SIGNAL DETECTION IN GAUSSIAN NOISE

The Gaussian noise model is the most common noise model in detection problems.

We will study the detection problem

$$H_1 : \underline{x} = \underline{s}_1 + \underline{w}$$

$$\vdots$$

$$H_M : \underline{x} = \underline{s}_M + \underline{w}$$

where

$$\underline{w} \sim N(\underline{0}, R).$$

and R is symmetric, positive definite.

R and $\underline{s}_1, \dots, \underline{s}_M$ are assumed <u>known</u>.

IID Gaussian Noise

For now, assume $R = \sigma^2 I_{N \times N}$.

Let's also assume $M=2$. and $\underline{s}_0 = \underline{0}$.

$$H_0 : \underline{x} = \underline{w}$$

$$H_1 : \underline{x} = \underline{s} + \underline{w}$$

The optimal detector is

$$\Lambda(\underline{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

where

$$\Lambda(\underline{x}) = \frac{(2\pi\sigma^2)^{-\frac{N}{2}} \exp\left\{\frac{-1}{2\sigma^2}(\underline{x}-\underline{s})^T(\underline{x}-\underline{s})\right\}}{(2\pi\sigma^2)^{-\frac{N}{2}} \exp\left\{\frac{-1}{2\sigma^2}\underline{x}^T\underline{x}\right\}}$$

$$= \exp\left\{\frac{-1}{2\sigma^2}\left[(\underline{x}-\underline{s})^T(\underline{x}-\underline{s}) - \underline{x}^T\underline{x}\right]\right\}$$

In terms of the log likelihood ratio, we have

$$\log \Lambda(\underline{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \log \eta$$

where

$$\log \Lambda(\underline{x}) = \frac{-1}{2\sigma^2}\left[(\underline{x}-\underline{s})^T(\underline{x}-\underline{s}) - \underline{x}^T\underline{x}\right]$$

$$= \frac{1}{\sigma^2}\left[\underline{s}^T\underline{x} - \frac{\underline{s}^T\underline{s}}{2}\right]$$

Rearranging terms, we have

$$\underline{s}^T\underline{x} \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \log \eta + \frac{\underline{s}^T\underline{s}}{2} \equiv \gamma$$

Remarks

1. If we rewrite the detection problem

$$H_0 : \underline{x} \sim \mathcal{N}(\theta \underline{s}, \sigma^2 \underline{\underline{I}}), \quad \theta = 0$$

$$H_1 : \underline{x} \sim \mathcal{N}(\theta \underline{s}, \sigma^2 \underline{\underline{I}}), \quad \theta = 1$$

Then $\underline{s}^T \underline{x}$ is a <u>sufficient statistic</u> for θ.

2. $\underline{s}^T \underline{s} = \sum_{n=0}^{N-1} s(n)^2 =$ <u>signal energy</u>.

3. The operator

$$\underline{x} \longmapsto \underline{s}^T \underline{x} = \sum_{n=0}^{N-1} x(n) s(n)$$

is called a <u>correlator</u>.

Projection Interpretation

We may rewrite the LRT as

$$\frac{\underline{s}^T \underline{x}}{\underline{s}^T \underline{s}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2 \log \eta}{\underline{s}^T \underline{s}} + \frac{1}{2} = \gamma'$$

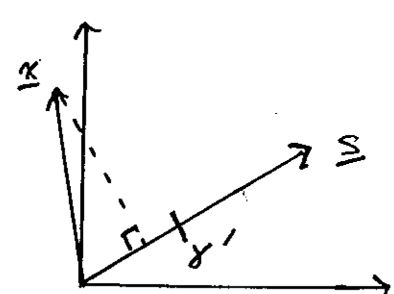
Let $S = \langle \underline{s} \rangle$. Then

$$\Pi_S(\underline{x}) = \underline{s}(\underline{s}^T\underline{s})^{-1}\underline{s}^T\underline{x}$$

$$= \left(\frac{\underline{s}^T \underline{x}}{\underline{s}^T \underline{s}}\right)\underline{s}$$

← Coefficient given by pseudo-inverse $(\underline{s}^T\underline{s})^{-1}\underline{s}^T\underline{x}$

So the LR detector depends only on the projection of the data onto the signal subspace. Other components of $\underline{x}$ are "filtered out" and do not factor into the decision.



Special case: DC signal

$$H_0: \quad x(n) = w(n), \quad n = 0, 1, \ldots, N-1$$

$$H_1: \quad x(n) = A + w(n), \quad n = 0, 1, \ldots, N-1$$

$$w(n) \sim \mathcal{N}(0, \sigma^2)$$

$$\underline{s} = [A \ A \ \cdots \ A]^T$$

Optimal detector:

$$\underline{s}^T \underline{x} = A \sum_{n=0}^{N-1} x(n)$$

$$\gamma = \sigma^2 \log \eta + \frac{\underline{s}^T \underline{s}}{2} = \sigma^2 \log \eta + \frac{NA^2}{2}$$

$$\Longrightarrow \quad \frac{1}{N} \sum_{n=0}^{N-1} x(n) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{NA} \log \eta + \frac{A}{2}$$

Just what we derived earlier.

Let's consider the slightly more general problem

$$H_0 : \underline{x} = \underline{s}_0 + \underline{w}$$

$$H_1 : \underline{x} = \underline{s}_1 + \underline{w} \qquad \underline{w} \sim N(\underline{0}, \sigma^2 \underline{I})$$

Now the transmitted signal is <u>nonzero</u> under <u>both</u> hypotheses.

<u>Log-likelihood ratio</u>:

$$\log \Lambda(\underline{x}) = \frac{-1}{2\sigma^2}\left[(\underline{x}-\underline{s}_1)^T(\underline{x}-\underline{s}_1) - (\underline{x}-\underline{s}_0)^T(\underline{x}-\underline{s}_0)\right]$$

$$= \frac{1}{\sigma^2}\left[(\underline{s}_1-\underline{s}_0)^T\underline{x} - \frac{\underline{s}_1^T\underline{s}_1}{2} + \frac{\underline{s}_0^T\underline{s}_0}{2}\right]$$

so the optimal test is:

$$(\underline{s}_1 - \underline{s}_0)^T\underline{x} \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \log(\eta) + \frac{\underline{s}_1^T\underline{s}_1}{2} - \frac{\underline{s}_0^T\underline{s}_0}{2} \equiv \gamma$$

Projection Interpretation

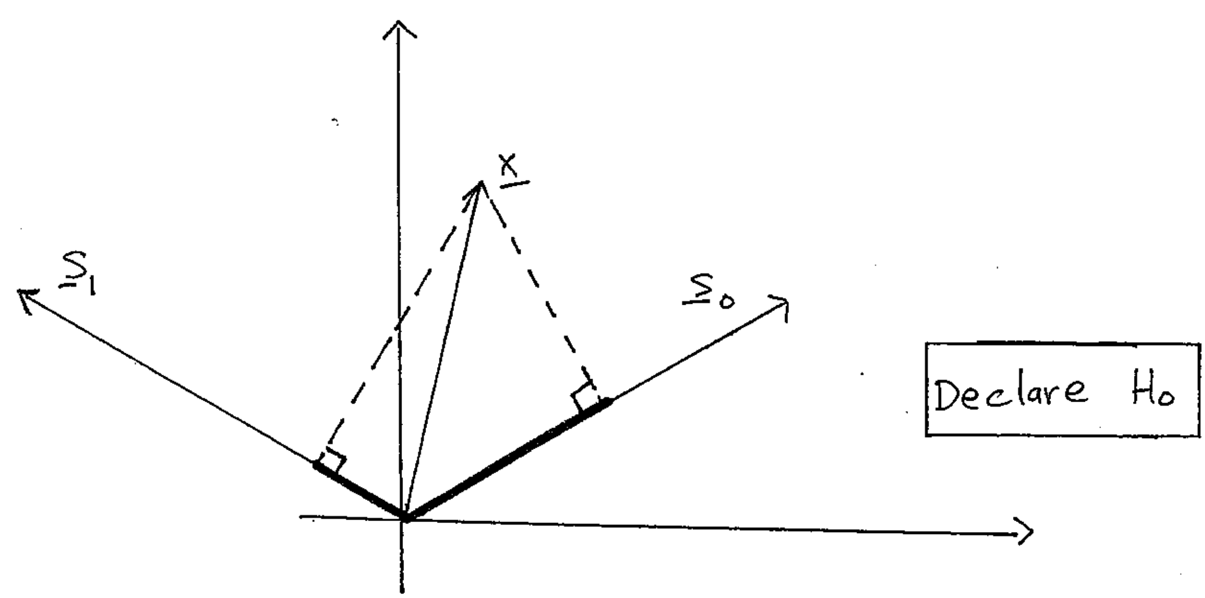
Consider a minimum probability of error detector. Assume

- $\pi_0 = \pi_1 = \frac{1}{2} \quad (\eta = 1)$
- $\|s_0\|^2 = \|\underline{s}_1\|^2$

Then the detector reduces to

(a) $$\mathop{\gtrless}_{H_0}^{H_1}$$

Recall $\underline{s}_i^T \underline{x} \propto$ coefficient of $\underline{x}$ projected onto $\underline{s}_i$.



Exercise | Performance analysis.

(a) Calculate P_E, P_F, P_M as functions of γ

(b) Determine γ in terms of desired false alarm rate α.

(c) Express P_D as function of P_F. (d) Identify SNR.

Solution Recall $\underline{X} \sim N(\underline{s}_i, \sigma^2 I)$ under H_i.

Under H_0,

$$(\underline{s}_1 - \underline{s}_0)^T \underline{X} \sim N\left((\underline{s}_1 - \underline{s}_0)^T \underline{s}_0, \ \sigma^2 \|\underline{s}_1 - \underline{s}_0\|^2\right)$$

Under H_1

$$(\underline{s}_1 - \underline{s}_0)^T \underline{X} \sim N\left((\underline{s}_1 - \underline{s}_0)^T \underline{s}_1, \ \sigma^2 \|\underline{s}_1 - \underline{s}_0\|^2\right)$$

(a) $$P_F = P\left((\underline{s}_1 - \underline{s}_0)^T \underline{X} > \gamma \mid H_0 \right)$$

$$= Q\left(\frac{\gamma - (\underline{s}_1 - \underline{s}_0)^T \underline{s}_0}{\sigma \|\underline{s}_1 - \underline{s}_0\|} \right)$$

$$P_M = P\left((\underline{s}_1 - \underline{s}_0)^T \underline{X} < \gamma \mid H_1 \right)$$

$$= 1 - Q\left(\frac{\gamma - (\underline{s}_1 - \underline{s}_0)^T \underline{s}_1}{\sigma \|\underline{s}_1 - \underline{s}_0\|} \right)$$

$$P_E = \pi_0 P_F + \pi_1 P_M, \quad \eta = \frac{\pi_0}{\pi_1}$$

(b) $$\gamma = \sigma \|\underline{s}_1 - \underline{s}_0\| \cdot Q^{-1}(\alpha) + (\underline{s}_1 - \underline{s}_0)^T \underline{s}_0$$

(c) $$P_D = P\left((\underline{s}_1 - \underline{s}_0)^T \underline{X} > \gamma \mid H_1 \right)$$

$$= Q\left(\frac{\gamma - (\underline{s}_1 - \underline{s}_0)^T \underline{s}_1}{\sigma \|\underline{s}_1 - \underline{s}_0\|} \right)$$

$$= Q\left(\frac{\sigma \|\underline{s}_1 - \underline{s}_0\| \cdot Q^{-1}(\alpha) + (\underline{s}_1 - \underline{s}_0)^T \underline{s}_0 - (\underline{s}_1 - \underline{s}_0)^T \underline{s}_1}{\sigma \|\underline{s}_1 - \underline{s}_0\|} \right)$$

$$= Q\left(Q^{-1}(\alpha) - \frac{\| \underline{s}_0 - \underline{s}_1 \|}{\sigma} \right)$$

(d) $$\text{SNR} = \frac{\| \underline{s}_0 - \underline{s}_1 \|^2}{\sigma^2}$$

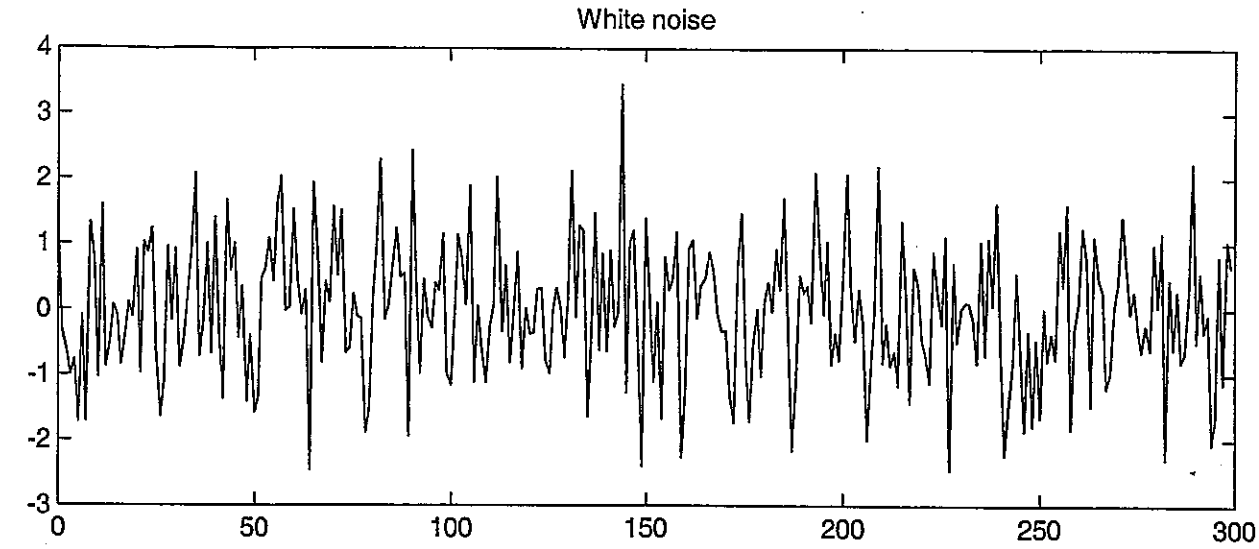
The Gaussian Assumption

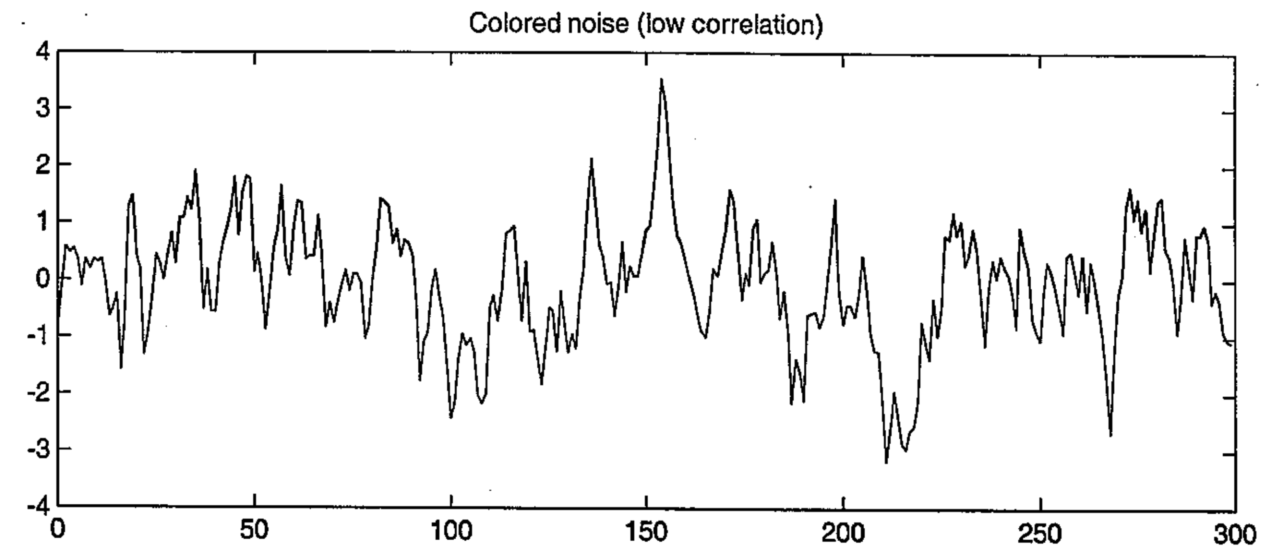
Is real world noise Gaussian? Is it white?

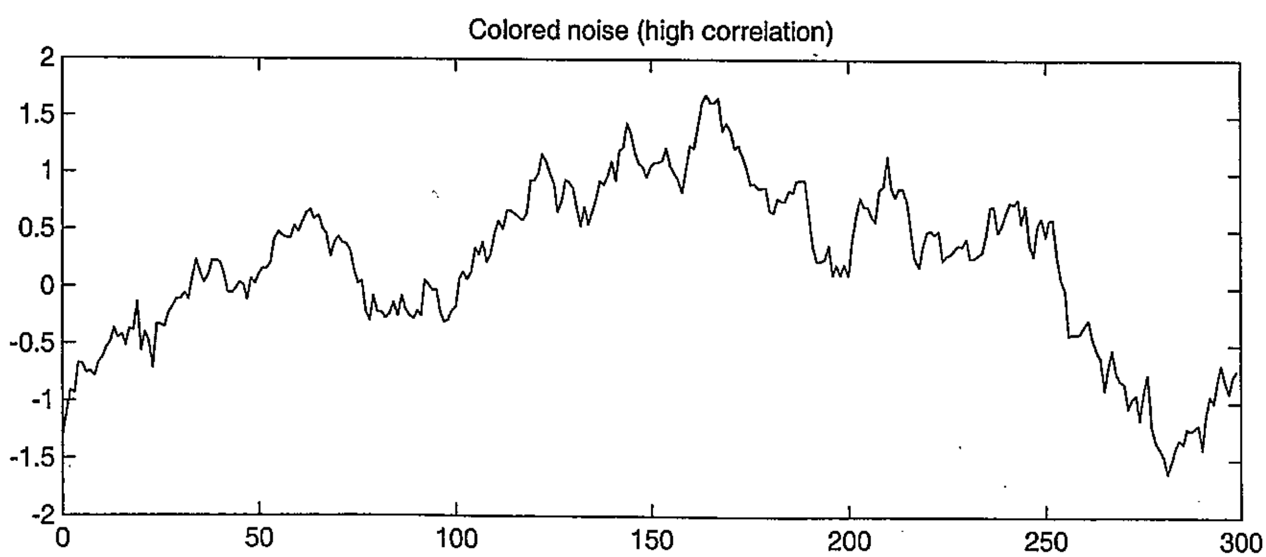
If the noise arises from a large number of random events, the central limit theorem suggests that the noise will be Gaussian.

<u>Example</u> In communication systems, electronic noise is due to the aggregate effect of huge numbers of charge carriers undergoing random motion.

However, the "whiteness" assumption is often violated. That is, errors tend to be <u>correlated</u> from sample to sample.


White noise
4
3
2
1
0
-1
-2
-3
0
50
100
150
200
250
300


Colored noise (low correlation)
4
3
2
1
0
-1
-2
-3
-4
0
50
100
150
200
250
300


Colored noise (high correlation)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
50
100
150
200
250
300

```
clear all
close all

% White versus colored Gaussian noise

N = 300;

% white noise
w = randn(N,1);

subplot(3,1,1);
plot(0:N-1,w);
title('White noise')

% colored noise (low correlation)
q=.8;
r=q.^(0:N-1);
R = Toeplitz(r); % covariance matrix

[U,D]=eig(R);
w=U*sqrt(D)*randn(N,1); % randn generates IID samples.
                        % this correlates the samples
subplot(3,1,2)
plot(0:N-1,w);
title('Colored noise (low correlation)');

% colored noise (high correlation)
q=.99;
r=q.^(0:N-1);

R = Toeplitz(r); % covariance matrix

[U,D]=eig(R);
w=U*sqrt(D)*randn(N,1); % randn generates IID samples.
                        % this correlates the samples
subplot(3,1,3)
plot(0:N-1,w);
title('Colored noise (high correlation)');

orient tall
```

Colored Gaussian Noise

$$H_0 : \underline{x} = \underline{s}_0 + \underline{w}$$

$$H_1 : \underline{x} = \underline{s}_1 + \underline{w}$$

$$\underline{W} \sim N(\underline{0}, R)$$

$$f(\underline{w}) = \frac{1}{(2\pi)^{\frac{N}{2}} |R|^{\frac{1}{2}}} e^{-\frac{1}{2}\underline{w}^T R^{-1} \underline{w}}$$

Log LRT

$$(\underline{x} - \underline{s}_1)^T R^{-1} (\underline{x} - \underline{s}_1) - (\underline{x} - \underline{s}_0)^T R^{-1} (\underline{x} - \underline{s}_0) \underset{H_0}{\overset{H_1}{\gtrless}} 2 \log \eta$$

or

$$(\underline{s}_1 - \underline{s}_0)^T R^{-1} \underline{x} \underset{H_0}{\overset{H_1}{\gtrless}} \log \eta + \frac{\underline{s}_1^T R^{-1} \underline{s}_1}{2} - \frac{\underline{s}_0^T R^{-1} \underline{s}_0}{2} \equiv \gamma$$

↑ test statistic: t

Performance :

$$H_0 : \quad t \sim N\left((\underline{s}_1 - \underline{s}_0)^T R^{-1} \underline{s}_0 , \; (\underline{s}_1 - \underline{s}_0)^T R^{-1} (\underline{s}_1 - \underline{s}_0) \right)$$

$$H_1 : \quad t \sim N\left((\underline{s}_1 - \underline{s}_0)^T R^{-1} \underline{s}_1 , \; (\underline{s}_1 - \underline{s}_0)^T R^{-1} (\underline{s}_1 - \underline{s}_0) \right)$$

$$P_F = Q\left(\frac{\gamma - (\underline{s}_1 - \underline{s}_0)^T R^{-1} \underline{s}_0}{\left[(\underline{s}_1 - \underline{s}_0)^T R^{-1} (\underline{s}_1 - \underline{s}_0) \right]^{1/2}} \right)$$

$$P_D = Q\left(\frac{\gamma - (\underline{s}_1 - \underline{s}_0)^T R^{-1} \underline{s}_1}{\left[(\underline{s}_1 - \underline{s}_0)^T R^{-1} (\underline{s} - \underline{s}_0) \right]^{1/2}} \right)$$

$$= Q\left(Q^{-1}(P_F) - \sqrt{SNR} \right)$$

where $\quad SNR \equiv (\underline{s}_1 - \underline{s}_0)^T R^{-1} (\underline{s}_1 - \underline{s}_0)$

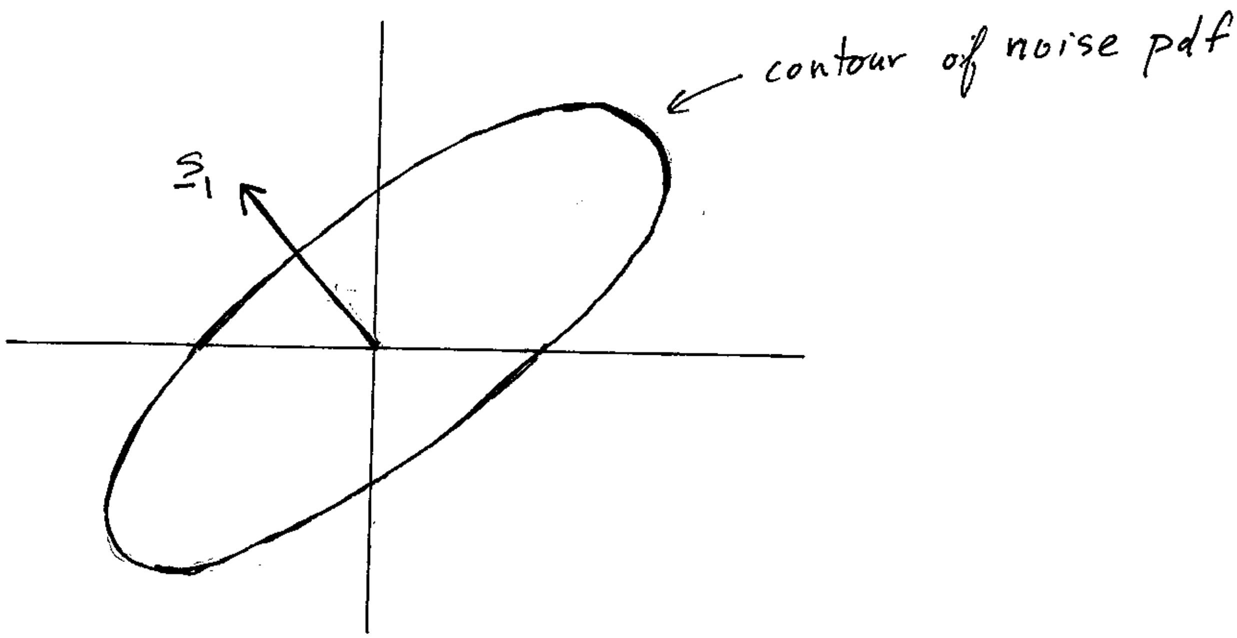
All of the detectors studied so far (today) can be deduced as special cases of this general form.

Signal Design

Suppose $\underline{s}_0 = \underline{0}$. How should we choose $\underline{s}_1$ to maximize SNR?

$$\underline{s}_1 = \arg\max_{\underline{s}} \frac{\underline{s}^T R^{-1} \underline{s}}{\underline{s}^T \underline{s}}$$ ← Rayleigh quotient

$\Longrightarrow$ $\underline{s}_1$ = eigenvector associated to smallest eigenvalue of R.



Prewhitening

Suppose we have colored noise

$$\underline{w} \sim \mathcal{N}(\underline{0}, R) \quad , \quad R \text{ known}$$

Since R is symmetric, we can write

$$R = \mathcal{U} \Lambda \mathcal{U}^T$$

where $\mathcal{U}\mathcal{U}^T = \mathcal{U}^T\mathcal{U} = I_{N\times N}$ and

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

Since R is positive definite, $\lambda_i > 0 \;\; \forall i$.

Then we can write

$$\tilde{\mathcal{U}} R \tilde{\mathcal{U}}^T = I$$

where

$$\tilde{\mathcal{U}} = \Lambda^{-\frac{1}{2}} \mathcal{U}^T \quad , \quad \Lambda^{-\frac{1}{2}} = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{\lambda_N}} \end{pmatrix}$$

Suppose we modify our observation according to

$$\tilde{\underline{x}} = \tilde{\mathcal{U}}\,\underline{x}$$

If $\underline{X} \sim \mathcal{N}(\underline{s}_i, R)$ under H_i,

then $\tilde{\underline{X}} \sim \mathcal{N}(\tilde{\underline{s}}_i, I)$, were $\tilde{\underline{s}}_i = \tilde{\mathcal{U}}\,\underline{s}_i$

In other words, if we know the covariance matrix R, we can reduce the problem of detecting a signal in <u>colored noise</u> to the problem of detecting a signal in <u>IID</u> noise

This process is called <u>prewhitening</u>, and the transformation $\tilde{\mathcal{U}}$ is called the <u>whitening filter</u>.

So we can reduce the colored noise problem to

$$H_0: \underline{x} = \underline{s}_0 + \underline{w}$$
$$H_1: \underline{x} = \underline{s}_1 + \underline{w}, \quad \underline{w} \sim N(\underline{0}, I).$$

Can we make any more reductions?

Define $\underline{\tilde{x}} = \underline{x} - \underline{s}_0$. Then the problem reduces to

$$H_0: \underline{\tilde{x}} = \underline{w}$$
$$H_1: \underline{\tilde{x}} = \underline{\tilde{s}} + \underline{w}$$

where $\underline{\tilde{s}} = \underline{s}_1 - \underline{s}_0$. This was the first problem we considered.

Conclusion Any binary detection problem involving Gaussian noise can be reduced to a "signal present vs. signal absent" problem in white Gaussian noise.

Multiple Hypotheses

Consider the problem of detecting M hypotheses in additive Gaussian noise.

$$H_1: \underline{x} = \underline{s}_1 + \underline{w}$$

$$H_2: \underline{x} = \underline{s}_2 + \underline{w}$$

$$\vdots$$

$$H_M: \underline{x} = \underline{s}_M + \underline{w}$$

where $\underline{w} \sim N(\underline{0}, \underline{I})$ ← Assume data is prewhitened

Recall the MAP detector:

Choose H_i such that $\pi_i f_i(\underline{x})$ is maximal

For simplicity, assume $\pi_k = \frac{1}{M}$, $k = 1, 2, \ldots, M$.

Then the minimum error probability is achieved by the maximum likelihood detector:

Choose H_i such that $f_i(\underline{x})$ is maximal

Now

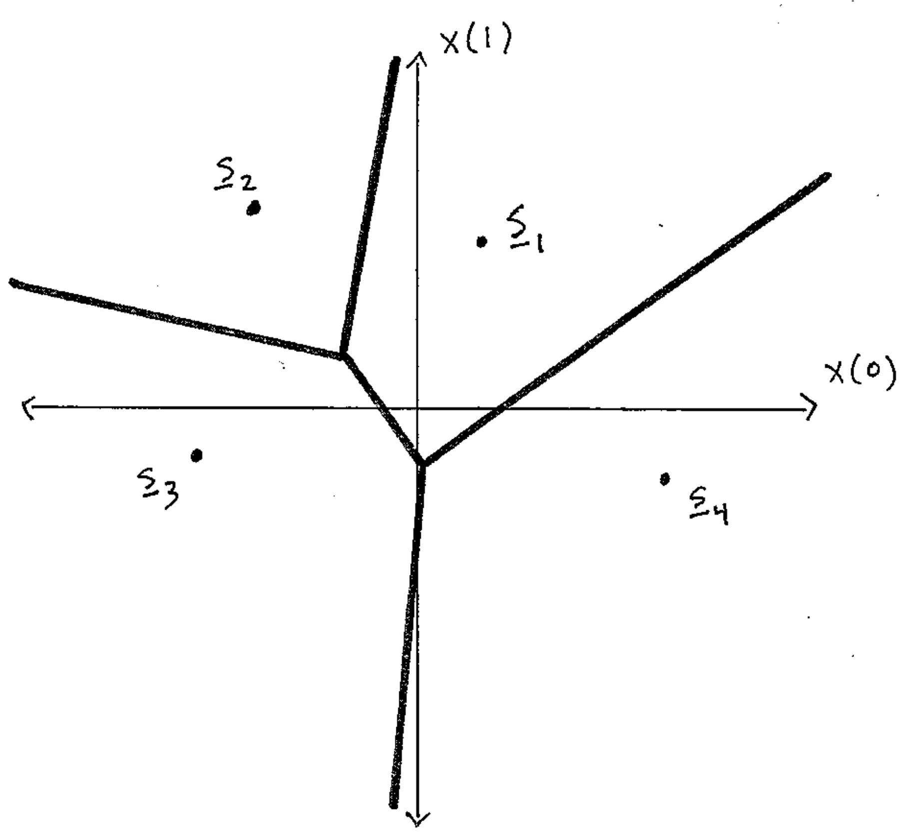
$$f_i(\underline{x}) = \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{1}{2}(\underline{x}-\underline{s}_i)^T(\underline{x}-\underline{s}_i)}$$

So maximizing $f_i(\underline{x})$ is equivalent to minimizing

$$(\underline{x}-\underline{s}_i)^T(\underline{x}-\underline{s}_i) = \|\underline{x}-\underline{s}_i\|^2$$

and the optimal detector reduces to a <u>nearest-neighbor detector</u>:

Choose H_i such that $\|\underline{x}-\underline{s}_i\|^2$ is minimal

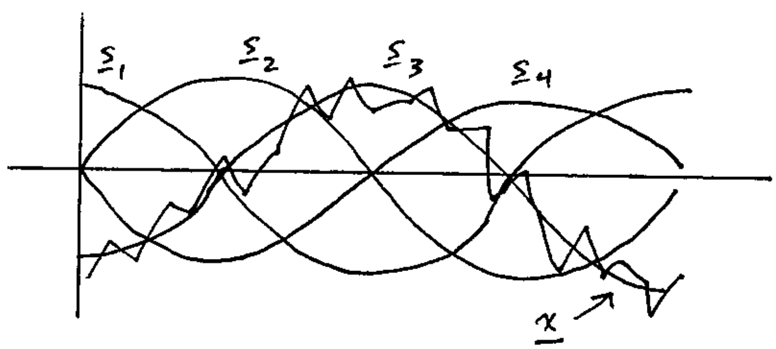


N=2

Equivalenty, this may be thought of as a <u>maximum correlation detector</u>:

Choose H_i such that $\underline{s}_i^T \underline{x}$ is maximal

provided all $\underline{s}_i$ have the same energy.



$\Rightarrow$ declare H_3

Summary

- Detection in Gaussian noise $\Longrightarrow$ many intuitive rules and interpretations:
  - max correlation
  - matched filter
  - projection
  - nearest neighbor

- Prewhitening reduces colored noise problem to white noise problem.

Key

a. $\underline{s}_1^T \underline{x} \underset{H_0}{\overset{H_1}{\gtrless}} \underline{s}_0^T \underline{x}$