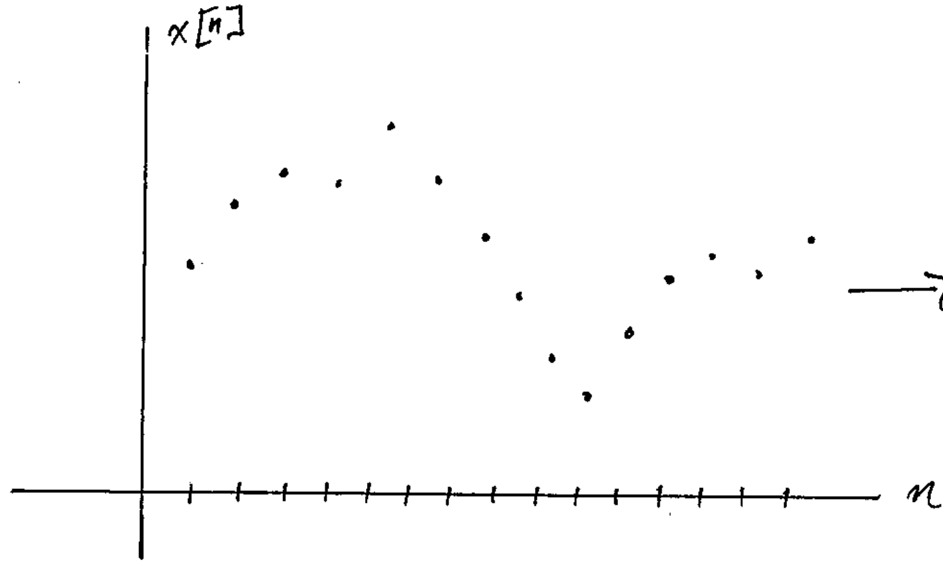
FILTERING

Thus far in our study of estimation the data have been <u>static</u>: All the data are available at once, and the number of measurements N is not so large that R_{xx} is difficult to invert.

Now we turn our attention to the situation where data are <u>dynamic</u>, that is, we observe a "stream" $\{x[n]\}$ of measurements one value at a time.



$$x[n] = s[n] + w[n]$$

The following problems are of interest:

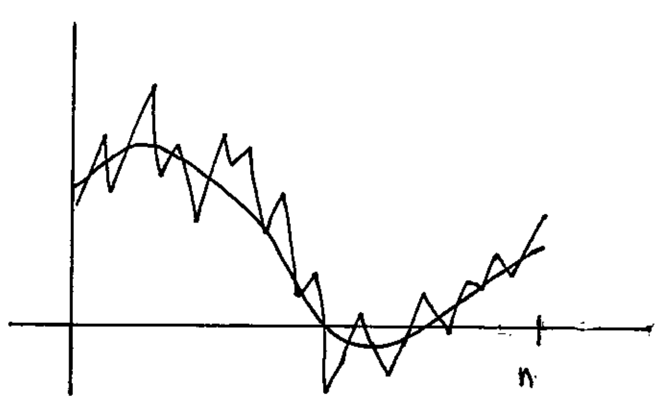
1. <u>Filtering</u>: Given $x[n], x[n-1], \cdots,$
   estimate $s[n]$.

2. <u>Smoothing</u>: Given $x[n], x[n-1], \ldots$
   estimate $s[m], \quad m < n$.

3. <u>Signal prediction</u>: Given $x[n], x[n-1], \ldots$
   estimate $s[m], \quad m > n$.

4. <u>Measurement prediction</u>: Given $x[n], x[n-1], \ldots$
   estimate $x[m], \quad m > n$

5. <u>Interpolation</u>: Given $\{x[n]\}_{n \in I}$
   estimate a missing value $x[m], \quad m \notin I$



To address these problems we need computationally efficient estimators that can be efficiently <u>updated</u> as new data arrive:

Our approach includes

- linear estimators
- signal models that lead to "structured" covariance matrices whose inverses are easier to compute and update than typical matrices.

<u>Example</u> If $\{x[n]\}$ is wide-sense stationary (WSS) with auto-correlation function (assuming zero mean)

$$r_{xx}[k] : E\{x[n]\,x[n+k]\},$$

and we observe $\underline{x} = [x[n-1] \; \cdots \; x[n-p]]^T$ then the data autocovariance matrix

$$R_{xx} = \begin{bmatrix} r_{xx}[0] & r_{xx}[-1] & & \\ r_{xx}[1] & r_{xx}[0] & r_{xx}[-1] & \\ & r_{xx}[1] & \ddots & \ddots \\ & & \ddots & \end{bmatrix}$$

is a <u>Toeplitz</u> matrix.

The inverse of a Toeplitz matrix can be computed in $O(p^2)$ operations as opposed to the usual $O(p^3)$. More on this later.

<u>Note</u> We will adhere to the convention of always using <u>lowercase</u> letters to denote random processes, regardless of whether we mean a random variable or a realization.

WSS Random Processes

Since one of our major assumptions (for linear prediction and Wiener filtering) will be wide-sense stationarity, let's look at some examples:

1. White noise: A white noise process $w[n]$ satisfies three properties:

   (a) values at different times are uncorrelated

   (b) $E\{w[n]\} = 0 \quad \forall n \in \mathbb{Z}$

   (c) $Var\{w[n]\} = \sigma_w^2 \quad \forall n \in \mathbb{Z}$

An important special case is Gaussian white noise,

$$w[n] \overset{iid}{\sim} N(0, \sigma_w^2).$$

Why do you think "white" is used to describe such a process?

Notation: $wn(\sigma_w^2)$

2. <u>Moving average process</u> : a linear combination of terms in a white noise process,

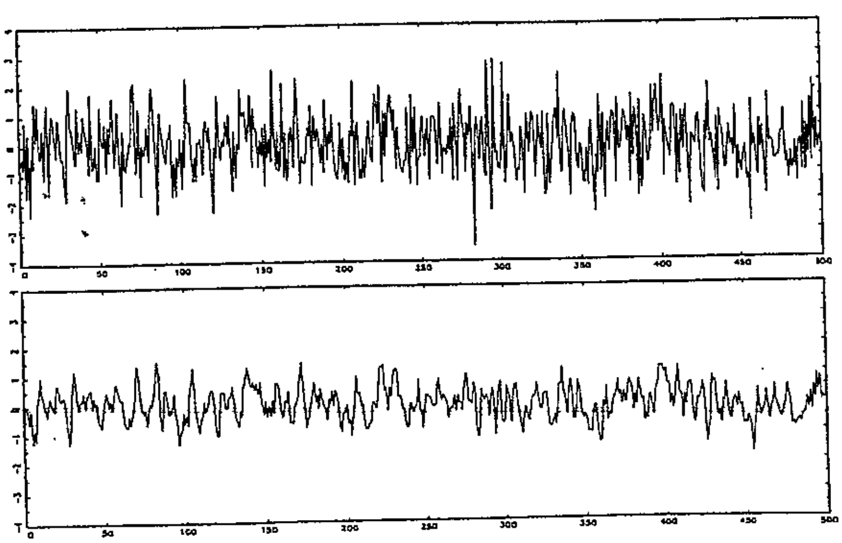
$$x[n] = \sum_{k=0}^{q} b[k]\, w[n-k]$$

such as

$$x[n] = \frac{1}{3} w[n] + \frac{1}{3} w[n-1] + \frac{1}{3} w[n-2].$$

In general, an MA process is obtained by passing white noise through a finite impulse response (FIR) filter

$$w[n] \longrightarrow \boxed{\;b[k]\;} \longrightarrow x[n].$$



white noise

filtered

Notation : MA(q)

3. <u>Autoregressive process</u>

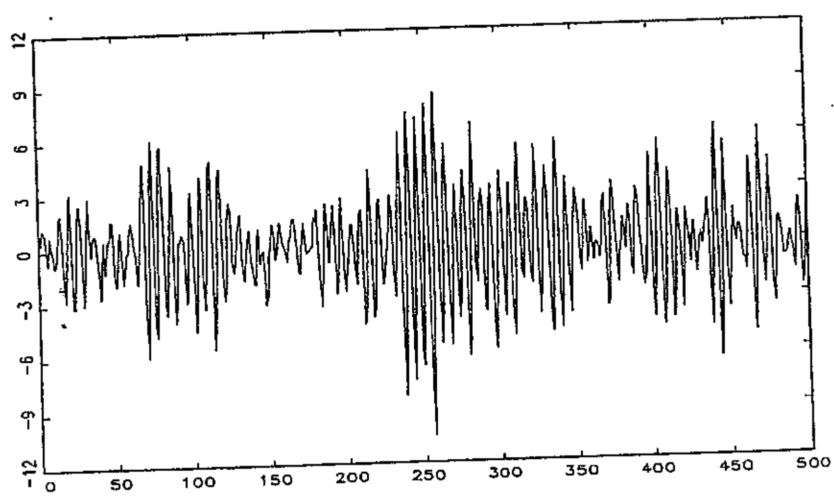
$$x[n] = -\sum_{k=1}^{p} a[k]\, x[n-k] + w[n]$$

where $w[n]$ is white noise. An AR process is obtained by passing white noise through an IIR filter

$$w[n] \longrightarrow \boxed{\text{IIR}} \longrightarrow x[n].$$

Simple example:

$$x[n] = x[n-1] - .9\, x[n-2] + w[n].$$



Notation: AR(p)

4. <u>ARMA process</u> An auto-regressive moving average process of orders p and q obeys

$$x[n] = -\sum_{k=1}^{p} a[k]\, x[n-k] + \sum_{k=0}^{q} b[k]\, w[n-k]$$

where $w[n]$ is white noise.

Notation: ARMA(p,q)

Fact: Essentially any discrete-time WSS RP can be approximated arbitrarily well by a

- MA(q) model, $q \to \infty$
- AR(p) model, $p \to \infty$
- ARMA(p,q) model, p and/or $q \to \infty$

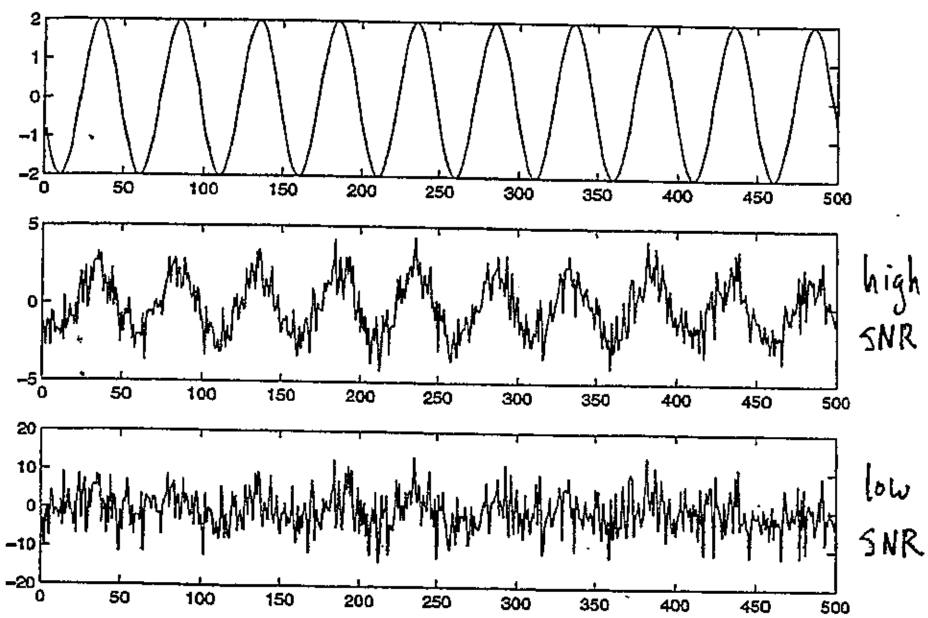
5. <u>Sinusoid</u> in white noise, uniform phase

$$x[n] = s[n] + w[n]$$

where

$$s[n] = A\cos(2\pi f n + \phi)$$

$$\phi \sim \text{unif}[0, 2\pi)$$



ACF / Spectrum Estimation

Throughout our discussion of filtering we will assume knowledge of first and second order moments. In practice, however, these may also need to be estimated.

There is an extensive literature on how to estimate an ACF or its Fourier transform, the power spectral density (PSD).

These are important topics that are beyond the scope of this discussion.