Sinusoidal

Parametric

Modeling

Sinusoidal Parametric Modeling

Assume the observed signal is a linear combination of p complex sinusoids in white complex Gaussian noise:

$$x[n] = \sum_{j=1}^{n} A_j \exp\left\{ i \left(2\pi f_j n + \phi_j \right) \right\} + z[n]$$

where

$A_j, f_j \longrightarrow$ unknown and desired

$\phi_j \longrightarrow$ unknown and possibly desired.

$z[n] \longrightarrow$ CWGN with variance σ_z^2.

Like ARMA Modeling :

This is a parametric model. If we can estimate the parameters of the model, then the corresponding spectral estimate is

$$\hat{P}_{SIN}(f) =$$

$$\hat{P}_{SIN}(f) = \sum_{j=1}^{p} A_j^2 \delta(f - f_j) + \sigma_z^2$$

assuming ϕ_i are unif $[0, 2\pi)$ and independent.

(The ACV is

$$\gamma_x[k] = \sum_{j=1}^{p} A_j^2 \exp\{2\pi i f_j k\} + \sigma_z^2 \delta[k] \quad)$$

Unlike ARMA Modeling:

The model needs to be very accurate for the method to be effective. Whereas ARMA models can approximate arbitrary spectra to any specified accuracy by taking sufficiently high order models, the same is not true for the sinusoidal model. An arbitrary spectrum cannot be written as a countably infinite ($p = \infty$) sum of delta functions.

Sinusoidal Parameter Estimation

Two cases:

$\boxed{A}$ $f_1, \dots, f_p$ known $\Longrightarrow$ MLE has nice form $\Longrightarrow$ easy

$\boxed{B}$ $f_1, \dots, f_p$ unknown $\Longrightarrow$ MLE requires difficult, nonlinear optimization $\Longrightarrow$ hard.

Case $\boxed{B}$ is most common in practice.
Therefore, the primary challenge is <u>frequency estimation</u>.
We will discuss three classes of methods for this:

1. Tackle the nonlinear MLE problem
2. Fit an AR model and find the peaks
3. If $p < N$, then $x[n]$ belongs to a <u>signal subspace</u> that is characterized by the eigenvectors of the ACV matrix.

Once the frequencies have been estimated, we're in the easy case $\boxed{A}$ and can apply MLE.

Maximum Likelihood Estimation

Let's first consider the case of a <u>single sinusoid</u>

$$x[n] = A \exp\{i(2\pi f n + \phi)\} + z[n]$$

In vector notation,

$$\underline{x} = A_c \cdot \underline{e}(f) + \underline{z}$$

where

$$A_c = A e^{i\phi}$$

$$\underline{e}(f) = [1 \quad e^{2\pi i f} \quad e^{4\pi i f} \quad \cdots \quad e^{2\cdot(N-1)\pi i f}]^T.$$

The likelihood function for $\underline{\theta} = [A_c, f]^T$ is

$$l(A_c, f) = \frac{1}{(\pi \sigma_z^2)^N} \exp\left\{-\frac{(\underline{x} - A_c \underline{e}(f))^H (\underline{x} - A_c \underline{e}(f))}{\sigma_z^2}\right\}.$$

Therefore, we need to minimize

$$(\underline{x} - A_c \underline{e}(f))^H (\underline{x} - A_c \underline{e}(f))$$

with respect to $A_c \in \mathbb{C}$, $f \in [-\frac{1}{2}, \frac{1}{2})$.

Previously we saw that for fixed f, the minimizing A_c is

$$\hat{A}_c(f) = \frac{\underline{e}(f) \cdot \Gamma_{zz}^{-1} \, \underline{x}}{\underline{e}(f) \, \Gamma_{zz}^{-1} \, \underline{e}(f)} = \frac{\underline{e}(f) \, \underline{x}}{\underline{e}(f) \, \underline{e}(f)}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \exp(-2\pi i f n)$$

Exercise

① What are the MLEs of A, ϕ, assuming f is known?

② What is $(\underline{x} - \hat{A}_c \, \underline{e}(f))^H (\underline{x} - \hat{A}_c \, \underline{e}(f))$?

Solution

(i) $$\hat{A} = |\hat{A}_c| = \left| \frac{1}{N} \sum_{n=0}^{N-1} x[n] \exp\{-2\pi i \hat{f} n\} \right|$$

$$\hat{\phi} = \arg \hat{A}_c = \tan^{-1}\left[\frac{\mathrm{Im}\left(\frac{1}{N} \sum x[n] \exp\{-2\pi i \hat{f} n\} \right)}{\mathrm{Re}\left(\frac{1}{N} \sum x[n] \exp\{-2\pi i \hat{f} n\} \right)} \right]$$

(2) $$(\underline{x} - \hat{A}_c \underline{e}(f))^H (\underline{x} - \hat{A}_c \underline{e}(f))$$

$$= \underline{x}^H \cdot (\underline{x} - \hat{A}_c \underline{e}(f)) - \underbrace{\hat{A}_c^* \underline{e}(f)^H (\underline{x} - \hat{A}_c \underline{e}(f))}$$

$$\boxed{\begin{aligned} &= \hat{A}_c^* \cdot (N \cdot \hat{A}_c) - \hat{A}_c^* \cdot \hat{A}_c \cdot N \\ &= 0 \end{aligned}}$$

$$= \underline{x}^H \underline{x} - \hat{A}_c \cdot \underline{x}^H \underline{e}(f)$$

$$= \underline{x}^H \underline{x} - \frac{1}{N} |\underline{e}(f)^H \underline{x}|^2$$

<u>Conclusion</u>: The MLE of f is obtained by maximizing

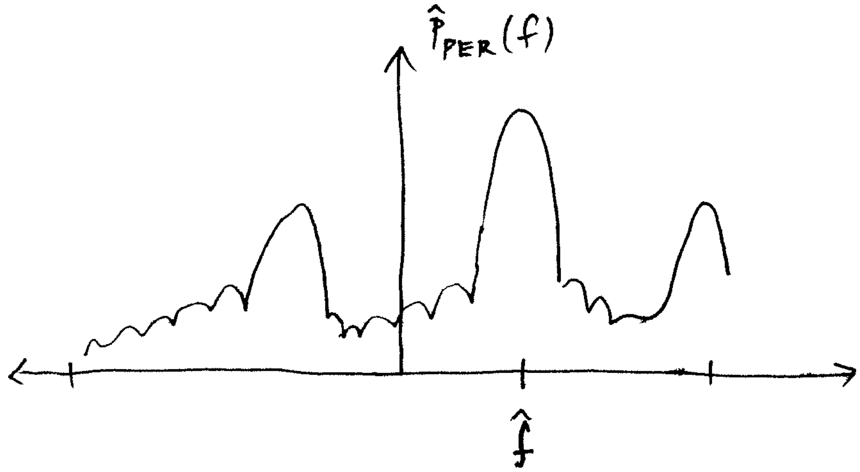
$$\frac{1}{N}\left| \underline{e}(f)^H \underline{x} \right|^2$$

$$= \frac{1}{N}\left| \sum_{n=0}^{N-1} x[n] \exp\{-2\pi i f n\} \right|^2$$

$$= \hat{P}_{PER}(f).$$

In other words,

$$\boxed{\hat{f}_{MLE} = \arg\min_{f \in [-\frac{1}{2}, \frac{1}{2})} \hat{P}_{PER}(f)}$$



Multiple Sinusoids

The preceeding method can be generalized.

Assume

$$x[n] = \sum_{j=1}^{P} A_c^j \exp\{2\pi i f_j n\} + z[n]$$

In vector notation:

$$\underline{x} = \sum_{j=1}^{P} A_c^j \, \underline{e}(f_j) + \underline{z}$$

where

$$A_c^j = A_j e^{i\phi_j}$$

$$\underline{e}(f_j) = [1 \quad e^{2\pi i f_j} \quad \cdots \quad e^{2\pi i f_j (N-1)}]^T$$

In matrix notation:

$$\underline{x} = E \cdot \underline{A}_c + \underline{z}$$

where

$$E = \begin{bmatrix} \text{---} \ \underline{e}(f_1) \ \text{---} \\ \vdots \\ \text{---} \ \underline{e}(f_p) \ \text{---} \end{bmatrix}$$

transpose

$$\underline{A}_c = [A_c^1 \ \cdots \ A_c^p]^T.$$

Again the MLE is obtained by minimizing

$$(\underline{x} - E\underline{A}_c)^H (\underline{x} - E\underline{A}_c)$$

w.r.t. $\underline{A}_c$. This is a standard least squares problem and the solution is

$$\hat{\underline{A}}_c(f) = (E^H E)^{-1} E^H \underline{x}$$

The minimum value of the squared error is

$$\underline{x}^H \underline{x} - \underline{x}^H E (E^H E)^{-1} E^H \underline{x}$$

and therefore the MLE is obtained by maximizing

$$\underline{x}^H E (E^H E)^{-1} E^H \underline{x}$$

w.r.t. $f_1, \ldots, f_p$.

$\boxed{\text{nonlinear in } f_1, \ldots, f_p}$

Summary of discussion in Kay:

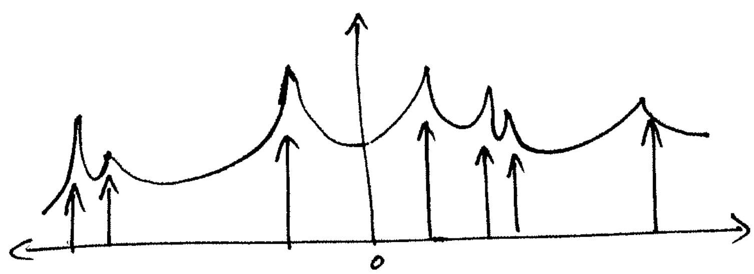
- Objective function is highly nonlinear and very difficult to maximize when the true frequencies are very close together

- When true frequencies are separated by $\geq \frac{1}{N}$, a good approximate MLE is to take the p largest peaks of the periodogram, subject to those peaks being $\geq \frac{1}{N}$ apart.

Frequency Estimation via High-Resolution S. E.

Since the periodogram cannot reliably resolve frequencies differing by $\le \frac{1}{N}$, an alternative is the identify the frequencies using a high resolution spectral estimator.

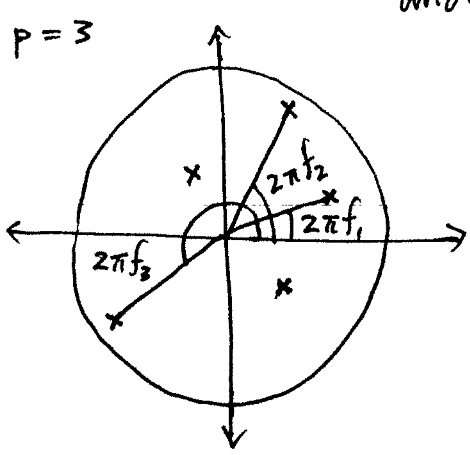
The AR SE is a popular choice



Two approaches

1. Apply AR SE method of order $p' \geq p$, such as the modified covariance method, and choose the frequencies corresponding to the p poles closest to the unit circle.

$p = 3$



2. Constrain AR model so that poles lie <u>on</u> the unit circle, giving rise to pure sinusoidal components.

The resulting constrained optimization problem is difficult.

An efficient but suboptimal algorithm called the iterative filtering algorithm (IFA) is discussed in Kay.

<u>Remarks</u>: AR methods are good at low noise levels but suffer when noise is strong.

Frequency Estimation via Eigenanalysis

<u>Idea</u>: It turns out (we will show) that the linear combo of sinusoids lives in a subspace of dimension p.

This subspace is the span of the p most significant components (the <u>principal components</u>) of the ACV Γ_{xx}.

We can use this fact to our advantage, to modify existing spectral estimators and derive new ones.

In short, we can use the signal subspace property to do <u>high resolution</u> spectral estimation that has <u>higher noise tolerance</u> than AR and other methods studied previously.

To even talk about ACV matrices, we need $x[n]$ to be WSS which means we need to assume

$$\boxed{\begin{array}{l} \phi_j \sim \text{unif}\,[0, 2\pi)\,, \quad j = 1, \dots, p \\ \phi_j, \phi_k \text{ independent}, \ j \neq k \end{array}}$$

Under this assumption, if

$$x[n] = \sum_{j=1}^{p} A_j \exp\left\{ i \left(2\pi f_j n + \phi_j \right) \right\} + z[n]$$

then

$$\gamma_x[k] = \sum_{j=1}^{p} A_j^2 \exp\left\{ 2\pi f_j k \right\} + \sigma_z^2 \delta[k]$$

The $M \times M$ autocorrelation matrix Γ_{xx} is

$$\Gamma_{xx} = \sum_{j=1}^{p} A_j^2 \, \underline{e}(f_j) \, \underline{e}(f_j)^H + \sigma_z^2 I_M$$

M x M identity matrix

$$= \Gamma_{ss} + \Gamma_{zz}$$

<u>Exercise</u> ① What are the eigenvalues/vectors of Γ_{zz}? ② Show that if $\underline{v}$ is an eigenvector of Γ_{ss} with eigenvalue λ, then $\underline{v}$ is also an eigenvector of Γ_{xx} and find the corresponding eigenvalue.
③ Can you guess the eigenvectors of Γ_{ss}?

Solution 1 ① Every vector is an eigenvector of Γ_{ZZ}, and the eigenvalue is σ_Z^2:

$$\Gamma_{ZZ} \cdot \underline{v} = \sigma_Z^2 \cdot I \cdot \underline{v} = \sigma_Z^2 \cdot \underline{v}$$

② If

$$\Gamma_{SS} \cdot \underline{v} = \lambda \underline{v}$$

then

$$\Gamma_{XX} \underline{v} = \Gamma_{SS} \cdot \underline{v} + \Gamma_{ZZ} \underline{v}$$

$$= \lambda \underline{v} + \sigma_Z^2 \underline{v}$$

$$= \underbrace{(\lambda + \sigma_Z^2)}_{\text{eigenvalue}} \underline{v}$$

③ In general, there is no nice formula for the eigenvectors of Γ_{ss}.

Eigenvectors of Γ_{ss}

Observe

$$\Gamma_{ss} = \sum_{j=1}^{p} A_j^2 \, \underline{e}(f_j) \, \underline{e}(f_j)^H$$

$$= \text{sum of } p \text{ rank 1 matrices}$$

$$\Rightarrow \text{rank}(\Gamma_{ss}) \leq p.$$

$\boxed{\text{Note: } \Gamma_{zz} \text{ is full rank, and hence so is } \Gamma_{xx}}$

Let $\underline{v}_1, \ldots, \underline{v}_M$ be an <u>orthonormal basis</u> of $\mathbb{C}^M$ consisting of eigenvectors of Γ_{ss}. Let $\lambda_1, \ldots, \lambda_M$ be the corresponding eigenvalues. Assume the eigenvectors are ordered such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M \geq 0$$

why?

Since $\operatorname{rank}(\Gamma_{ss}) \leq p$ we know

$$\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_M = 0.$$

Assuming $f_1, \ldots, f_p$ are distinct, we also have $\operatorname{rank}(\Gamma_{ss}) = p$, in which case

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0.$$

$\boxed{\text{SUPER USEFUL FACT}}$

$$\operatorname{span}\{\underline{e}(f_1), \ldots, \underline{e}(f_p)\} = \operatorname{span}\{\underline{v}_1, \ldots, \underline{v}_p\}$$

Now define the

<u>signal subspace</u> $:= \operatorname{span}\{\underline{v}_1, \ldots, \underline{v}_p\}$

<u>noise subspase</u> $:= \operatorname{span}\{\underline{v}_{p+1}, \ldots, \underline{v}_M\}.$

Why is it called the "signal subspace?"

Because we are interested in signals of the form

$$\underline{x} = \sum_{j=1}^{p} A_c^j \, \underline{e}(f_j)$$

$$\in \text{span}\{\underline{e}(f_1), \ldots, \underline{e}(f_p)\}$$

$$= \text{span}\{\underline{v}_1, \ldots, \underline{v}_p\}$$

<u>Exercise</u>: Show that signal vectors are orthogonal to all vectors in the noise subspace.

<u>Solution</u> Let

$$\underline{s} = \sum_{j=1}^{p} A_c^j \, \underline{e}(f_j)$$

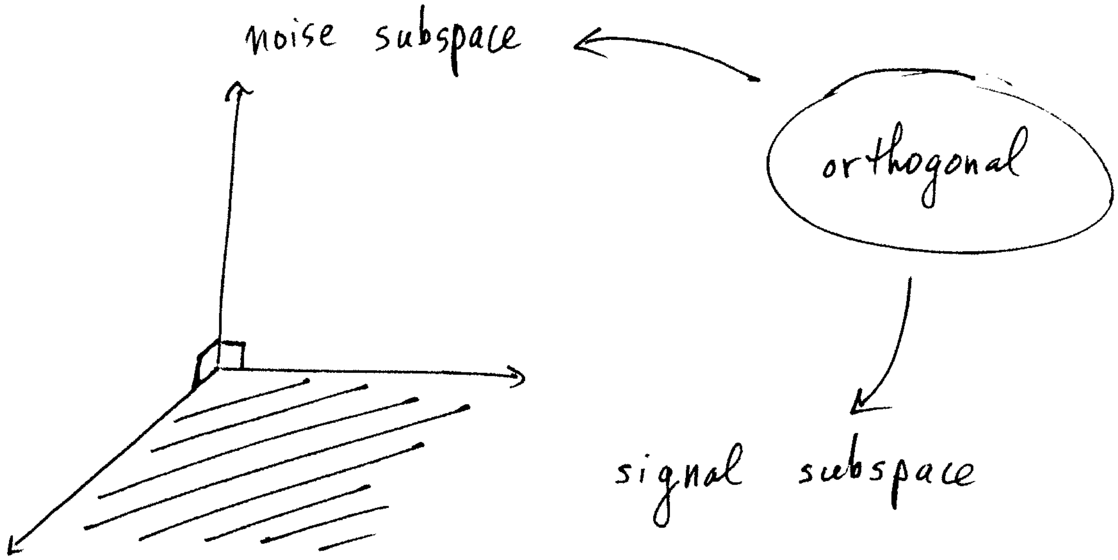
$$= \sum_{j=1}^{p} \alpha_j \underline{v}_j \quad \in \text{signal subspace}$$

and

$$\underline{t} = \sum_{j=p+1}^{M} \beta_j \underline{v}_j \quad \in \text{noise subspace.}$$

Then $$\underline{s}^H \underline{t} = \sum_{j=1}^{p} \sum_{k=p+1}^{M} \alpha_j^* \beta_k \underbrace{\underline{v}_j^H \underline{v}_k}_{=0}$$

$$= 0$$



Key point

- There is **no** signal in the noise subspace
- There is **some** noise in the signal subspace.

Terminology:

$\underline{v}_1, \dots, \underline{v}_p$ are called principal components or principal eigenvectors.

Signal Subspace (Principal Component) Frequency Estimation

Basic Idea: Several spectral estimators can be expressed in terms of Γ_{xx}:

→ AR

$$\underline{\hat{a}} = -\hat{\Gamma}_{xx}^{-1} \, \underline{\hat{\gamma}}_x$$

→ Capon

$$\hat{P}_{CAPON}(f) = \frac{M}{\underline{e}(f)^H \cdot \hat{\Gamma}_{xx}^{-1} \, \underline{e}(f)}$$

→ Blackman Tukey with Bartlett window.

$$\hat{P}_{BAR}(f) = \frac{1}{M} \underline{e}(f)^H \, \hat{\Gamma}_{xx} \, \underline{e}(f)$$

Let's replace standard estimates of Γ_{xx} with estimates that use the signal subspace property.

Exercise Verify that

$$\Gamma_{xx} = \sum_{j=1}^{p} (\lambda_j + \sigma_z^2)\, \underline{v}_j \underline{v}_j^H + \sum_{j=p+1}^{M} \sigma_z^2\, \underline{v}_j \underline{v}_j^H$$

Solution We know $\Gamma_{xx} \underline{v}_j = (\lambda_j + \sigma_z^2)\, \underline{v}_j$

Let

$$V := \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_M \\ | & & | \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1+\sigma_z^2 & & & & \\ & \ddots & & & \\ & & \lambda_p+\sigma_z^2 & & \\ & & & \sigma_z^2 & \\ & & & & \ddots \\ & & & & & \sigma_z^2 \end{bmatrix}$$

Then $\Gamma_{xx} V = V \cdot \Lambda$,

$$\Longrightarrow \Gamma_{xx} = V \cdot \Lambda \cdot V^H$$

$$= \sum_{j=1}^{M} (\lambda_j + \sigma_z^2)\, \underline{v}_j \underline{v}_j^H$$

$$= \sum_{j=1}^{p} (\lambda_j + \sigma_z^2)\, \underline{v}_j \underline{v}_j^H + \sum_{j=1} \sigma_z^2\, \underline{v}_j \underline{v}_j^H$$

Principal Component Bartlett Spectral Estimator

Recall the Blackman-Tukey (lag window) S.E.

$$\hat{P}_{BT}(f) = \sum_{k=-M}^{M} w[k]\,\hat{\gamma}_{xx}[k]\,e^{-2\pi i f k}$$

Using the Bartlett window $w[k] = M - |k$

$$w[k] = \begin{cases} \dfrac{M-|k|}{M} & |k| \le M \\ 0 & |k| > M \end{cases}$$

we have

$$\hat{P}_{BAR}(f) = \frac{1}{M}\sum_{k=-(M-1)}^{M-1} (M-|k|)\,\hat{\gamma}_{xx}[k]\,e^{-2\pi i f k}$$

$$= \frac{1}{M}\sum_{m=0}^{M-1}\sum_{n=0}^{M-1} \hat{\gamma}_{xx}[m-n]\,e^{-2\pi i (m-n) f}$$

$$= \frac{1}{M}\,\underline{e}(f)^H\,\hat{\Gamma}_{xx}\,\underline{e}(f)$$

where we have used the identity

$$\sum_{k=-(M-1)}^{M-1} (M-|k|)\,g[k] = \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} g[m-n]$$

The principal component Bartlett SE replaces

$$\hat{\Gamma}_{xx} = \sum_{i=1}^{M} \hat{\lambda}_i \, \hat{\underline{v}}_i \, \hat{\underline{v}}_i^H$$

with

$$\hat{\Gamma}_{xx} = \sum_{i=1}^{p} \hat{\lambda}_i \, \hat{\underline{v}}_i \, \hat{\underline{v}}_i^H$$

uses only the principal components

This results in

$$\hat{P}_{BAR-PC}(f) = \frac{1}{M} \underline{e}(f)^H \left[\sum_{i=1}^{p} \hat{\lambda}_i \, \hat{\underline{v}}_i \, \hat{\underline{v}}_i^H \right] \underline{e}(f)$$

$$= \frac{1}{M} \sum_{i=1}^{p} \hat{\lambda}_i \left| \underline{e}(f)^H \hat{\underline{v}}_i \right|^2 \qquad (*)$$

Contrast with the original Bartlett S.E:

$$\hat{P}_{BAR}(f) = \frac{1}{M} \sum_{i=1}^{p} \hat{\lambda}_i \left| \underline{e}(f)^H \hat{\underline{v}}_i \right|^2 + \frac{1}{M} \sum_{i=p+1}^{M} \hat{\lambda}_i \left| \underline{e}(f)^H \hat{\underline{v}}_i \right|^2$$

projections onto noise subspace

$\Longrightarrow$ spurious peaks

__Algorithm__

1. Estimate Γ_{xx} using standard method, such as modified covariance method.

2. Find principal eigenvectors and eigenvalues of $\hat{\Gamma}_{xx}$ (__eig__ function in Matlab)

3. Plug in to $(*)$

__Note__: Here I'm using λ_i to denote an eigenvalue of Γ_{xx}. Previously I used λ_i to denote an eigenvalue of Γ_{ss}. Kay does the same.

Principal Component Capon Spectral Estimator

If

$$\Gamma_{xx} = V \cdot \Lambda \cdot V^H, \qquad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_M \end{bmatrix}$$

then

$$\Gamma_{xx}^{-1} = V \Lambda^{-1} V^H$$

$$= \sum_{i=1}^{M} \frac{1}{\lambda_i} \underline{v}_i \underline{v}_i^H$$

This suggests the following modification of Capon's method:

$$\hat{P}_{CAP-PC}(f) = \frac{M}{\underline{e}(f)^H \left[\sum_{i=1}^{P} \frac{1}{\hat{\lambda}_i} \hat{\underline{v}}_i \hat{\underline{v}}_i^H \right] \underline{e}(f)}$$

$$= \frac{M}{\sum_{i=1}^{P} \frac{1}{\hat{\lambda}_i} \left| \underline{e}(f)^H \hat{\underline{v}}_i \right|^2}$$

Principal Component AR Spectral Estimator

The AR spectral estimator computes

$$\hat{\underline{a}} = -\hat{\Gamma}_{xx}^{-1} \underline{\gamma}_x$$

and then

$$\hat{P}_{AR}(f) = \frac{1}{|\hat{A}(f)|^2}$$

where

$$\hat{A}(f) = 1 + \sum_{k=1}^{M} a[k] e^{-2\pi i f k}$$

The principal component version of the AR S.E. replaces $\hat{\underline{a}}$ by

$$\hat{\underline{a}}_{PC} = -\left[\sum_{i=1}^{p} \frac{1}{\hat{\lambda}_i} \hat{\underline{v}}_i \hat{\underline{v}}_i^H \right] \underline{\gamma}_x$$

Noise Subspace (Nonprincipal Component) Frequency Estimation

The PC version of Capon's method discussed earlier seems reasonable at first glance. Recall

$$\hat{P}_{CAP}(f) = \frac{M}{\underbrace{\sum_{i=1}^{p} \frac{1}{\hat{\lambda}_i} |\underline{e}(f)^H \hat{\underline{v}}_i|^2}_{A} + \underbrace{\sum_{i=p+1}^{M} \frac{1}{\hat{\lambda}_i} |\underline{e}(f)^H \hat{\underline{v}}_i|^2}_{B}}$$

We replaced B with 0. This was consistent with our knowledge that Γ_{XX} can be expressed using only the principal components. This approach reflects a desire to improve the spectral estimator as a whole.

Instead, what if we focus on frequency estimation. How should we modify $\hat{P}_{CAP}(f)$ so that it has sharp peaks at $f = f_i$, $i = 1, \ldots, n$?

Eigenvector (EV) Method

$$\hat{P}_{EV}(f) = \frac{M}{\sum_{i=p+1}^{M} \frac{1}{\hat{\lambda}_i} |\underline{e}(f)^H \underline{\hat{v}}_i|^2}$$

If $\underline{\hat{v}}_i$ are the true $\underline{v}_i$, then

$$\hat{P}_{EV}(f_i) =$$

In practice, $\underline{\hat{v}}_i \neq \underline{v}_i$ and $\hat{P}_{EV}(f)$ will have peaks at the f_i.

MUSIC (MUltiple SIgnal Classification)

→ clever acronym, but does <u>not</u> do classification

→ set $\hat{\lambda}_i = 1$ in EV method

$\boxed{\text{why is this reasonable?}}$

$$\hat{P}_{MUSIC}(f) = \frac{M}{\sum_{i=p+1}^{M} |\underline{e}(f)^H \underline{\hat{v}}|^2}$$

Setting $\hat{\lambda}_i = 1$, $i = p+1, \dots, n$, is reasonable because

(a) $\lambda_i = \sigma_z^2$ for $i = p+1, \dots, n$

(b) we are only interested in frequency <u>locations</u>, not <u>amplitudes</u>

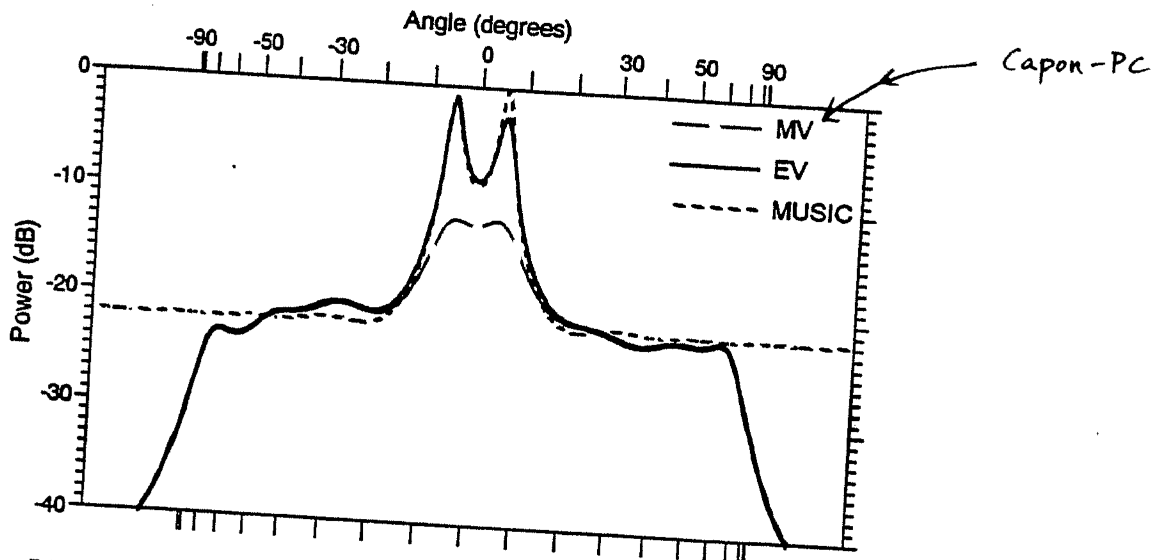


Figure 7.14 The eigenvector (*EV*) and *MUSIC* variations of truncating the eigenexpansions of the spatial correlation matrix's inverse are shown in the context of the minimum variance algorithm. Two signals are propagating from −5° and 5° in the presence of spherically isotropic noise. The signal-to-noise ratio for each signal is 0 dB, and the time-bandwidth product of the spatial correlation matrix's estimate equals 100. The minimum variance algorithm alone does not have sufficient resolving power to distinguish the two signals. Either of the eigenanalysis-based methods can be at about the same level of performance. The greatest difference between the methods occurs in the invisible region; the eigenvector method yields the same results as minimum variance while *MUSIC* yields a flattened (whitened) result.

[From: D. H. Johnson, D. E. Dudgeon, <u>Array Signal Processing</u>, Prentice Hall, 1993]

<u>Comments</u>:

- CAPON-PC does not have as sharp peaks as EV + MUSIC
- MUSIC has flat baseline which results from setting $\hat{\lambda}_i = 1$.

Pisarenko Harmonic Decomposition (PHD)

If we choose $M = p+1$ then the dimension of the noise subspace is ____.

Let $\underline{v}_{p+1}$ be the non principal component spanning the noise subspace.

Then $\underline{v}_{p+1} \perp \underline{e}(f_j)$ for $j = 1, \ldots, p$.

$$0 = \sum_{n=1}^{p+1} [\underline{v}_{p+1}] \cdot e^{-2\pi i f_j (n-1)}$$

$$= \sum_{n=1}^{p+1} [\underline{v}_{p+1}]_n \cdot z_j^{(n-1)} \qquad \boxed{z_j = e^{-2\pi i f_j}}$$

That is, $z_1, \ldots, z_p$ are zeros of the polynomial

$$\sum_{n=1}^{p+1} [\underline{v}_{p+1}]_n \cdot z^{n-1}$$

This suggests the following algorithm:

1. Compute $\hat{\underline{v}}_1, \ldots, \hat{\underline{v}}_{p+1}$

2. Find the zeros of the polynomial

$$\sum_{n=1}^{p+1} [\hat{\underline{v}}_{p+1}]_n \, z^n$$

3. Let $\hat{z}_1, \ldots, \hat{z}_p$ be the zeros closest to the unit circle.

4. Set $\hat{f}_j = (\arg(\hat{z}_j^{-1}))$, Avoid this

Do this first

Remarks

- If we compute $\hat{\underline{v}}_j$ using the biased sample ACV for $\hat{\Gamma}_{xx}$, then we retain the properties of the theoretical ACV matrix. In particular, $\hat{z}_1, \ldots, \hat{z}_p$ will lie <u>on</u> the unit circle.

- Extension to $M > p+1$ discussed in Kay.