (ACF).

<u>Exercise</u>: Verify the following properties

(a) $\gamma_x[0] = E\{(x[n]-\mu)^2\} \geq 0$

(b) $|\gamma_x[k]| \leq \gamma_x[0]$ for all k

(c) $\gamma_x[-k] = \gamma_x[k]$

(d) $\gamma_{xy}[-k] = \gamma_{yx}[k]$

Complex Random Processes

Sometimes it is convenient to consider random processes taking on complex values. The definitions for γ_x, ρ_x, γ_{xy}, ρ_{xy} can be easily extended to complex RPs by introducing complex conjugation appropriately. For example,

$$\gamma_x[m,n] = E\left\{ (x[m]-\mu[m])^* (x[n]-\mu[n]) \right\}.$$

The following properties still hold:

(a) $\gamma[0] \geq 0$

(b) $|\gamma[k]| \leq \gamma[0]$

(c) $\gamma[-k] = \gamma[k]^*$

(d) $\gamma_{xy}[-k] = \gamma_{yx}[k]^*$

Notice: Figures borrowed from Shumway + Stoffer (2006).