CFAR DETECTION

Signal Detection in Unknown Noise Level

When the noise variance or covariance is not known, the signal detection problem can become significantly more difficult.

We can still apply the GLRT to obtain a detector, but the test statistic will depend on the noise under H_0 and H_1, which can make it difficult to set a meaningful threshold.

In certain special cases, however, it is possible to derive tests such that P_F is the same regardless of the noise.

<u>Def</u> | If the distribution of a test statistic under H_0 is independent of the noise variance (or other unknown parameter), then the detector is called a <u>constant</u> <u>false</u> <u>alarm</u> <u>rate</u> (CFAR) detector.

For a CFAR detector with a fixed threshold, P_F is constant as a function of the unknown parameter.

Consider

$$H_0: \quad \underline{X} = \underline{W}$$

$$H_1: \quad \underline{X} = A\underline{s} + \underline{W}$$

where

$$\underline{W} \sim N(\underline{0}, \sigma^2 R)$$

Here R is normalized such that $\mathrm{tr}(R) = N$.

Assume $\underline{s}$, R are <u>known</u>, A, σ^2 are <u>unknown</u>.

Essentially we are assuming we know the shape of the correlation function but not the overall magnitude, which is σ^2.

Does a UMP test exist?

No. The LRT reduces to

$$A \cdot \underline{x}^T R^{-1} \underline{s} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

but the sign of A is unknown (two-sided test).
Even if we knew A, the distribution of

$$\underline{x}^T R^{-1} \underline{s}$$

is dependent on σ^2 under H_0 (and H_1), so we cannot set the threshold.

Let's try the GLRT.

GLRT

$$\tilde{\Lambda}(\underline{x}) = \frac{\max_{A,\sigma^2} f_1(\underline{x}; A, \sigma^2)}{\max_{\sigma^2} f_0(\underline{x}; \sigma^2)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Denominator

$$f_0(\underline{x}; \sigma^2) = (2\pi\sigma^2)^{-\frac{N}{2}} |R|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2} \underline{x}^T R^{-1} \underline{x}\right\}$$

$$\log f_0(\underline{x}; \sigma^2) = -\frac{N}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \underline{x}^T R^{-1} \underline{x} + C$$

$$\frac{\partial}{\partial(\sigma^2)}\left(\downarrow \right) = 0 \Rightarrow \widehat{\sigma_0^2} = \frac{x^T R^{-1} x}{N}$$

Numerator

$$f_1(\underline{x}; A, \sigma^2) = (2\pi\sigma^2)^{-\frac{N}{2}} |R|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma} (\underline{x} - A\underline{s})^T R^{-1} (\underline{x} - A\underline{s})\right\}$$

$$\log f_1(\underline{x}; A, \sigma^2) = -\frac{N}{2} \log \sigma^2 - \frac{1}{2\sigma^2} (\underline{x} - A\underline{s})^T R^{-1} (\underline{x} - A\underline{s}) + C$$

$$\frac{\partial}{\partial A}\left(\downarrow \right) = 0 \Rightarrow \hat{A} = \frac{\underline{s}^T R^{-1} \underline{x}}{\underline{s}^T R^{-1} \underline{s}}$$ independent of σ^2

$$\frac{\partial}{\partial(\sigma^2)} \log f_1(\underline{x}; \hat{A}, \sigma^2) = 0$$

$$\Rightarrow \widehat{\sigma_1^2} = \frac{(\underline{x} - \hat{A}\underline{s})^T R^{-1} (\underline{x} - \hat{A}\underline{s})}{N}$$

$$= \frac{\left(\underline{x} - \frac{\underline{s}^T R^{-1} \underline{x}}{\underline{s}^T R^{-1} \underline{s}} \cdot \underline{s}\right)^T R^{-1} \left(\underline{x} - \frac{\underline{s}^T R^{-1} \underline{x}}{\underline{s}^T R^{-1} \underline{s}} \underline{s}\right)}{N}$$

The GLRT simplifies to

$$\left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)^{\frac{N}{2}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \iff \frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} \underset{H_0}{\overset{H_1}{\gtrless}} \eta^{2/N}$$

Consider the test statistic

$$\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} = \frac{\underline{x}^T R^{-1} \underline{x}}{\left(\underline{x} - \frac{\underline{s}^T R^{-1} \underline{x}}{\underline{s}^T R^{-1} \underline{s}} \cdot \underline{s}\right)^T R^{-1} \left(\underline{x} - \frac{\underline{s}^T R^{-1} \underline{x}}{\underline{s}^T R^{-1} \underline{s}} \cdot \underline{s}\right)}$$

Under H_0, $\underline{X} = \sigma \underline{\tilde{W}}$, $\underline{\tilde{W}} \sim N(\underline{0}, R)$, so

$$\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} = \frac{\sigma^2 \underline{\tilde{w}} R^{-1} \tilde{W}}{\left(\sigma \underline{\tilde{W}} - \sigma \frac{\underline{s}^T R^{-1} \underline{\tilde{W}}}{\underline{s}^T R^{-1} \underline{s}} \underline{s}\right)^T R^{-1} \left(\sigma \underline{\tilde{W}} - \sigma \frac{\underline{s}^T R^{-1} \underline{\tilde{W}}}{\underline{s}^T R^{-1} \underline{s}} \underline{s}\right)}$$

Lo and behold, the σ's cancel. What remains is independent of σ^2. That is, the GLRT is a CFAR detector!

Remarkably, the GLRT is not CFAR if A is known. In that case, the GLRT reduces to

$$\frac{\underline{x}^T R^{-1} \underline{x}}{(\underline{x}-\underline{s})^T R^{-1} (\underline{x}-\underline{s})} \underset{H_0}{\overset{H_1}{\gtrless}} \eta^{2/N}$$

and the previous argument breaks down (the σ's don't cancel).

Setting the Threshold

To set the threshold we need to determine the distribution of the test statistic under H_0.

For simplicity, let's assume $R = I$.

Test statistic: $(R = I)$

$$\frac{\underline{x}^T \underline{x}}{\left(\underline{x} - \frac{\underline{s}^T \underline{x}}{\underline{s}^T \underline{s}} \underline{s}\right)^T \left(\underline{x} - \frac{\underline{s}^T \underline{x}}{\underline{s}^T \underline{s}} \underline{s}\right)}$$

Note

$$\frac{\underline{s}^T \underline{x}}{\underline{s}^T \underline{s}} \underline{s} = \underline{s} \frac{\underline{s}^T \underline{x}}{\underline{s}^T \underline{s}} = \underbrace{\frac{\underline{s}\,\underline{s}^T}{\underline{s}^T \underline{s}}}_{\text{does this look familiar?}} \cdot \underline{x}$$

The test statistic is written as

$$\frac{\underline{x}^T \underline{x}}{(\underline{x} - P_s \underline{x})^T (\underline{x} - P_s \underline{x})}$$

Look at the denominator:

$$(\underline{x} - P_s \underline{x})^T (\underline{x} - P_s \underline{x}) =$$

(a)

$$=$$

We now have

$$= \frac{\underline{x}^T \underline{x}}{\underline{x}^T \underline{x} - \underline{x}^T P_s \underline{x}}$$

$$= \frac{\underline{x}^T (I - P_s) \underline{x} + \underline{x}^T P_s \underline{x}}{\underline{x}^T (I - P_s) \underline{x}}$$

$$= 1 + \frac{\underline{x}^T P_s \underline{x}}{\underline{x}^T (I - P_s) \underline{x}}$$

This gives us a modified test

$$t(\underline{x}) = \frac{\underline{x}^T P_s \underline{x}}{\underline{x}^T (I - P_s) \underline{x}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta^{2/N} - 1 = \gamma$$

Ok. So what? Well let's try to determine the distribution of the numerator and denominator of $t(\underline{x})$.

<u>Proposition</u> If P is a rank r projection matrix and $\underline{X} \sim N(\underline{0}, \sigma^2 I)$, then $\frac{\underline{X}^T P \underline{X}}{\sigma^2} \sim \chi^2_r$

Proof:

$$P = U \cdot \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix} U^T$$

$$\Rightarrow \quad P = \sum_{i=1}^{r} \underline{u}_i \underline{u}_i^T$$

$$\Rightarrow \quad \underline{X}^T P \underline{X} = \sum_{i=1}^{r} \underline{X}^T \underline{u}_i \underline{u}_i^T \underline{X} = \sum_{i=1}^{r} (\underline{u}_i^T \underline{X})^2$$

Now $\underline{u}_i^T \underline{X} \sim N(0, \sigma^2)$ since $\underline{u}_i^T \underline{u}_i = 1$.

In addition, if $i \neq j$, $\underline{u}_i^T X$ and $\underline{u}_j^T \underline{X}$ are uncorrelated (and hence independent) because

(b) $$E[(\underline{u}_i^T \underline{X})(\underline{u}_j^T \underline{X})] =$$

Def | If $u \sim \chi^2_p$, $V \sim \chi^2_q$ are independent, and $Z = \dfrac{u/p}{V/q}$, we say Z has an F-distribution with p, q degrees of freedom.

Recall our test statistic $t(\underline{x}) = \dfrac{\underline{x}^T P \underline{x}}{\underline{x}^T (I-P)\underline{x}} = \dfrac{\underline{x}^T P \underline{x} / \sigma^2}{\underline{x}^T (I-P) \underline{x} / \sigma^2}$

Clearly the numerator and denominator are chi-squared RVs. Are they independent? Writing

$$I - P = \sum_{i=r+1}^{N} \underline{u}_i \underline{u}_i^T$$

we can argue that $\underline{u}_i^T \underline{x}$, $i \le r$, and $\underline{u}_j^T \underline{x}$, $j \ge r+1$, are independent (as before), and therefore the two chi-squared RVs are independent.

In our case, $r = 1$ $\left(P = P_s = \dfrac{\underline{s}\,\underline{s}^T}{\underline{s}^T \underline{s}} \right)$.

Therefore, under H_0

$$(N-1)\, t(\underline{x}) \sim F_{1, N-1}$$

$$\Longrightarrow \gamma = \frac{1}{N-1} Q^{-1}_{F_{1,N-1}}(\alpha)$$

ensures $P_F = \alpha$ regardless of σ^2.

Summary:

CFAR: test statistic's distribution under H_0 is independent of unknown parameters $\Rightarrow$ constant false alarm rate

Most tests are not CFAR.

In special cases, like the unknown noise variance and unknown signal amplitude scenario, the GLRT is CFAR and F-distributed.

This allows us to design a detector and set a threshold to achieve a desired P_F even though σ^2 is unknown.

Key

a. $(\underline{x} - P_s \underline{x})^T (\underline{x} - P_s \underline{x})$

$$= \underline{x}^T \underline{x} - \underline{x}^T P_s \underline{x} - x^T P_s^T x + \underline{x}^T P_s^T P_s \underline{x}$$

$$= \underline{x}^T \underline{x} - \underline{x}^T P_s \underline{x}$$

[$P = P^T$ and $P = P^2$ for projections]

$$= \underline{x}^T (I - P_s) \underline{x}$$

b. $E[(\underline{u}_i^T \underline{x})(\underline{u}_j^T \underline{x})]$

$$= E[\underline{u}_i^T \underline{x} \cdot \underline{x}^T \underline{u}_j]$$

$$= \underline{u}_i^T \cdot \sigma^2 I \cdot \underline{u}_j$$

$$= 0$$

since $\underline{u}_i^T \underline{u}_j = \delta_{ij}$