The Power Spectral Density: Properties and Interpretations.

The Power Spectral Density

Let $x[n]$ be a stationary random process. The <u>power spectral density</u> (PSD, spectral density, power spectrum, spectrum) is the DTFT of $\gamma_x[k]$:

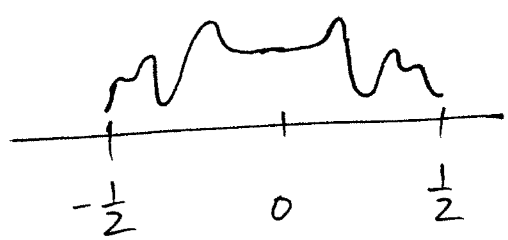
$$P_x(f) = \sum_{k=-\infty}^{\infty} \gamma_x[k]\, e^{-2\pi i f k}, \qquad -\tfrac{1}{2} \le f \le \tfrac{1}{2}$$

provided $\sum_{k=-\infty}^{\infty} |\gamma[k]| < \infty$.

Basic Properties

1. $P_x(f)$ is periodic with period 1.

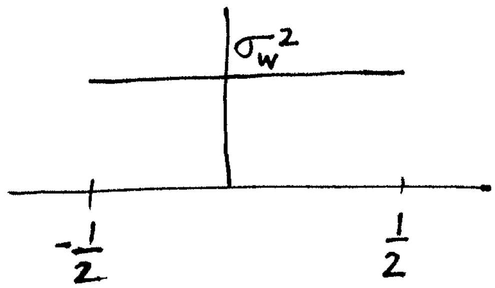
2. If $x[n]$ is real-valued, then $P_x(f)$ is even, i.e. $P_x(-f) = P_x(f)$.



For these reasons, it is common to plot the PSD on $[0, 1/2]$.

3. If $w[n]$ is white noise with variance σ_w^2, then

$$P_w(f) = \gamma_w[0] = \sigma_w^2$$



So where does the name "power spectral density" come from?

Density

The name is appropriate because of two properties:

1. $\gamma_x[0] = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_x(f)\,df$

2. $P_x(f) \geq 0$ for all f

To establish the first property, recall the inverse DTFT:

$$\gamma[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_x(f) e^{2\pi i f k} \, df.$$

Now evaluate at $k=0$.

Establishing the second is more involved. It follows from two results:

A. For any stationary RP, $\gamma[k]$ is non-negative definite.

B. A complex-valued function $\gamma[k]$ is non-negative definite if and only if its Fourier transform (DTFT) is non-negative everywhere.

Non-negative definite means that for any $k_1, \ldots, k_T$ and $a_1, \ldots, a_T$,

$$\sum_{i=1}^{T} \sum_{j=1}^{T} a_i^* \gamma[k_i - k_j] a_j \geq 0.$$

This is true for autocovariance functions of WSS RPs because

$$\sum_i \sum_j a_i^* \gamma[k_i - k_j] a_j = E\left\{ \left| \sum_{i=1}^{T} a_i^* x[k_i] \right|^2 \right\} \geq 0.$$

This establishes A. B. is known as Herglotz's Theorem and is proved in Brockwell and Davis or Shumway and Stoffer.

Spectral

Theorem: If $\sum_{k=-\infty}^{\infty} |\gamma_x[k]| < \infty$ and $\mu[n] = 0$ then

$$P_x(f) = \lim_{M \to \infty} E\left\{ \frac{1}{2M+1} \left| \sum_{k=-M}^{M} x[k] e^{-2\pi i f k} \right|^2 \right\}.$$

Thus, the PSD is like the "expected magnitude-squared of the Fourier transform of $x[n]$."

Proof: RHS (right-hand side)

$$= \lim_{M\to\infty} E\left\{ \frac{1}{2M+1} \sum_{m=-M}^{M} \sum_{n=-M}^{M} x[m]\, x[n]^* e^{-2\pi i f(m-n)} \right\}$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_{m=-M}^{M} \sum_{n=-M}^{M} \gamma_x[m-n]\, e^{-2\pi i f(m-n)}$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_{k=-2M}^{2M} (2M+1-|k|)\, \gamma_x[k]\, e^{-2\pi i f k}$$

$$= \lim_{M\to\infty} \sum_{k=-2M}^{2M} \left(1 - \frac{|k|}{2M+1}\right) \gamma_x[k]\, e^{-2\pi i f k}$$

$$= \sum_{k=-\infty}^{\infty} \gamma_x[k]\, e^{-2\pi i f k}$$

$$= P_x(f) \qquad \square.$$

Power

We saw earlier that $\gamma_x[0] = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_x(f)\, df$, so the expected power of $x[n]$ (i.e., the variance) is given by the area under the PSD.

But what about the power at a certain frequency?

Suppose $y[k] = h[k] * x[k]$, where $h[k]$ is a filter (deterministic). It can be shown that

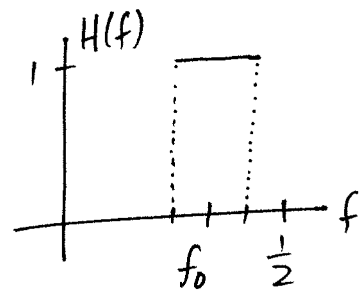
$$\gamma_y[k] = h[k] * h^*[-k] * \gamma_x[k]$$

and

$$P_y(f) = |H(f)|^2 \cdot P_x(f)$$

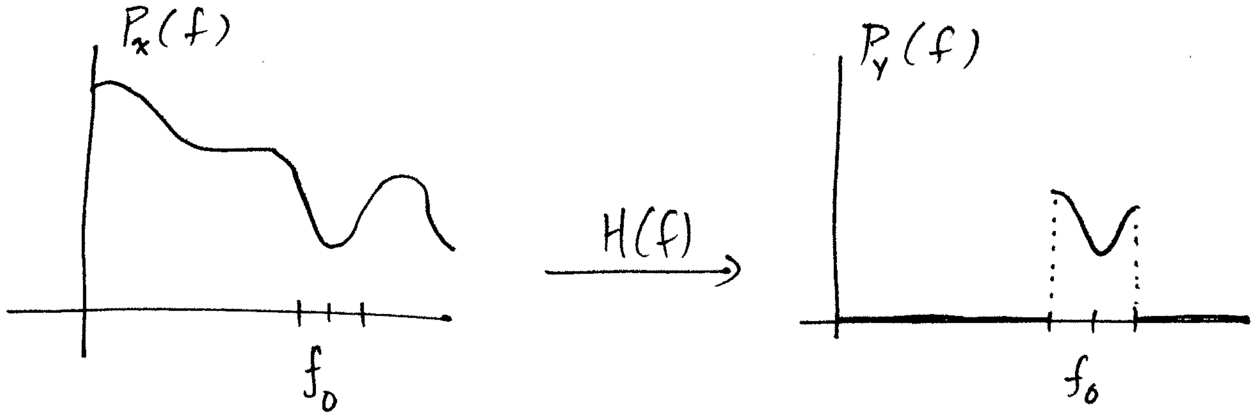
Now take $h[k]$ to be a narrowband filter with center frequency f_0 and bandwidth B:

$$H(f) = \begin{cases} 1 & |f - f_0| \le \frac{B}{2} \\ 0 & \text{otherwise} \end{cases}$$



Then the "power" of $y[k]$, which is the power in $x[n]$ near the frequency f_0, is

$$\sigma_y^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_y(f)\,df = \int_{f_0 - \frac{B}{2}}^{f_0 + \frac{B}{2}} P_x(f)\,df$$



This is another way to understand how the PSD captures the frequency content of a RP.

We can also apply this filtering idea to moving average models. In particular, if

$$w[n] \longrightarrow \boxed{h[k]} \longrightarrow x[n]$$

where $w[n] \sim wn(0, \sigma_w^2)$ and $h[k]$ is FIR, so that $x[n]$ is MA, then

$$P_x(f) = |H(f)|^2 \cdot \sigma_w^2.$$

This idea will be key in our study of parametric spectral estimation using ARMA models.

Cross-Spectrum

Technically, we should call the PSD the auto-PSD. If $x[n]$, $y[n]$ are two RPs, their cross-PSD is defined as

$$P_{xy}(f) = \sum_{k=-\infty}^{\infty} \gamma_{xy}[k] e^{-2\pi i f k}.$$

Unlike the PSD, the cross-PSD is in general complex-valued.

From Kay: "The magnitude of the cross-PSD describes whether frequency components in $x[n]$ are associated with large or small amplitudes at the same frequency in $y[n]$, and the phase of the cross-PSD indicates the phase lag or lead of $x[n]$ with respect to $y[n]$ for a given frequency component."