

EIGENDECOMPOSITIONS & THE SPECTRAL THEOREM

The Spectral Theorem

Definition If $U \in \mathbb{C}^{N \times N}$ is such that

$$U^H U = U U^H = I_{N \times N}$$

then U is said to be unitary.

If $U \in \mathbb{R}^{N \times N}$ is such that

$$U^T U = U U^T = I_{N \times N}$$

then U is said to be orthogonal

↑
[slightly confusing since the columns of U are in fact orthonormal]

Intuitively, such matrices are distance preserving since

$$\begin{aligned} \|U\underline{x} - U\underline{y}\|^2 &= (U\underline{x} - U\underline{y})^H (U\underline{x} - U\underline{y}) \\ &= (\underline{x} - \underline{y})^H U^H U (\underline{x} - \underline{y}) \\ &= \|\underline{x} - \underline{y}\|^2 \end{aligned}$$

Unitary and orthogonal matrices effect a change of coordinate system.

Theorem | (Spectral Theorem)

If $A \in \mathbb{C}^{N \times N}$ is Hermitian, then there exist a unitary matrix U and a real diagonal matrix Λ such that

$$A = U\Lambda U^H.$$

If $A \in \mathbb{R}^{N \times N}$ is symmetric, the same result holds where now U is orthogonal.

Proof | See Moon and Stirling, Mathematical Methods and Algorithms for Signal Processing.

Eigenvalues and eigenvectors

Suppose A is Hermitian/symmetric. Write

$$A = U \Lambda U^H$$

according to the spectral theorem. Let

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}, \quad U = \begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \dots & \underline{u}_N \end{bmatrix}$$

Since $AU = U\Lambda$, we conclude that the λ_i are eigenvalues with \underline{u}_i the associated eigenvector:

$$A \underline{u}_i = \lambda_i \underline{u}_i, \quad i = 1, \dots, N.$$

The spectral theorem also gives rise to following spectral decomposition of A :

$$A = \sum_{i=1}^N \lambda_i \underline{u}_i \underline{u}_i^H$$

Positive (semi-) definite matrices

Let A be a Hermitian / symmetric $N \times N$ matrix.

We say A is positive definite (PD) if

$$\underline{x} \neq 0 \implies \underline{x}^H A \underline{x} > 0.$$

We say A is positive semi-definite (PSD) if

$$\forall \underline{x} \quad \underline{x}^H A \underline{x} \geq 0.$$

[PSD is also called nonnegative definite]

Exercise | Show that A is PD (PSD)

iff. the eigenvalues of A are positive (nonnegative).

Solution

By the spectral theorem, $\underline{u}_1, \dots, \underline{u}_N$ is an orthonormal collection (a basis, in fact) such that

$$A \underline{u}_i = \lambda_i \underline{u}_i.$$

For each $i = 1, \dots, N$ we have

$$\lambda_i = \lambda_i \cdot \underline{u}_i^H \underline{u}_i$$

$$= \underline{u}_i^H \cdot \lambda_i \underline{u}_i$$

$$= \underline{u}_i^H A \underline{u}_i \quad \begin{cases} > 0 & \text{if } A \text{ is PD} \\ \geq 0 & \text{if } A \text{ is PSD.} \end{cases}$$

Conversely, suppose $\lambda_i > 0$ (≥ 0) $\forall i$.

Then for $\underline{x} \neq 0$ we have

$$\underline{x}^H A \underline{x} = \sum_{i=1}^n \lambda_i \underline{x}^H \underline{u}_i \underline{u}_i^H \underline{x}$$

$$= \sum_{i=1}^n \lambda_i \|\underline{u}_i^H \underline{x}\|^2$$

$$\begin{cases} > 0 & \text{if } \lambda_i > 0 \quad \forall i \\ \geq 0 & \text{if } \lambda_i \geq 0 \quad \forall i \end{cases}$$