Parametric

Modeling

Parametric Modeling

- Nonparametric (Classical) S.E.:
  - → No assumptions about random process
- Parametric S.E.
  - → Assumes specific parametric model
  - → Better resolution if model is accurate
  - → Worse performance if model is off.

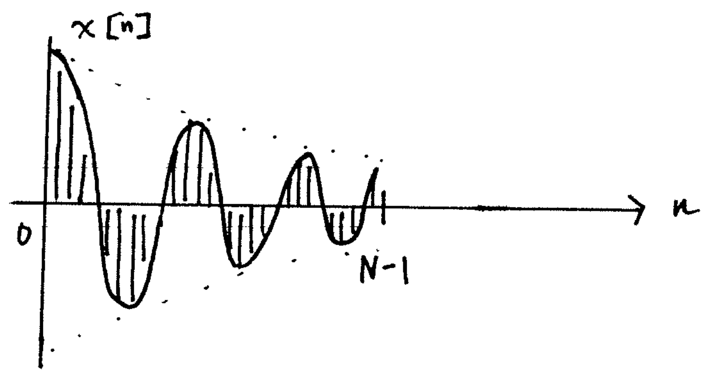
<u>Intuition</u>: Extending the data

Classical S.E. involves windowing the data or sample ACF (even basic periodogram corresponds to rectangular window)

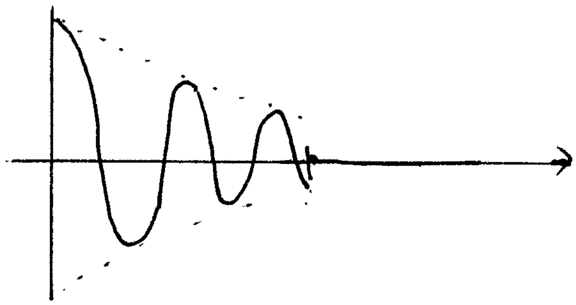
$\Rightarrow$ signal is assumed $= 0$ off of $0, 1, \ldots, N-1$

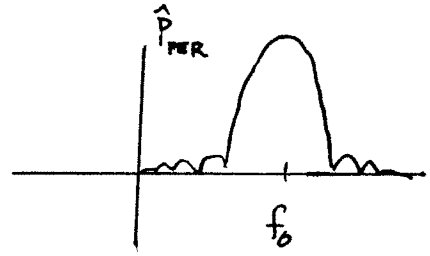
In many cases, however, zero padding is unreasonable.

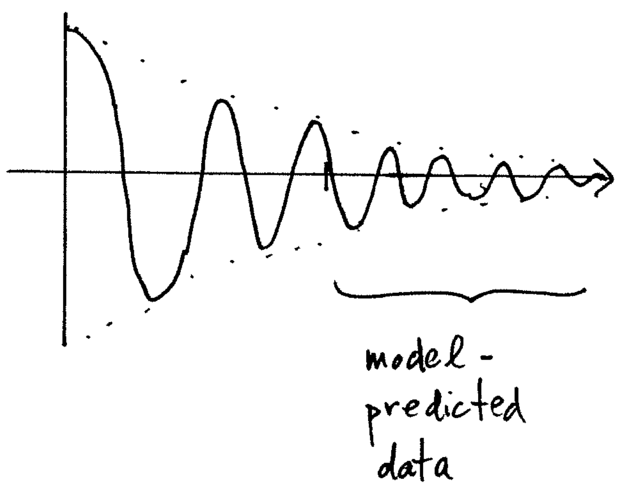
Suppose we observe

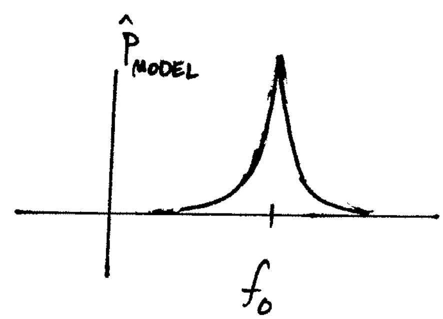


Now suppose we had the knowledge that $x[n]$ is the amplitude of a damped, resonant system:









Windowing / zero-padding $\Longrightarrow$ smearing of PSP

$$E\{\hat{P}_{PER}(f)\} = P_x(f) \star W(f)$$

Accurate modeling $\Longrightarrow$ greater resolution

Inaccurate modeling $\Longrightarrow$ systematic bias

In practice, parametric S.E. entails three steps:

STEP 1 Select a model described by a finite number of parameters $\theta = (\theta_1, \dots, \theta_p)$.

Factors to consider:

(a) Size of model: large $\Longrightarrow$ good accuracy but harder to estimate

(b) type of model: some are easier to fit than others

STEP 2 Use available data to estimate parameters in the model. This is often challenging and the subject of Kay 6–10.

STEP 3 Given the parameters, compute the PSD of the model. This step is straightforward.

Recall

If $w[n]$ is white noise and

$$w[n] \longrightarrow \boxed{H(f)} \longrightarrow x[n]$$

then $P_x(f) = |H(f)|^2 \sigma_w^2$.

If H is taken from a sufficiently flexible class, it can approximate arbitrary PSDs.

We will study models corresponding to <u>rational transfer functions</u>, which are parametrized by $a[1], \ldots, a[p]$, $b[0], b[1], \ldots, b[q]$

$$\boxed{H(z) = \frac{\sum_{k=0}^{q} b[k] z^{-k}}{1 + \sum_{k=1}^{p} a[k] z^{-k}} = \frac{B(z)}{A(z)}}$$

quotient of two polynomials in z^{-1}

ARMA (p,q) Models

Let $w[n]$ be white noise. An <u>autoregressive moving average</u> process of order p, q, is generated by

$$x[n] = -\sum_{k=1}^{p} a[k]\,x[n-k] + \sum_{k=0}^{q} b[k]\,w[n-k].$$

Take z-transform of both sides:

$$X(z) = -\sum_{k=1}^{p} a[k]\,X(z)\,z^{-k} + \sum_{k=0}^{q} b[k]\,W(z)\,z^{-k}$$

$$\Rightarrow \quad X(z) = H(z)\cdot W(z)$$

where

$$H(z) = \frac{\sum_{k=0}^{q} b[k]\,z^{-k}}{1 + \sum_{k=1}^{p} a[k]\,z^{-k}} \quad \leftarrow \boxed{\text{rational transfer function}}$$

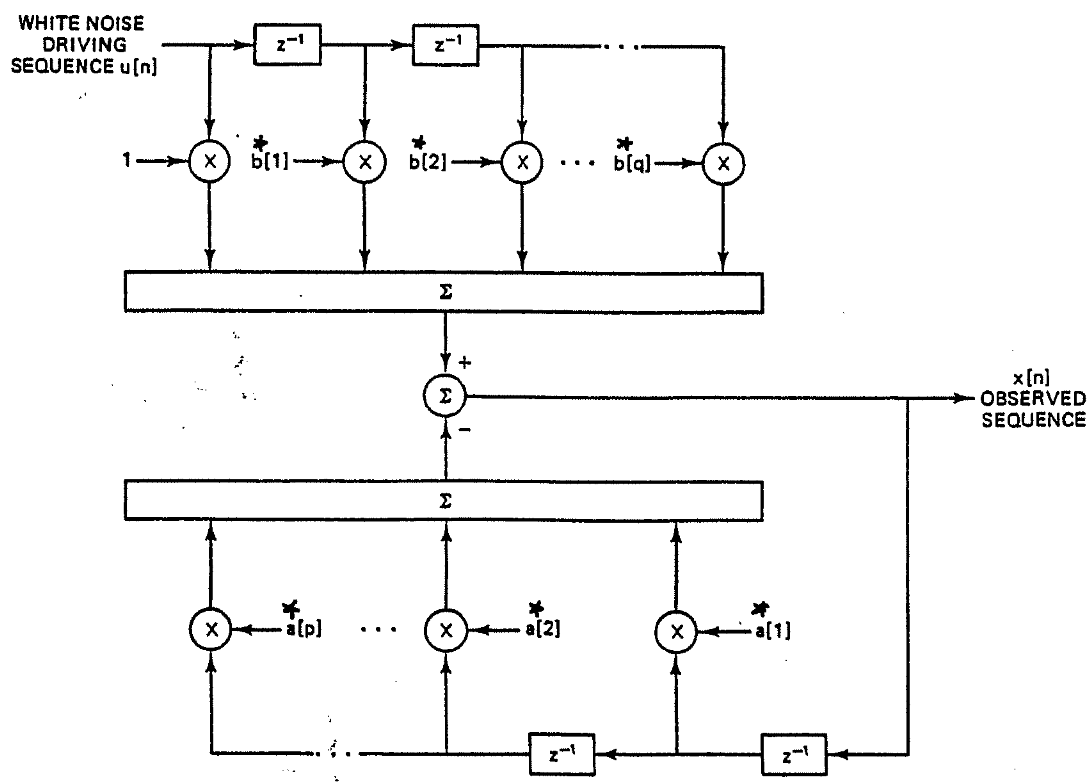
Hence, an ARMA (p,q) model is the result of passing white noise through an LTI system with a rational transfer function.

Is $x[n]$ WSS?

Special Cases:

- $p = 0 \implies$ Moving average process of order q ($MA(q)$ process) $\implies$ FIR filter

- $q = 0 \implies$ Autoregressive process of order p ($AR(p)$ process) $\implies$ IIR filter



From: S. M. Kay, Modern Spectral Estimation, Prentice-Hall, 1988.

Remark: Do not confuse $w[n]$, the "driving noise," with "observation noise," which is unwanted.

PSD of ARMA(p,q)

If $x[n]$ is ARMA(p,q), then

$$P_x(f) = \sigma_w^2 \cdot |H(f)|^2 = \sigma_w^2 \left|\frac{B(f)}{A(f)}\right|^2$$

where
$$B(z) = \sum_{k=0}^{q} b[k] z^{-k}$$
$$A(z) = 1 + \sum_{k=0}^{p} a[k] z^{-k}$$

and
$$B(f) \to B(z) \text{ with } z = e^{2\pi i f}$$
$$A(f) \to A(z) \text{ with } z = e^{2\pi i f}$$

Remark: By convention we will take $b[0] = 1$ since any gain can be incorporated into σ_w^2.

Poles and Zeros

A general ARMA(p,q) model is sometimes called a <u>pole-zero model</u> because $H(z)$ has both poles and zeros.

$$H(z) = \frac{B(z)}{A(z)}$$

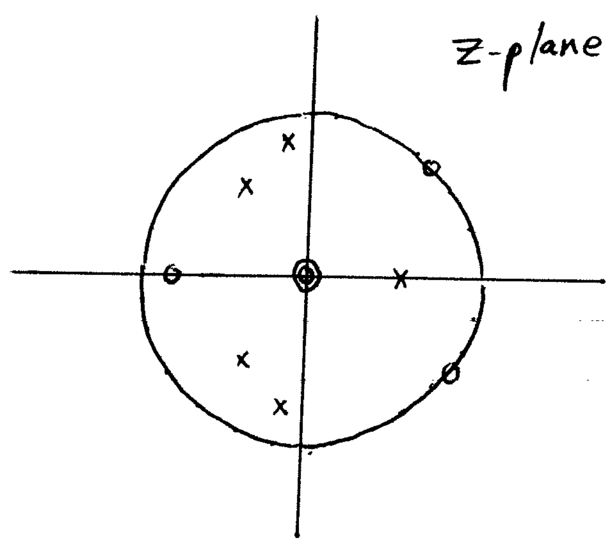
$$= \frac{(1-\beta_1 z^{-1}) \cdots (1-\beta_q z^{-1})}{(1-\alpha_1 z^{-1}) \cdots (1-\alpha_p z^{-1})}$$

$$= z^{-q+p} \frac{(z-\beta_1) \cdots (z-\beta_q)}{(z-\alpha_1) \cdots (z-\alpha_p)}$$

$\beta_1, \dots, \beta_q \implies$ zeros of H

$\alpha_1, \dots, \alpha_p \implies$ poles of H

Exercise: Sketch PSD



Review the following facts:

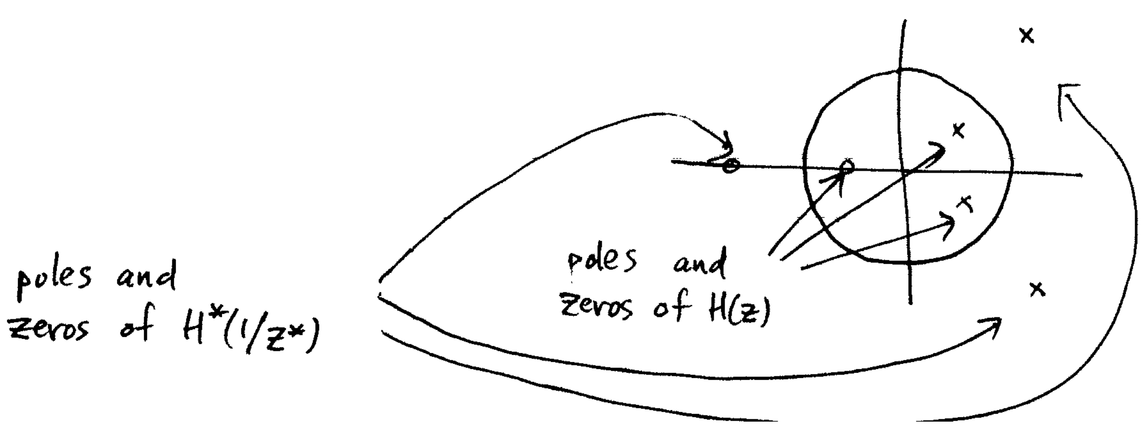
1. If $b[1], \ldots, b[q]$ are real, then $\beta_1, \ldots, \beta_q$ occur in conjugate pairs. Similarly for the poles.

2. $H(z)$ is stable and causal iff $|\alpha_i| < 1 \quad \forall i$

3. $H(z)$ is minimum phase iff $|\beta_i| \leq 1 \quad \forall i$. The inverse filter $\dfrac{A(z)}{B(z)}$ is also stable and causal if strict inequality holds $\forall i$.

4. $P_X(z) = H(z) \cdot H^*(1/z^*)$

5. Given any rational spectrum $P_X(z)$, there is a unique representation of P_X in terms of a stable, causal, and minimum phase filter H. This is given by the spectral factorization of $P_X(z)$



All-Pole Models

An AR(p) model is sometimes called an <u>all-pole</u> model because

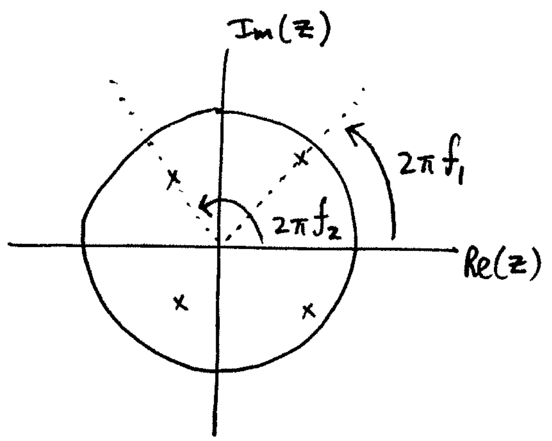
$$H(z) = \frac{1}{A(z)}$$

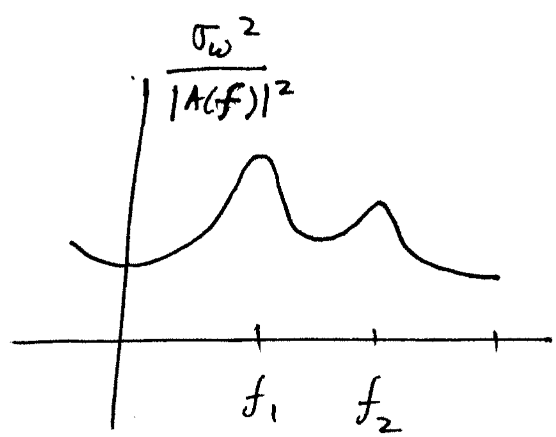
trivial zeros at origin

$$= \frac{1}{z^{-p}(z-\alpha_1)\cdots(z-\alpha_p)}$$

$\Longrightarrow$ $P_x(f)$ is a series of <u>peaks</u>

<u>Example</u>





Peak location $\longrightarrow$ angle of pole

Peak height $\longrightarrow$ inversely proportional to distance of pole to unit circle

All-Zero Models

An MA(q) model is also called an <u>all-zero model</u>:

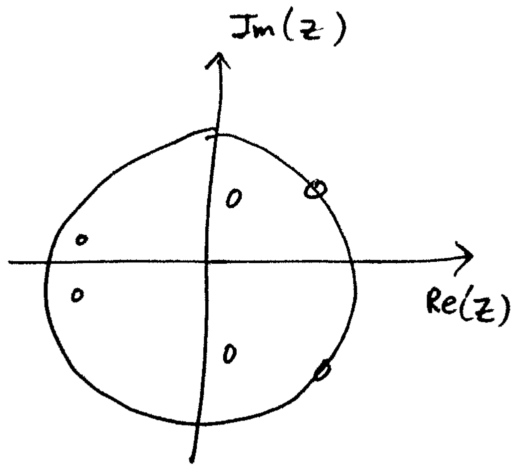
$$H(z) = B(z)$$

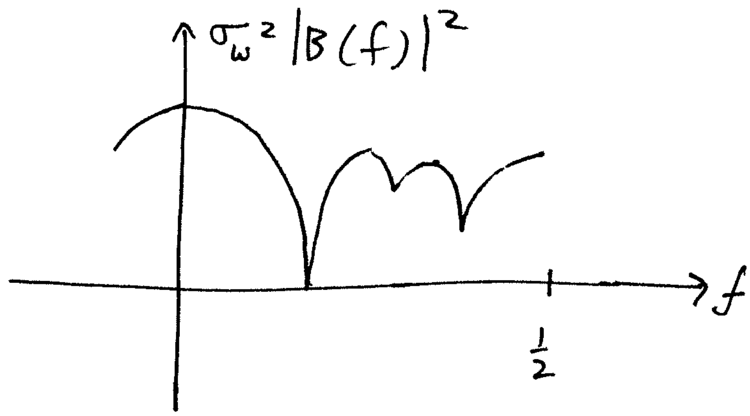
$$= z^{-q}(z-\beta_1)\cdots(z-\beta_q)$$

$\boxed{\text{trivial poles at origin}}$

$\Longrightarrow$ $P_x(f)$ is a series of <u>nulls</u>

<u>Example</u>





null location $\longrightarrow$ angle of zero

null depth $\longrightarrow$ inversely proportional to distance of zero to unit circle.

Pole-Zero Models $\Longrightarrow$ both peaks and nulls
(ARMA(p,q))

Theory

Any ARMA(p, q) process can be represented by a unique

(a) MA process of infinite order

(b) AR process of infinite order

(Equivalently, given an infinite filter you can match any desired frequency response).

Implication: Even if we specify the wrong model, we may still have a decent approximation if the order is large enough

This is important because

AR Models :	easy to fit
MA Models :	hard to fit

Exercise / Write $\dfrac{1+b[1]z^{-1}}{1+a[1]z^{-1}}$ as an MA (∞) process.

$\uparrow$ ARMA (1, 1)

Solution / We want to solve for $d[k]$ where

$$\frac{1+b[1]z^{-1}}{1+a[1]z^{-1}} = 1+d[1]z^{-1}+d[2]z^{-2}+\cdots$$

Recall the identity

$$\frac{1}{1-x} = 1+x+x^2+\cdots$$

Let $x = -a[1]z^{-1}$

$$\Rightarrow \frac{1+b[1]z^{-1}}{1+a[1]z^{-1}} = (1+b[1]z^{-1})(1-a[1]z^{-1}+a[1]^2z^{-2}-\ldots)$$

$$= 1+(b[1]-a[1])z^{-1}$$

$$+(b[1]-a[1])(-a[1])z^{-2}$$

$$\vdots$$

$$\Rightarrow d[k] = \begin{cases} 1 & k=0 \\ (b[1]-a[1])(-a[1])^{k-1} & k\geq 1 \end{cases}$$

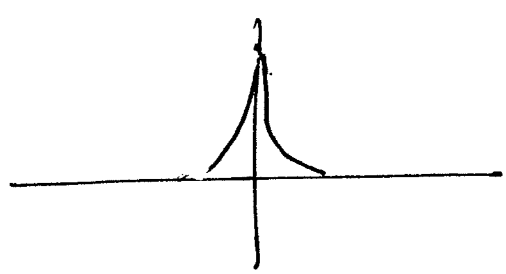
What happens (in the example) as $a[1] \to 1$?

Then $d[k]$ decays more slowly, meaning we need a larger q to have a good approximation.

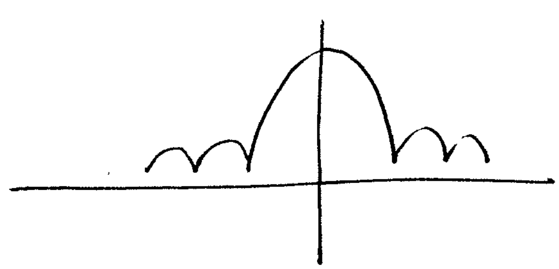
This makes sense, because as $a[1] \to 1$, the peak at 0 gets sharper and sharper,

(assuming $a[1]$ is real and positive)

So it gets more difficult to approximate



with a superposition of things that look like



Autocovariance of ARMA(p,q)

Let $x[n]$ be an ARMA(p,q) process:

$$x[n] = -\sum_{\ell=1}^{p} a[\ell]\, x[n-\ell] + \sum_{\ell=0}^{q} b[\ell]\, w[n-\ell]$$

Now multiply both sides by $x^*[n-k]$ and take $E\{\cdot\}$

$$\gamma_x[k] = -\sum_{\ell=1}^{p} a[\ell]\, \gamma_x[k-\ell] + \sum_{\ell=0}^{q} b[\ell]\, \gamma_{xw}[k-\ell]$$

For $k > 0$ we have

$$\gamma_{xw}[k] = 0 \qquad (\text{Why?})$$

For $k \le 0$ we have

$$\begin{aligned}
\gamma_{xw}[k] &= E\left\{ w[n]\, x^*[n-k] \right\} \\
&= E\left\{ w[n] \cdot \sum_{\ell=0}^{\infty} h^*[\ell]\, w^*[n-k-\ell] \right\} \\
&= \sum_{\ell=0}^{\infty} h^*[\ell]\, \gamma_{ww}[k+\ell] \\
&= h^*[-k] \cdot \sigma_w^2
\end{aligned}$$

causal filter corresponding to $H(z)$

Therefore

$$\sum_{l=0}^{q} b[l]\,\gamma_{xw}[k-l] = \sum_{l=k}^{q} b[l]\,\gamma_{xw}[k-l]$$

$$= \sigma_w^2 \sum_{l=k}^{q} b[l]\,h^*[l-k]$$

$$= \sigma_w^2 \sum_{l=0}^{q-k} b[l+k]\,h^*[l]$$

In summary, we get

$$\boxed{\gamma_{xx}[k] = \begin{cases} -\sum_{l=1}^{p} a[l]\,\gamma_{xx}[k-l] + \sigma_w^2 \sum_{l=0}^{q-k} b[l+k]\,h^*[l], & 0 \le k \le q \\ -\sum_{l=1}^{p} a[l]\,\gamma_{xx}[k-l] & k \ge q+1 \\ \gamma^*[-k] & k < 0 \end{cases}}$$

where $h[n] = \mathcal{Z}^{-1}\{H(z)\}$.

Note that $h[k]$ is a function of $\{a[k]\}, \{b[k]\}$:

$$\frac{\sum_{k=0}^{q} b[k] z^{-k}}{1 + \sum_{k=1}^{p} a[k] z^{-k}} = \sum_{k=0}^{\infty} h[k] z^{-k}$$

Thus, for ARMA and MA processes $(q > 0)$, these equations are <u>nonlinear</u> in the model parameters.

However, for AR models $(q = 0)$, the equations are <u>linear</u>. This makes it much easier to fit AR models.

$$\boxed{\gamma_x[k] = \begin{cases} -\sum_{\ell=1}^{p} a[\ell]\gamma_x[k-\ell] & k \geq 1 \\ -\sum_{\ell=1}^{p} a[\ell]\gamma_x[-\ell] + \sigma_w^2 & k = 0 \\ \gamma_x^*[-k] & k < 0 \end{cases}}$$

These are the <u>Yule-Walker equations</u>.

<u>Exercise</u> Consider an AR(1) process.

Express the ACV and PSD as functions of $a[1]$.

Sketch the ACV for real $a[1]$, and the PSD for complex $a[1]$.

Solution / For $k \geq 1$

$$\gamma_x[k] = -a[1]\gamma_x[k-1]$$

$$= (-a[1])^2 \gamma_x[k-2]$$

$$\vdots$$

In general,

$$\gamma_x[k] = \gamma_x[0] \cdot (-a[1])^{|k|}$$

We also know

$$\gamma_x[0] = -a[1]\gamma_x[-1] + \sigma_w^2$$

$$= -a[1]\gamma_x^*[1] + \sigma_w^2$$

$$= -a[1](-a^*[1]\gamma_x[0]) + \sigma_w^2$$

$$\Longrightarrow \gamma_x[0] = \frac{\sigma_w^2}{1-|a[1]|^2}$$

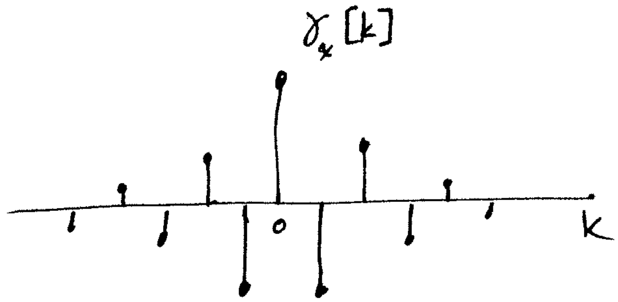
$$\Longrightarrow \gamma_x[k] = \frac{\sigma_w^2}{(1-|a[1]|^2)}(-a[1])^{|k|}$$

PSD:

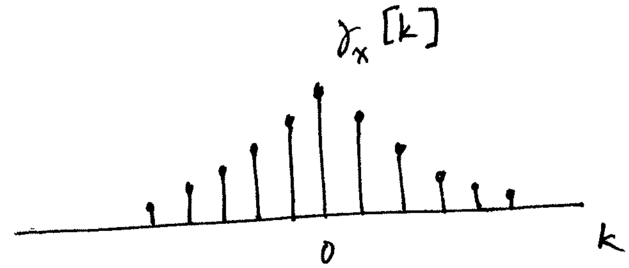
$$P_x(f) = \frac{\sigma_w^2}{|A(f)|^2} = \frac{\sigma_w^2}{\left|1 + a[1]e^{-2\pi i f}\right|^2}$$

For real $a[1]$:

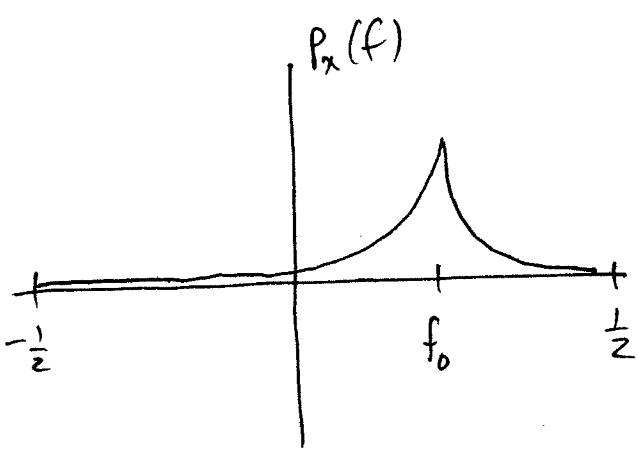
$\boxed{a[1] > 0}$

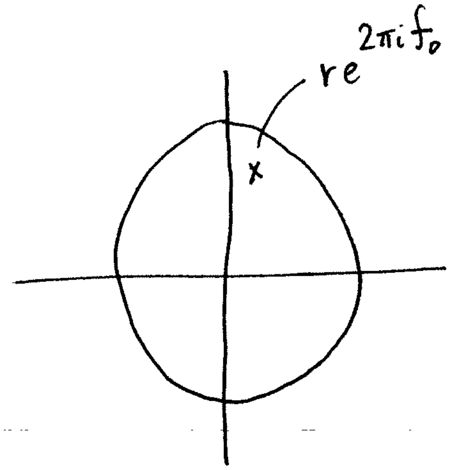


$\boxed{a[1] < 0}$



PSD:





pole-zero diagram

since $a[1]$ is complex,
no symmetry of PSD