Notes on MRI, Part 1

Overview

Magnetic resonance imaging (MRI) – Imaging of magnetic moments that result from the quantum mechanical property of nuclear spin. The average behavior of many spins results in a net *magnetization* of the tissue.

The spins possess a natural frequency that is proportional to the magnetic field. This is called the Larmor relationship:

$$w = gB$$

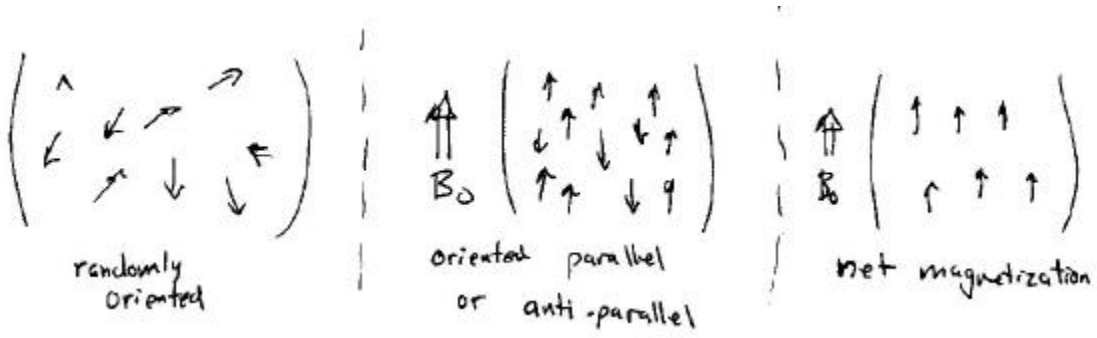
Any magnetization that is transverse (perpendicular) to an applied magnetic field **B** will precess around that **B** field at the Larmor frequency.

In MRI there are 3 kinds of magnetic fields:

1. B_0 – the main magnetic field
2. B_1 – an RF field that excites the spins
3. G_x, G_y, G_z – the gradient fields that provide localization

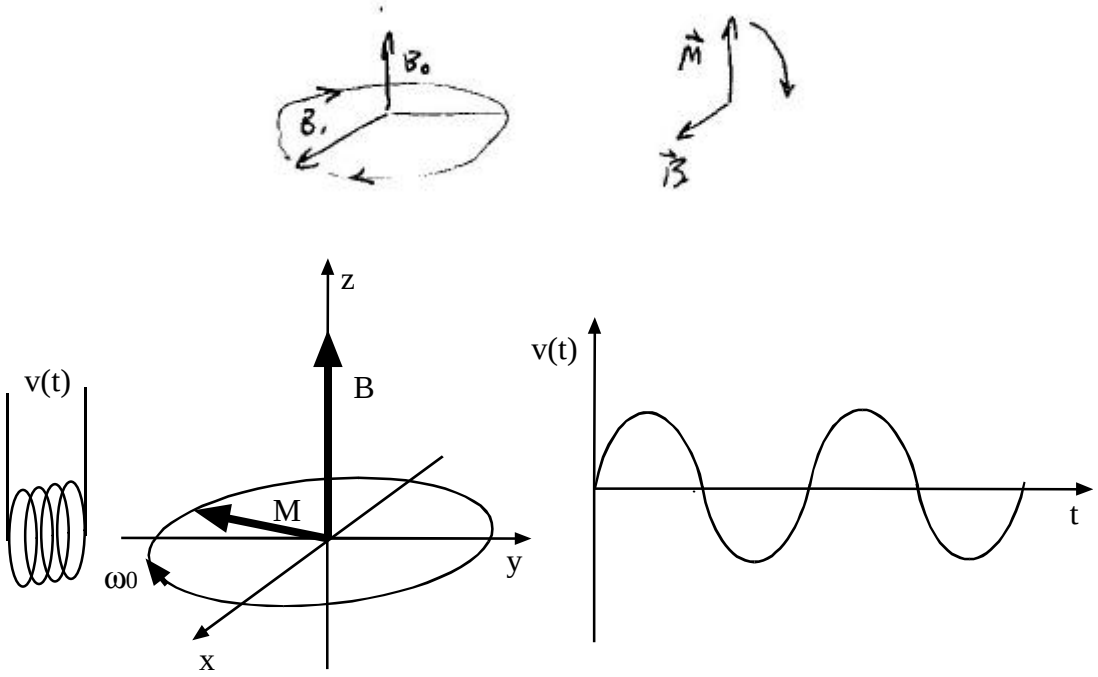
The major steps in a 1D MRI experiment are (we'll do 2 and 3 acquisitions later):

1. Object to be imaged is placed into the main field, B_0. Subsequently, the object develops a distribution of magnetization, $m_0(x,y,z)$, that is to be imaged. This magnetization is aligned with B_0 (in the z-direction).



2. A rotating RF magnetic field, B_1, is applied to tip the magnetization into the plane that is transverse to B_0. While in this plane, the magnetization precesses about the main field at a

frequency proportional to the strength of the main field ($w = gB$). This precessing magnetization creates a voltage in a receive coil, which is acquired for subsequent processing.

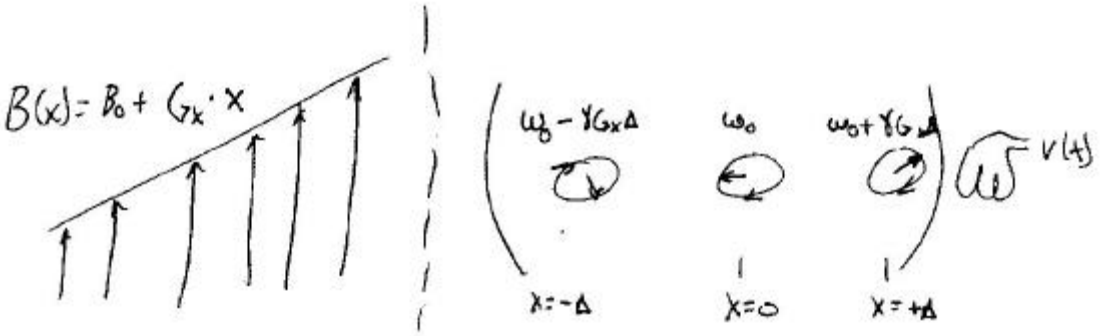


3. Gradient magnetic fields are applied to set-up a one-to-one correspondence between spatial position and frequency. For example, if we apply an *x* gradient, G_x, the magnetic field distribution is: $B(x) = B_0 + G_x \cdot x$, and thus:

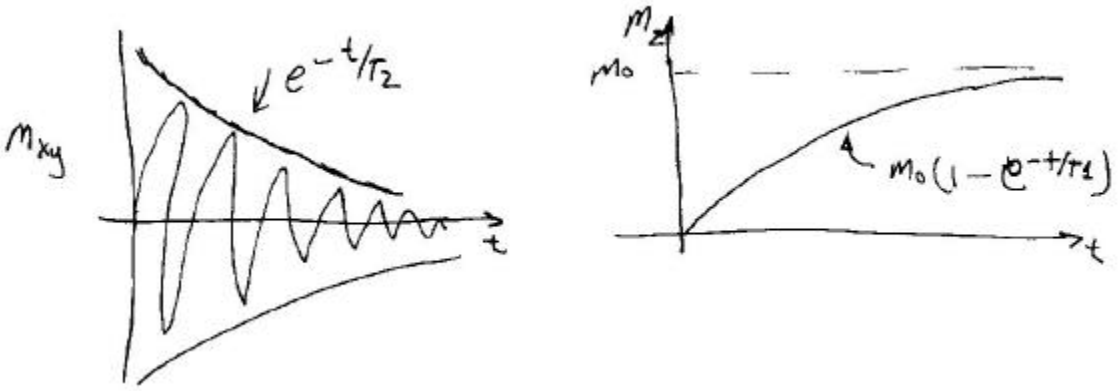
$$w(x) = w_0 + gG_x \cdot x.$$

By performing Fourier analysis on the received signal we can localize the magnetization in 1D:

$$m(x) = \iint m_0(x, y, z)dydz = F\{s(t)\}\Big|_{x=(2pf - w_0)/gG_x}$$



4. Following excitation, the magnetization in the transverse plane (*x*-*y*)decays away with time constant T2, e.g. $m_{xy}(t) = m_0 e^{-t/T2}$, and the z-component recovers with time constant T1, e.g. $m_z(t) = m_0(1 - e^{-t/T1})$. After this, the steps is repeated many times.



NMR Physics

The physical basis of Nuclear Magnetic Resonance (NMR) centers around the concept of a nuclear "spin," its associated angular momentum and its magnetic moment.

What is nuclear spin? "Spin" is a purely quantum mechanical quantity with no direct classical analogue (though we will talk about one anyway). We call it spin because this quantity give nuclei a net angular momentum (it also gives a nucleus its magnetic moment as well).

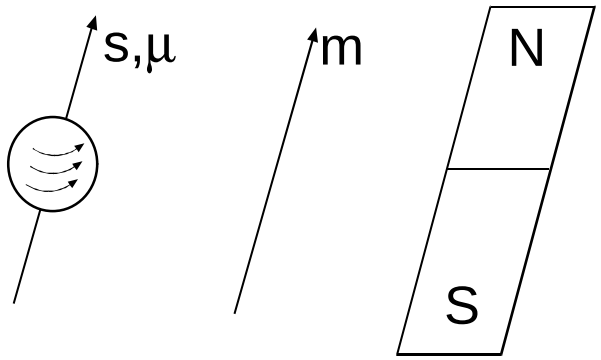
Consider a proton or hydrogen (^{1}H) nucleus. Spin will give this nucleus a "spin angular momentum," **s**, and a magnetic moment, $\boldsymbol{\mu}$, which are related though a proportionality constant, γ, in the following equation:

$$\boldsymbol{\mu} = \gamma \mathbf{s}$$

s and $\boldsymbol{\mu}$ are vector quantities and like many things in quantum mechanics, they can only take on discrete values.

This analogy is suspect, but I'll give it anyway. The classical analogue to the nuclear spin is a small charged sphere (representing a proton). The mass of the spinning particle give the angular

momentum and the charge on the surface give the net magnetic moment. The net magnetic moment can be viewed as a small magnetic dipole or bar magnet.



What nuclei exhibit this magnetic moment (and thus are candidates for NMR)?

Nuclei with: odd number of protons
odd number of neutrons
odd number of both

Magnetic moments: ^{1}H, ^{2}H, ^{3}He, ^{31}P, ^{23}Na, ^{17}O, ^{13}C, ^{19}F

No magnetic moment: ^{4}He, ^{16}O, ^{12}C

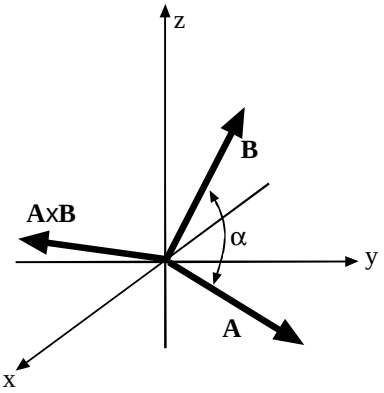
Spin Physics

Before talking about spins in a magnetic field, it is useful to review the behavior of a top in a gravitational field. And before talking about that, let's review the cross product operator.

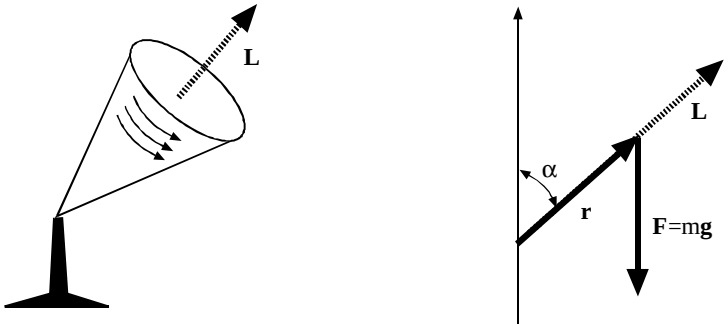
Cross-product. We start with a brief introduction to the cross-product operator. Given 2 vectors, **A** and **B**, the cross product can be written as:

$$\mathbf{A}\times\mathbf{B} = AB\sin a\hat{n}$$

where $\hat{n}$ is the unit vector perpendicular to **A** and **B**. The sign of $\hat{n}$ is determined by the "right hand rule."



Equations of motion for a top (gyroscope) in a gravitational field



In this drawing, the force generated by the mass of the top and the gravitational field (**F** = m**g**) appears to be acting at the center of mass of the top, which is located at position **r**, a distance *r* from the tip of the top. The angular momentum of the top is **L** (**F**, **L**, **g**, and **r** are all vector quantities). The simplified equation of motion for this top, describes the torque on the angular momentum:

$$\mathbf{T} = \frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}$$

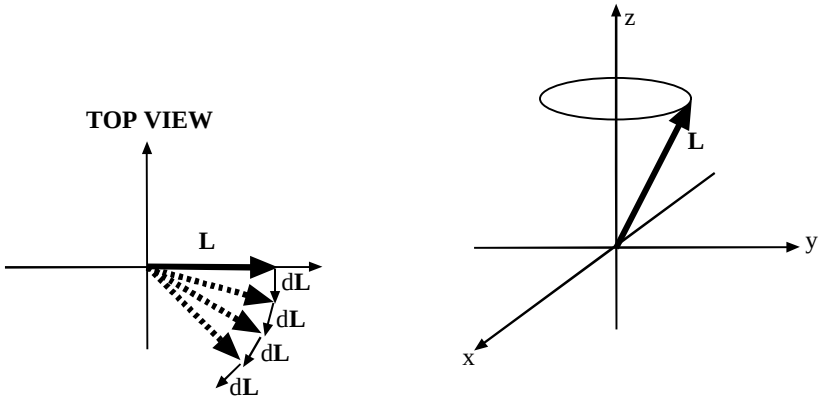
$$\frac{d\mathbf{L}}{dt} = r\hat{n}_r \times \mathbf{F} = r\frac{\mathbf{L}}{L} \times m\mathbf{g}$$

$$\frac{d\mathbf{L}}{dt} = \mathbf{L} \times \left(\frac{rm}{L} \right) \mathbf{g}$$

The tip of the angular momentum vector move at a speed given by:

$$\left|\frac{d\mathbf{L}}{dt}\right| = rmg\cos a$$

where a is the angle between the axis of the top and the direction of the gravitational field (vertical axis). The direction the tip moves is perpendicular to the plane containing the axes of both **L** and **g** (the top and gravitational field). This is always true and the thus as the position of the top changes, so does the direction of movement. The locus of points traced out by the tip of the **L** vector form a circle.



These relationship works out so that the top precesses around the gravitational field. It can be shown that the precession frequency is:

$\Omega = (rmg)/L$ (units are radians per second)

Thus the top will precess around g at a rate proportional to the mass of the top, the strength of the gravitational field, the distance from the tip to the center of mass and inversely proportional to the angular momentum (which is related to the distribution of mass).

Classical description of a spin in a magnetic field.

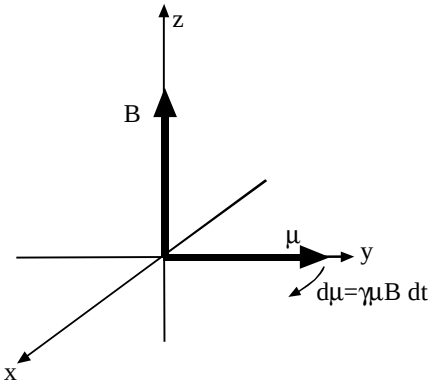
Since the spin had angular momentum, it does not just snap to alignment with the field (like the needle on a compass). This is much like a top in a gravitational field – the gravitational field exerts a torque on the top causing it to precess rather than fall in the direction of the gravitational field.

A spin (characterized by **s** and μ) in a magnetic field **B**, behaves as follows:

$$\frac{d\mathbf{s}}{dt} = \grave{\imath} \times \mathbf{B}$$

$$\frac{d\grave{\imath}}{dt} = g\grave{\imath} \times \mathbf{B}$$

The second expression follows from $\boldsymbol{\mu} = g\,\mathbf{s}$. For the case where $\boldsymbol{\mu}$ and **B** are perpendicular, then the magnitude of $d\boldsymbol{\mu}/dt$ (speed at which the tip of $\boldsymbol{\mu}$ moves) is $|g\boldsymbol{\mu}\mathbf{B}| = g\mu B$.



Given that the circumference of the circle here is $2\pi\mu$, the time for one cycle of precession is $2\pi\mu/g\mu B$, and the frequency of precession is thus, $f = gB/2\pi$ or $\omega = gB$. The latter is the most important relationship in NMR and MRI. It is known as the Lamor relationship:

$$\boxed{\omega = g\,B}$$

The parameter g is the "gyromagnetic ratio" and is nuclei dependent. For protons (^{1}H), $g/2\pi$ = 42.58 MHz/T (4.258e7 T^{-1} – I often use this notation for to mean 4.258 x10^7 T^{-1}).

A word about terminology. In MRI, the quantity, B, is usually called the "magnetic field strength," which engineers traditionally call "magnetic flux density." Units of flux density are Telsa (T) = 10^4 Gauss (g) = Webbers (Wb)/m^2, where Wb = Ampere-Henry (A H). The flux density is related to a quantity, H, known as "magnetic field intensity" in the following relationship:

$$\mathbf{B} = \mu_0\,\mathbf{H}$$

Where μ_0 is the permeability of free space ($\mu_0 = 4\pi$e-7 H/m) and **H** has units of A/m. In any substance other than free space (vacuum), we have to consider the magnetic susceptibility:

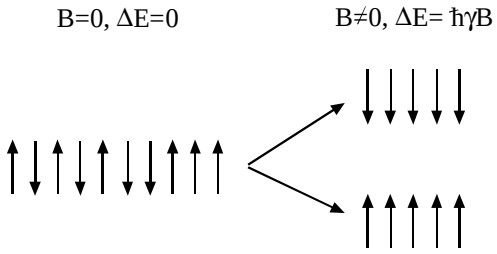
$$\mathbf{B} = \mu_0\,(1 + \chi_m)\mathbf{H}$$

Where χ_m is the magnetic susceptibility (unitless) – the ability of a substance to produce an internal magnetic field in response to an applied magnetic field. χ_m can be positive or negative (paramagnetic or diamagnetic).

Some useful units conversions: W=J/s (power), V=Wb/s, J/T = Am^2 (magnetic moment), Am^2/m^3 = A/m (magnetization), kg m /(A^2 s^2) = H/m (permeability), Wb = A H = J/A,
T = Wb/m^2 = AH/m^2 = $J/(A\ m^2)$.

Quantum mechanical (QM) description of a spin in a magnetic field.

With no applied magnetic field, all spins are in the same energy state (E=0). Their magnetic moments are randomly oriented are do not form any coherent magnetization. When placed in an applied magnetic field, the spin will tend to align with or opposite to the direction of applied magnetic field. These two states are known as "spin up" and "spin down," respectively. The spin-up state (in alignment) is slightly preferred, and thus has a lower energy level. The spin-down state is at a higher energy. A spin-up nuclei can absorb energy and transition to a spin-down and a spin-down nuclei can give up energy and transition to spin-up. These energy states are similar to electron energies in a neon atom, except here there are only two possible energy states.



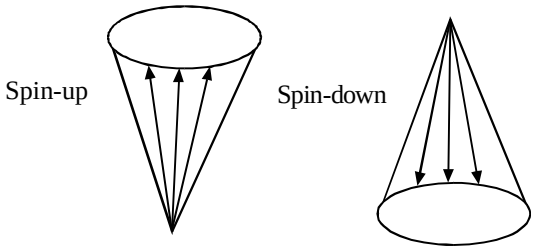
The energy difference between these states is determined by the strength of the applied magnetic field, which will we will call B_0, in the following relationship:

$$\Delta E = \hbar g B_0 = \hbar w_0 = h f_0$$

where *g* is the gyromagnetic ratio, *h* is Plank's constant (*h* = 6.63e-34 J s = 4.14e-15 eV s) and $\hbar = h / 2p$.

If we inject energy into this system (excite the system) at a frequency f_0, we should be able to induce spin-flip transitions between the two energy states. As we shall see later, this system is very selective to that specific energy level – higher and lower frequencies won't work. Excitation must be a this specific frequency in order to "resonate" with the nuclei this frequency selectivity is the origin of the term resonance in nuclear magnetic resonance.

The spin (and associated magnetic moment and angular momentum) is probabilistic in nature (much in the same way that electrons surrounding the nucleus travel in probabilistic volumes (or shells)). Thus, each spin doesn't really align with the **B**, but rather exists in a probabilistic cone and spin-up and spin-down implies that probabilistic cone faces up or down.



The spin and magnetic moment exist in all directions simultaneously, but average behavior is non-zero in only one of the directions:

$$\langle m_x \rangle = \langle m_y \rangle = 0; \langle m_z \rangle = \tfrac{1}{2}\hbar g; \left| \right| = \hbar g$$

Question: What is the population distribution (of nuclei) in these two energy states and how many more are in the lower state?

These are governed by thermal equilibrium condition, which are characterized by the Boltzmann distribution. Letting N+ be the higher energy state (spin-down) and N- be the lower energy state, Boltzmann dictates that:

$$\frac{N+}{N-} = e^{-\Delta E / kT}$$

where

k = Boltzmann's constant (8.62e-5 eV/K or 1.38e-23 J/K)

T = temperature (human body temperature = 310 K)

$\Delta E = \hbar g\ B_0$

In general, the exponent is extremely small and N+ and N- are nearly the same and approximately ½ of the total number of nuclei. Using the first two terms of the Taylor series expansion of the exponent, we get:

$$\frac{N+}{N-} \approx 1 - \frac{\Delta E}{kT}$$

$$\Delta N = (N-) - (N+) = \frac{\Delta E}{kT} N+ \approx \frac{\Delta E}{kT} \tfrac{1}{2} N_T$$

$$\Delta N = \frac{\hbar g B_0}{kT} \tfrac{1}{2} N_T$$

Important! Please note that ΔN, then number of excess nuclei in lower vs. upper energy states is proportional to B_0. It is also proportional to *g*. These excess nuclei are the source of magnetization for all MRI experiments. It follows then, that a larger magnetic field, B_0, will generate larger magnetization to perform our imaging experiments and different nuclei will develop differing amounts of magnetization depending on their concentration in the body (N_T) and their *g*.

What fraction are spin-up vs. spin-down? $\frac{\hbar g B_0}{kT} \approx$ 7e-6 (for 310K, B_0 = 1 T). That is, for every million nuclei in the spin-down state, there are about 1 million plus 7 extra nuclei in the spin-up state.

How big is N_T? Consider water – one gram of water contains 1/18 mole of water molecules and 1/9 mole of ^{1}H. Given Avogadro's number (6.023e23), for 1 cc (1 gm) or water, N_T = 6.68e22.

Thus, for every cc of water (tissue is mostly water) at 1 T, ΔN ≈ 2.2e17 (!).

Connection between QM and classical descriptions.

We cannot observe individual spins, only the ensemble average. Fortunately, it can be shown that the ensemble average equations of motion is:

$$\left\langle \frac{d\hat{\mathbf{i}}}{dt} \right\rangle = \frac{d}{dt} \langle \hat{\mathbf{i}} \rangle = g \langle \hat{\mathbf{i}} \rangle \times \mathbf{B}$$

We now define two more quantities. The "net magnetic dipole" is:

$$\mathbf{m} = \Delta N \langle \mathbf{i} \rangle$$

And the "magnetization" is the magnetic dipole/unit volume:

$$\mathbf{M} = \mathbf{m}/dV$$

Since only the z-component of the spins shows a preferential direction, the net magnetic dipole is created in this direction:

$$|\mathbf{m}| = \Delta N \langle m_z \rangle = \Delta N \tfrac{1}{2} \hbar g = \Delta N \text{ (1.4e-26 J/T)}$$

For, 1 g of water at 310K and 1 T, the net magnetic dipole is

$$|\mathbf{m}| = 3.1\text{e-}9 \text{ Am}^2$$

One gram of water occupies 1 cc (10^{-6} m^3), thus the nuclear magnetization of water is:

$$|\mathbf{M}| = 3.1\text{e-}3 \text{ A/m}$$

Important! This is the nuclear magnetization. There are other things (notably electrons) that lead to further magnetization of materials. It is the 3 unpaired electrons (not the nucleus) in iron and gadolinium that give these substances their very large magnetic properties.

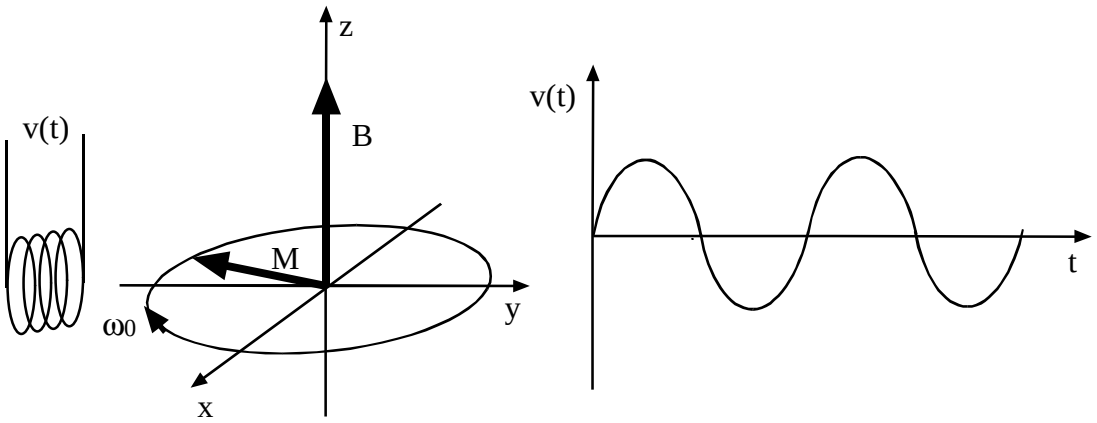
Behavior of magnetization in the presence of applied magnetic fields

The main result is "Bloch Equation" (named for Felix Bloch, the Nobel laureate who co-discovered MR in 1946):

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times g\mathbf{B}$$

which says that the magnetization **M** will precess around a **B** field at frequency $\omega = \gamma B$.

Now consider **M** lying a plane perpendicular to the main magnetic field **B**, which has strength B_0. We first define a coordinate system in which the applied field is assumed to be in the z-direction, thus $\mathbf{B} = B_0\mathbf{k}$, where (**i**, **j**, **k**) are the unit-length vectors in the (x, y, z) directions. For this system, **M** will precess in the x-y plane at frequency $\omega_0 = \gamma B_0$ as shown below:

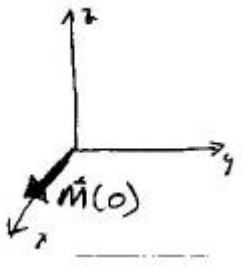


If we place a small loop of wire near this precessing magnetization, we will induce a voltage in the coil, $v(t)$, at frequency, $\omega_0 = \gamma B_0$.

Induction of a voltage in a coil from magnetization precessing in the x-y plane is the basis of signal reception in MRI.

Solutions to the Bloch Equation:

$\mathbf{M} = [m_x, m_y, m_z]$ and let the initial condition of $\mathbf{M}(0) = [m_0, 0, 0]$.



Let **i**, **j**, and **k** be the unit vectors in the *x*-, *y*- and *z*-directions. Thus:

$$\mathbf{B} = B_0\,\mathbf{k} \quad \text{and} \quad \mathbf{M}(0) = m_0\,\mathbf{i}$$

The Bloch equation then becomes:

$$\frac{d\mathbf{M}}{dt} = (m_x\mathbf{i} + m_y\mathbf{j} + m_z\mathbf{k}) \times gB_0\mathbf{k}$$

$$\frac{d}{dt}(m_x\mathbf{i} + m_y\mathbf{j} + m_z\mathbf{k}) = gB_0 m_x(\mathbf{i}\times\mathbf{k}) + gB_0 m_y(\mathbf{j}\times\mathbf{k}) + gB_0 m_z(\mathbf{k}\times\mathbf{k})$$

$$= gB_0 m_x(-\mathbf{j}) + gB_0 m_y(\mathbf{i}) + 0$$

This can be rewritten as a matrix differential equation:

$$\frac{d}{dt}\begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} = \begin{bmatrix} 0 & gB_0 & 0 \\ -gB_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} \text{ and } \begin{bmatrix} m_x(0) \\ m_y(0) \\ m_z(0) \end{bmatrix} = \begin{bmatrix} m_0 \\ 0 \\ 0 \end{bmatrix}$$

We can start out by solving the last row of this equation:

$$\frac{dm_z}{dt} = 0 \text{ and } m_z(0) = 0 \Rightarrow m_z(t) = 0$$

To solve the first two lines, we define a new term, $m_{xy} = m_x + i\, m_y$:

$$\frac{dm_{xy}}{dt} = \frac{dm_x}{dt} + i\frac{dm_y}{dt}$$

$$= gB_0 m_y - igB_0 m_x$$

$$= -igB_0(m_x + im_y)$$

$$= -igB_0 m_{xy}$$

$$= -iw_0 m_{xy} \quad \text{and} \quad m_{xy}(0) = m_0$$

The solution to this simple linear differential equation is:

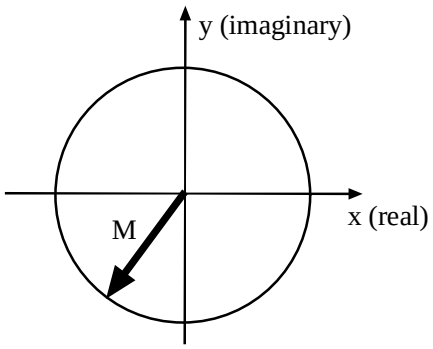
$$m_{xy}(t) = m_{xy}(0)e^{-iw_0 t} = m_0 e^{-iw_0 t} = m_0(\cos(w_0 t) - i\sin(w_0 t))$$

and thus:

$$\begin{bmatrix} m_x(t) \\ m_y(t) \\ m_z(t) \end{bmatrix} = \begin{bmatrix} m_0\cos(w_0 t) \\ -m_0\sin(w_0 t) \\ 0 \end{bmatrix}$$

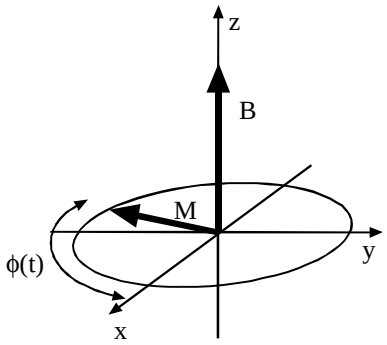
Here magnetization, m_0, precesses around B_0 at frequency $w_0 = gB_0$. The Bloch equations, have the Larmor relationship built right in!

The quantity, $m_{xy} = m_x + i\, m_y$, is a transformation the x-y components of **M** into the complex plane. This allows us to have a simplified expression for the magnetization:



Now, let's consider a non-constant **B**: $\mathbf{B}(t) = [B_0 + \Delta B(t)]\mathbf{k}$ (the **B** field is still applied along the z-axis). As before, **M** will still precess around B, but now the frequency of precession will vary with time:

$$\omega(t) = \gamma[B_0 + \Delta B(t)]$$



The direction that the **M** points (the phase of **M**) is given by the time integral of the frequency function:

$$f(t) = g\int_0^t [B_0 + \Delta B(t)]dt = w_0 t + g\int_0^t \Delta B(t)dt$$

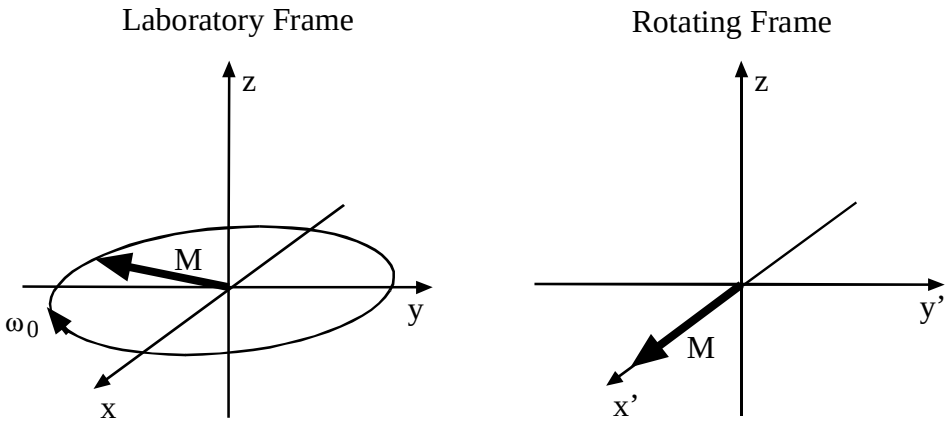
And thus,

$$m_{xy}(t) = m_0 e^{-i\left[w_0 t + g\int_0^t \Delta B(t)\,dt\right]}$$

Rotating Frame of Reference

One of the more useful tools in simplifying MRI concepts is the rotating frame of reference. Here we consider that our coordinate system for observation of the magnetization is rotating at a

frequency, $\omega_0 = \gamma B_0$. In particular, the coordinate system is rotating about the z-axis in the same direction that **M** rotates about **B**. The z coordinate does not change, but we now must define a new x and y coordinate system. The "laboratory" frame of reference is the usual frame of reference with coordinates (x, y, z). The "rotating" frame of reference has coordinates (x', y', z). If we have magnetization precessing at ω_0, it will appear to be stationary in the rotating frame of reference.



Conceptually, we can think of this as being similar to riding on a carousel. If we are on the carousel, other objects on the carousel appear stationary, but to someone on the ground, the objects are spinning by at $w_{\text{carousel}}=w_0$.

For a rotation frame at w_0, the coordinate axes are transformed in this way:

$$\mathbf{i}' = \mathbf{i}\cos(w_0 t) - \mathbf{j}\sin(w_0 t)$$

$$\mathbf{j}' = \mathbf{i}\sin(w_0 t) + \mathbf{j}\cos(w_0 t)$$

$$\mathbf{k}' = \mathbf{k}$$

Thus, when $\mathbf{B} = B_0\mathbf{k}$, the apparent **B** in the rotating frame is:

$$\mathbf{B}_{\mathbf{eff}} = (B_0 - \frac{w_{frame}}{g})\mathbf{k} = (B_0 - \frac{w_0}{g})\mathbf{k} = (B_0 - B_0)\mathbf{k} = 0$$

The x-y components of the magnetization are then:

$$m_{xy.rot}(t) = m_{xy}(t)\exp(i\, w_0\, t) = m_0$$

which is stationary. We now have a simple conversion of magnetization in the rotating frame and the lab frame. If $\mathbf{M} = [m_x, m_y, m_z]$ and $\mathbf{M}_{\mathbf{rot}} = [m_{x,rot}, m_{y,rot}, m_{z,rot}]$, then

$$m_{xy.rot} = m_{x,rot} + i\, m_{y,rot} = m_{xy}\exp(i\, w_0\, t)$$

$$m_{z,rot} = m_z$$

Let's now consider $\mathbf{B}(t) = [B_0 + DB(t)]\mathbf{k}$. Here the magnetization in the rotation frame will appear to be precessing at

$$w_{rot}(t) = g[B_0 + DB(t)] - w_0 = gDB(t)$$

Thus, the apparent **B** in the rotating frame ($w_{\text{frame}}=w_0$) is:

$$\mathbf{B}_{\mathbf{eff}} = \mathbf{B} - \frac{w_0}{g}\mathbf{k} = (B_0 + \Delta B(t) - B_0)\mathbf{k} = \Delta B(t)\mathbf{k}$$

The direction that the $\mathbf{M}_{\mathbf{rot}}$ points is given by the time integral of this frequency function:

$$f_{rot}(t) = g\int_0^t \Delta B(t')dt'$$

And thus,

$$m_{xy,rot}(t) = m_0 e^{-i\left[g\int_0^t \Delta B(t')dt'\right]}$$

The Bloch equation can now be rewritten for use in the rotating frame:

$$\frac{d\mathbf{M}_{rot}}{dt} = \mathbf{M}_{rot} \times g\mathbf{B}_{eff}$$

where **M** can be derived from $\mathbf{M}_{\mathbf{rot}}$ using:

$$m_{xy} = m_{xy,rot}\exp(-i\, w_0\, t)$$

$$m_z = m_{z,rot}$$

Excitation

The preceding discusses the behavior of **M** when it is a plane perpendicular to $\mathbf{B} = B_0\mathbf{k}$. That is, the magnetization is the plane transverse to the main field. Earlier, we described placing the spins in a magnetic field and developing a magnetization in the same direction as **B**. So the obvious questions is, how does one get the magnetization that points along the z-axis to lie in the plane perpendicular to this axis?

Answer: RF excitation.

RF (radiofrequency) magnetic fields are applied. These are rotating magnetic fields applied in the plane transverse to B_0**k.** This field is usually called $\mathbf{B_1}$ (B_0 is the "main magnetic field"). If the frequency of the RF pulse is w_{RF}, then the applied RF field can be written as:

$$B_{1x} = B_1 \cos(w_{RF}\, t) \text{ and } B_{1y} = -B_1 \sin(w_{RF}\, t)$$

$$\text{Equivalently: } B_{1xy} = B_1 \exp(-i\, w_{RF}\ \ t)$$

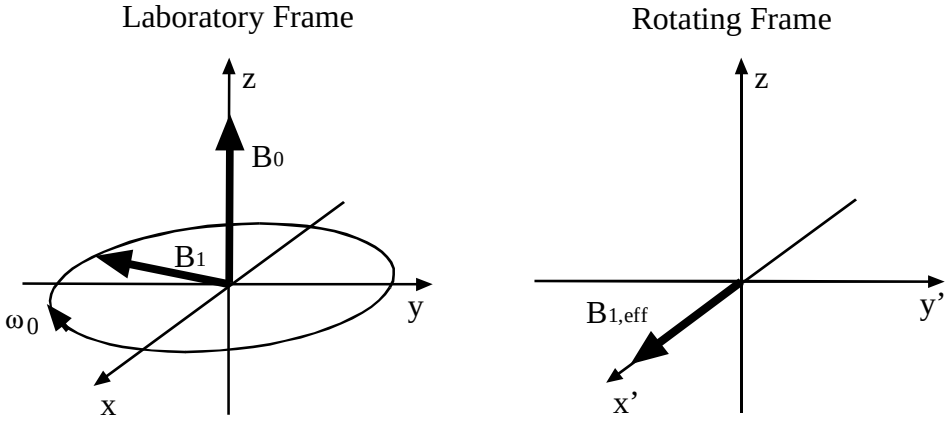
Let's look at a special case, where $w_{RF} = w_0$. Here, the total applied **B** field is:

$$\mathbf{B}(t) = \begin{bmatrix} B_1 \cos(w_0 t) \\ -B_1 \sin(w_0 t) \\ B_0 \end{bmatrix}$$

Again, in a frame rotating at w_0, $\mathbf{B_{1,eff}}$ will appear stationary. Thus:

$$\mathbf{B_{eff}}(t) = \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix}$$

Which is constant: no time dependent variations, rotations, etc.



Behavior of M **in the presence of** B_1

Recall, we said that the Bloch equation, which describes the motion of **M** in the presence of a **B** field, dictates that the magnetization will precess around the **B** field at frequency γB. Here, again, is the **B** field includes $B_0$and B_1:

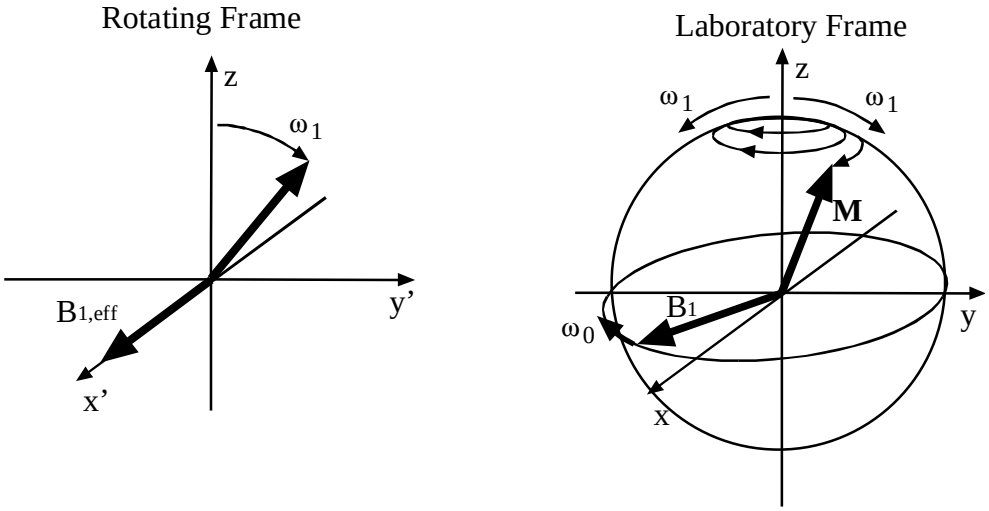
$$\mathbf{B}(t) = \begin{bmatrix} B_1 \cos(w_0 t) \\ -B_1 \sin(w_0 t) \\ B_0 \end{bmatrix}$$

As you might guess, determining the motion of **M** in the case can be quite difficult. But fortunately, we have a tool to make this analysis easier: the rotating frame and the rotating frame version of the Bloch equation:

$$\frac{d\mathbf{M}_{\mathbf{rot}}}{dt} = \mathbf{M}_{\mathbf{rot}} \times g\mathbf{B}_{\mathbf{eff}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & gB_1 \\ 0 & -gB_1 & 0 \end{bmatrix} \mathbf{M}_{\mathbf{rot}}$$

Also, let's consider the magnetization starting in its equilibrium position occurs from placing the object in the large magnetic field (aligned to the main magnetic field): $\mathbf{M}(0) = m_0\mathbf{k}$. The above matrix differential equation can be solved in a manner very similar to the case for **M** precessing around $B_0\mathbf{k}$, by creating $m_{yz} = m_{y,rot} + i\, m_z$ and solving for the solution of these linked terms. Since the $\mathbf{B_{1,eff}}$ is applied along the x' axis, $\mathbf{M_{rot}}$ will precess around x' in the z-y' plane and will precess at frequency $w_1 = gB_1$. Thus:

$$\mathbf{M}_{\mathbf{rot}}(0) = \begin{bmatrix} 0 \\ 0 \\ m_0 \end{bmatrix}; \quad \mathbf{M}_{\mathbf{rot}}(t) = \begin{bmatrix} 0 \\ m_0 \sin(w_1 t) \\ m_0 \cos(w_1 t) \end{bmatrix}$$



If we go back to the lab frame, then motion of M is rather complex – simultaneously precessing about $\mathbf{B_1}$ at ω_1 and about $B_0\mathbf{k}$ at ω_0. Using the relationships that related rotating frame to lab frame we get:

$$m_{xy,rot} = \mathrm{i}\, m_0 \sin(w_1 t) = m_{xy} \exp(i\, w_0 t)$$

$$m_{xy} = \mathrm{i}\, m_0 \sin(w_1 t) \exp(-i\, w_0 t)$$

And thus:

$$\mathbf{M}(t) = \begin{bmatrix} m_0 \sin(w_1 t)\sin(w_0 t) \\ m_0 \sin(w_1 t)\cos(w_0 t) \\ m_0 \cos(w_1 t) \end{bmatrix}$$

These equations for **M** trace out the path along the surface of a sphere that is spiraling downward as shown above. It can be shown that this **M** satisfies the Bloch equation:

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times g\,(B_0\mathbf{k} + \mathbf{B}_1)$$

Usually, B_1 is much smaller than B_0. Typical ranges of values: $\omega_1 \sim 1$ kHz and $\omega_0 \sim 10$'s to 100's of MHz, thus B_1 is about 5 orders of magnitude smaller than B_0.

Now, if we want the magnetization to end up in the transverse (x-y) plane, we can apply the B_1 field for a period of time and then stop. If we have a constant B_1 for a period of time T, then we want:

$$w_1 T = gB_1 T = \pi/2$$

This RF pulse is known as a $\pi/2$ or 90 degree pulse. Example – suppose B_1 = 0.2 g = 2e-5 T. Then

$$w_1 = gB_1 = 2\pi(852)\ \mathrm{s}^{-1}$$

For a 90 degree pulse, T = 294 μs.

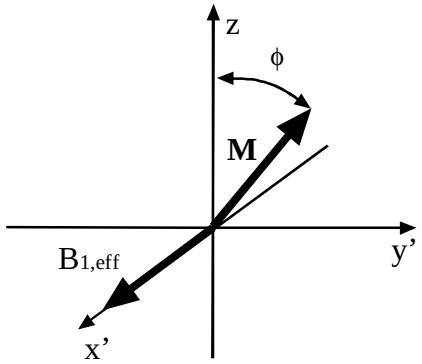
We don't have to just stop at 90 degrees indeed, we can stop at nearly any point along the way. The angle between the z axis and the magnetization after the RF pulse, ϕ, is called the "flip angle" or "tip angle" and is given by:

$$f = gB_1 T$$

or for the general case of a time varying $B_1(t)$, we have:

$$\mathbf{M}_{\mathbf{rot}}(t) = \begin{bmatrix} 0 \\ m_0 \sin\left(g \int_0^t B_1(t)\,dt \right) \\ m_0 \cos\left(g \int_0^t B_1(t)\,dt \right) \end{bmatrix}$$

$$f = g\int_0^T B_1(t)dt$$



Finally, we derive the Bloch equations, in the rotation frame for the general case of a time-varying B_1 and a non-zero field in the z-direction:

$$\mathbf{B}(t) = \begin{bmatrix} B_1(t)\cos(w_0 t) \\ -B_1(t)\sin(w_0 t) \\ B_0 + \Delta B \end{bmatrix}$$

which, in the rotation frame is:

$$\mathbf{B_{eff}}(t) = \begin{bmatrix} B_1(t) \\ 0 \\ \Delta B \end{bmatrix}$$

Here, the Bloch equation can be written as:

$$\frac{d\mathbf{M_{rot}}}{dt} = \mathbf{M_{rot}} \times g\mathbf{B_{eff}} = \begin{bmatrix} 0 & g\Delta B & 0 \\ -g\Delta B & 0 & gB_1(t) \\ 0 & -gB_1(t) & 0 \end{bmatrix} \mathbf{M_{rot}}$$

Later in the class we will work on solutions to this equation.

So why do we have RF pulses? We cannot detect **M** if it is aligned along B_0.

- It is not moving and thus does not induce voltage in a coil.
- It is small relative to B_0.
- Nuclear magnetization might be obscured by other magnetization (e.g. from electrons).

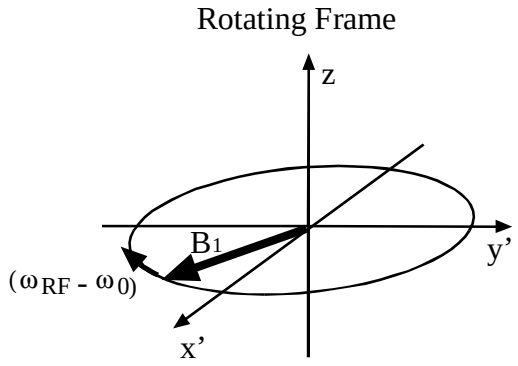
When **M** is in the transverse plane, it induces a voltage in a coil at ω_0 and the size of the magnetization is proportional to the size of the magnetization, m_0..

The process is goes by several names:

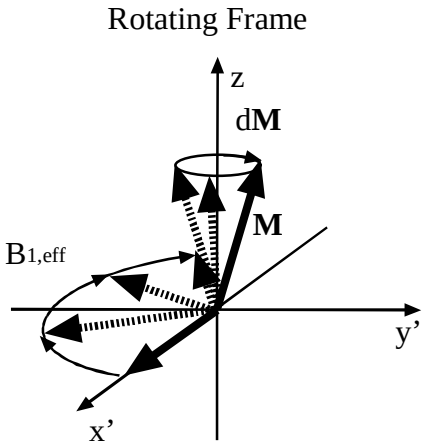
- RF pulses
- B1 fields
- Excitation
- Transmission (vs. detection)

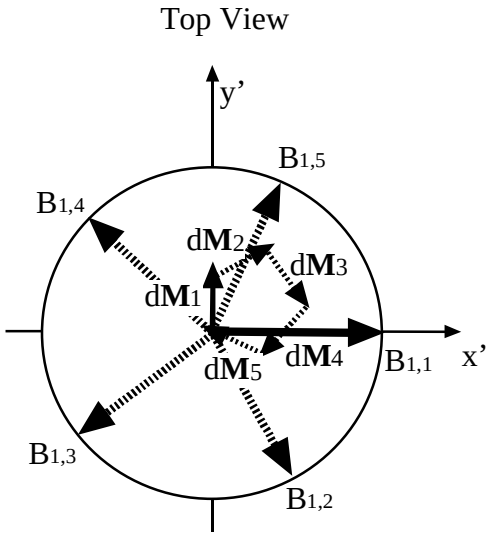
The resonance condition

What happens if $w_{RF} \neq w_0$? We now have the rotating frame version of $\mathbf{B_1}$ described as $B_{1xy,eff} = B_1 \exp(-i (w_{RF} - w_0) t)$, a more slowly rotating $\mathbf{B_1}$ vector.



In this case, as **M** gets tipped away from the z-axis $\mathbf{B_1}$ has moved relative the **M** and the axis of rotation has now changed.





Under this condition, the **M** vector never gets far from the z-axis because the $\mathbf{B_1}$ vector moves to a position that causes the change in **M** (e.g. d**M**) to move back towards the z-axis.

If excitation $\mathbf{B_1}$ occurs at a frequency that resonates with the magnetization **M**, then **M** is tipped from the z-axis into the transverse plane where it can be observed.

How close must ω_{RF} be to ω_0?

If $|\omega_{RF} - \omega_0| < \omega_1$, then excitation is effective.

If $|\omega_{RF} - \omega_0| >> \omega_1$, then no excitation occurs.

Comment. We've talked about tipping M into the transverse plane and making M precess faster or slower depending on $B_0 + \Delta B(t)$. All this was done using classical equations of motion. Please keep in mind that in the quantum mechanical description, all that is going on is flipping of the magnetization between energy states. This is done in a manner that preserves coherences in the magnetic dipoles to produce a net magnetization that behaves as described. Also bear in mind that if the applied RF is not at $\Delta E = \hbar g B_0$, then the RF will be very inefficient at flipping between energy states. This is another way to view the resonance condition requiring ω_{RF} to be close to ω_0.

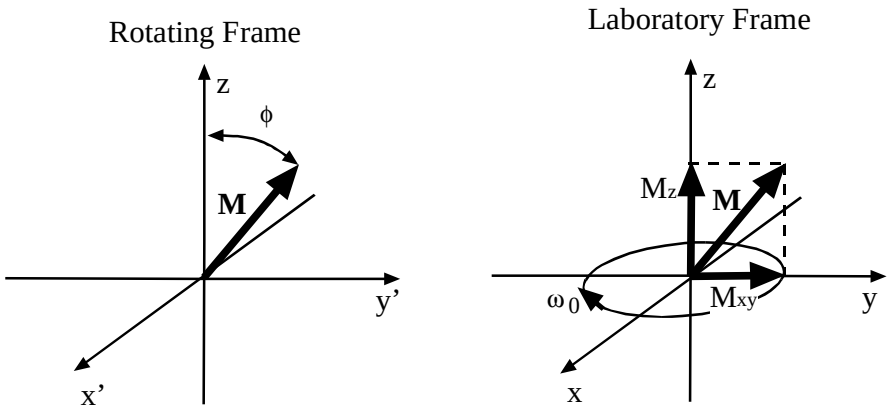
Other RF pulses.

1. Small flip angle pulses. We described a 90 degree or $\pi/2$ pulse above. If the flip angle is less than 90 degrees, is there still rotating magnetization that is detectable? Yes – the amount that is observable is the component in the transverse plane. Consider a flip angle of ϕ degrees. The magnetization can be describes as follows:

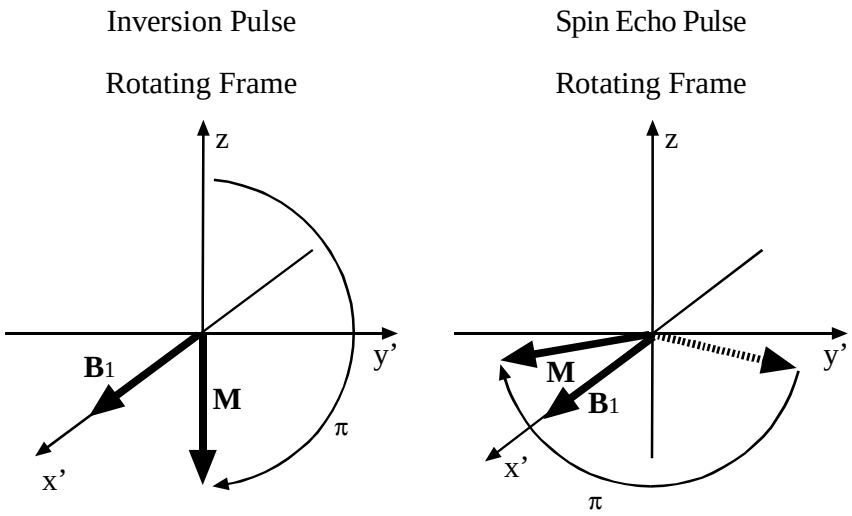
$$m_{xy} = \mathrm{i}\, m_0 \sin(f) \exp(-i\, w_0\, t)$$

$$m_z = m_0 \cos(f)$$

where m_{xy} is the detectable part.



2. 180 degree or π pulses. Here the RF pulses is applied for a duration and amplitude that leads to a precession angle, f, of 180 degrees. There are two variants of 180 degree pulses: inversion and spin-echo pulses. In an inversion pulse, **M** starts aligned to the z axis and is inverted to the -z axis. In an spin-echo pulse, **M** starts in the x'-y' plane and is flipped (around the axis of $\mathbf{B_1}$) to a new position in the x'-y' plane. We'll talk more about both of



Relaxation

So far, we've manipulated **M** as if it were a constant length vector at all times – in practice, it is not. There are thermal processes that will tend to bring **M** back to its equilibrium state (that is to the Boltzmann distribution in the spin-up/down energy states).

Consider the inversion pulse just described – the spin populations are all switched so that then higher energy state has a larger population than the lower energy state. By spins giving up energy (e.g. heat) into the surrounding molecular matrix, the spins will eventually return to the Boltzmann distribution.

In fact, there are two distinct processes going on:

1. Recovery of **M** back to m_0**k** (the thermal equilibrium state with the Boltzmann distribution).
2. Decay of m_{xy}.

"T1 relaxation" or "spin-lattice relaxation."

This is characterized by the growth of m_z towards m_0 with time constant T1. Examples:

- Polarization the tissue when place in B_0.
- Recovery from an inversion.
- Recovery from any reduction in m_z by RF excitation (including a 90 degree pulse which would make $m_z = 0$).

This is governed by the differential equation:

$$\frac{dm_z}{dt} = -\frac{(m_z - m_0)}{T1}$$

(This differential equation comes from relationship that *dN*, the number of state changes in interval *dt*, is proportional to the number of spins not in equilibrium, (*N* – *DN*), where *DN* corresponds to the equilibrium magnetization, m_0.)

The general solution to the differential equation is:

$$m_z(t) = m_0 + (m_z(0) - m_0)e^{-t/T1}$$

Specific cases:

1. After a 90 degree pulse:

$$m_z(0) = 0;\ m_z(t) = m_0\ (1 - e^{-t/T1})$$

2. After an inversion pulse:

$$m_z(0) = -m_0;\ m_z(t) = m_0\ (1 - 2e^{-t/T1})$$

3. After an α pulse:

$$m_z(0) = m_0 \cos\alpha;\ m_z(t) = m_0\,(1 - (1 - \cos\alpha)e^{-t/T1})$$

Recovery mechanism

- Spin gives up energy into the surrounding molecular matrix as heat
- Transitions from higher (spin-down) energy states to lower (spin-up) energy states (quantum mechanical view)

Spontaneous E state transitions are rare – usually these transitions need to be stimulated by something - in most cases, this is a fluctuating magnetic field. As nuclei tumble and move around, their local magnetic environment is always changing as electrons and other nuclei come in close proximity to the spin of interest.

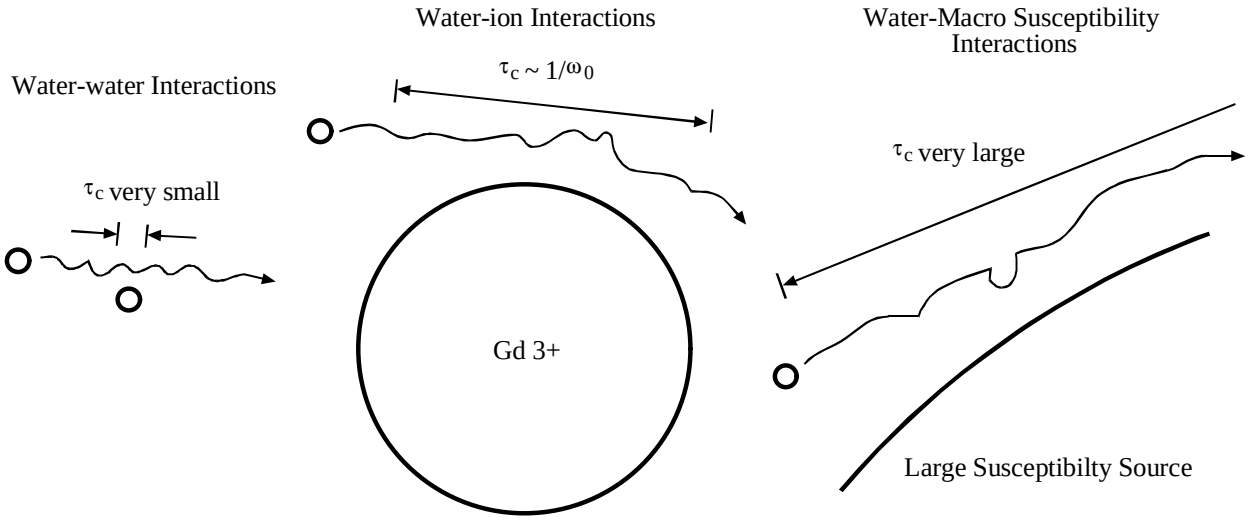
If the duration of these interactions has a frequency content near ω_0, then the probability of a transition is increased.

Correlation time. The correlation time, τ_c, describes the average length of time for an interaction between a nuclear spin and an external pertubation of the magnetic field. If $1/\tau_c$, the approximate frequency content of the interaction, is close to ω_0, then the probability of a transition is increased.

Examples:

a. Water-water interaction - $\tau_c \sim 10^{-12}$ s and thus $1/\tau_c >> \omega_0$. Poor efficiency at stimulating transitions resulting in long T1's.
b. We can help the process along by adding ions to the water (ions have unpaired electrons with large magnetic moments (an electron has a magnetic moment that is 700x larger than that of a nucleus). This skews the magnetic field over a much larger region increasing the efficiency of simulating an transitions. Thus, adding ions to water usually results in a shorter T1.
c. Extreme case – very large (macroscopic) pertubations of magnetic field. Suppose we have a large source of magnetic susceptibility that skews the field over a much larger region (e.g., like the iron in a large blood clot). Since the field pertubation is so large, the amount of field

fluctuation it can induce is at too low a frequency to stimulate E state transitions (T1 relaxation).



In general, T1 properties result from a complex interaction of different mechanisms with different kinds of spin motion. Here are some factors that influence T1:

1. Viscosity – affects τ_c
2. Temperature– affects τ_c
3. State (solid, liquid, gas) – affects τ_c
4. Ionic content – affects τ_c
5. B_0 – affects ω_0.

More examples:

d. Tissues with restricted diffusion of ^1H have longer affects τ_c's, which makes $1/\tau_c$' closer to ω_0, which results in a faster (shorter) T1's (e.g. white matter, fat)
e. Solids – very long T1's – no motion of nuclei

"T2 relaxation" or "spin-spin relaxation."

This is characterized by the decay of m_{xy} towards 0 with time constant T2.

This is governed by the differential equation:

$$\frac{dm_{xy,rot}}{dt} = -\frac{m_{xy,rot}}{T2}$$

(This differential equation comes from relationship that *dN*, the change in the number of exicited in interval *dt*, is proportional to the number of spins in the excited state, *N*.)

The general solution to the differential equation is:

$$m_{xy,rot}(t) = m_{xy,rot}(0)e^{-t/T2}$$

Specific cases:

1. After a 90 degree pulse:

$$m_{xy,rot}(0) = m_0;\ m_{xy,rot}(t) = m_0\ e^{-t/T2}$$

2. After an inversion pulse:

$$m_{xy,rot}(0) = 0;\ m_{xy,rot}(t) = 0$$

3. After an α pulse:

$$m_{xy,rot}(0) = m_0 \sin\alpha;\ m_{xy,rot}(t) = m_0 \sin\alpha\ e^{-t/T2}$$

Decay Mechanisms

1. The T1 component – the approach to thermal equilibrium reduces m_{xy}.
2. Phase incoherence – remember that the observable magnetization, **M**, is the ensemble average of all nuclei – if the little μ's get out of phase with respect to each other we get reduced signal.

The phase for a spin is:

$$f_{rot}(t) = g\int_0^t \Delta B(t)dt$$

where $\Delta B(t)$ represents the time varying, random field fluctuations generated by other nuclei, electrons, ions, and larger sources of magnetic field susceptibility. The signal is then the average across all spins.

$$s(t) = \int_V m_0 e^{if_{rot}(t)}d\mathbf{r}$$

Examples:

a. Spins are tumbling rapidly in a homogeneous media. Then $f_{rot}(t) \cong g\Delta\overline{B}t \cong 0$ for all spins. That is, the integral over time gives the time average of ΔB(t) which is nearly 0. In this case, there is very little field induced dephasing and thus, T2 ~ T1. (e.g. distilled water)
b. If large paramagnetic ions are present, then f_{rot} varies much more from spin to spin and the signal decays much more rapidly. Here T2 << T1. (e.g. water doped with ions)
c. Solids – there is virtually no tumbling which leads to a fixed relationship with the ΔB's. Here there are other mechanisms that can lead to T2 relaxation in addition to accumulation of phase from ΔB. In general, solids have T2's that are very small. Most solids c imaged with normal MRI techniques because the T2's are so small (e.g. μs regime).

In most biological tissues, T2 << T1, usually by an order of magnitude.

Full Bloch Equation with T1 and T2

The full Bloch equation with T1 and T2 is:

$$\frac{d}{dt}\begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} \times g \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} - \begin{bmatrix} m_x \\ m_y \\ 0 \end{bmatrix} \frac{1}{T_2} - \begin{bmatrix} 0 \\ 0 \\ m_z - m_0 \end{bmatrix} \frac{1}{T_1}$$

Pulsed NMR Experiments

The vast majority of MRI experiments use repeated pulsing of the spin system. Following each RF pulse the transverse signal behaves according to:

$$\frac{dm_{xy}}{dt} = -(iw_0 + \frac{1}{T_2})m_{xy} \quad \text{and} \quad m_{xy}(0) = m_0$$

and thus:

$$m_{xy}(t) = m_0 e^{-iw_0 t} e^{-t/T2}$$

This decaying oscillating signal is often known as the free induction decay ("free" no interference from other RF pulses, "induction" Bloch's original term for precession around B_0, and "decay" for, well, T2 decay). Since different biological tissues may have differe often useful to select an observation time following the RF pulse. This observation time is known as the "echo time" or TE. Looking in the rotating frame at this observation time we get:

$$m_{xy,rot}(TE) = m_0 e^{-TE/T2}$$

A long TE results in T2-weighted images. In T2-weighted images, tissues with long T2's appear bright while tissues with short T2's are dark (their signal has completely decayed away).

As described previously, the *z* component of the magnetization recovers after a 90 degree excitation pulse according to:

$$m_z(t) = m_0 (1 - e^{-t/T1})$$

The time between excitation pulses is referred to as the "repetition time" or TR. If TR is not long compared to T1, then all of the magnetization will not have recovered an the initial magnetization available to rotate into the transverse plane will not be m_0, but will be $m_0 (1 - e^{-TR/T1})$.

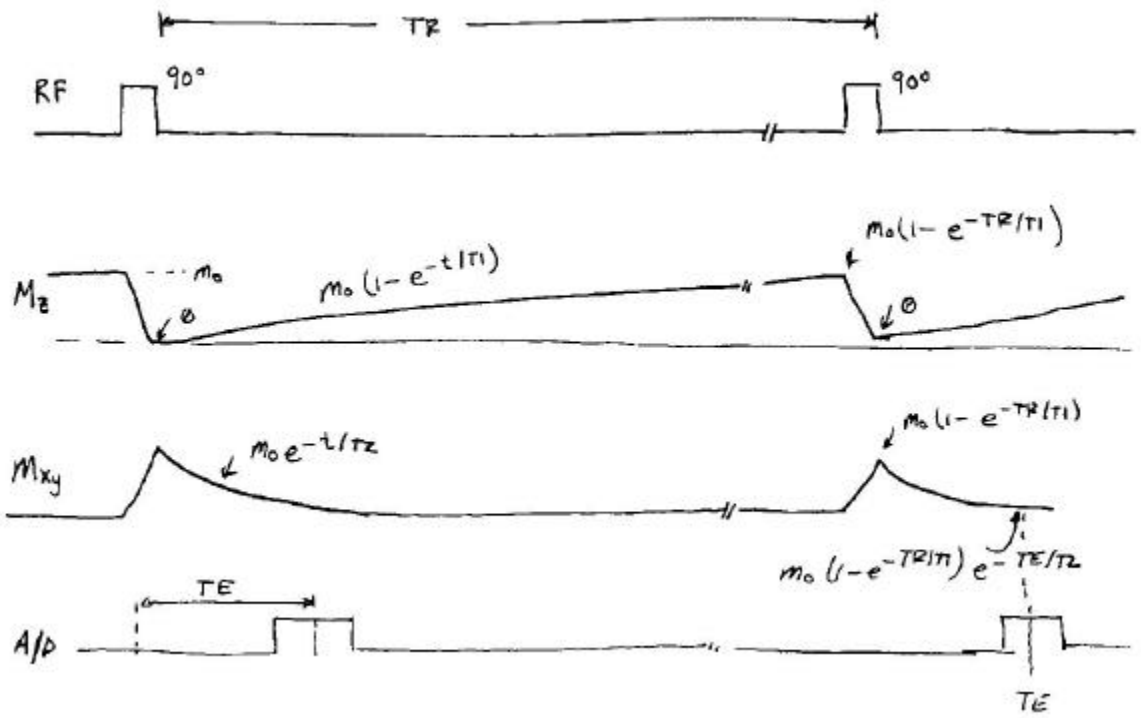
A short TR results in T1-weighted images. In T1-weighted images, tissues with short T1's appear bright while tissues with long T1's are dark (very little magnetization has recovered for the next excitation pulse).

Finally, we note that different tissues have differing concentrations of hydrogen, the m_0 is proportional to the hydrogen density, ρ. The signal intensity, for a particular tissue this thus a function of tissue parameters ρ, T1, and T2, and imaging parameters TE and TR:

$$\text{signal intensity} \propto r(1 - e^{-TR/T1})e^{-TE/T2}$$

Typical T1's, T2's and ρ's for Brain Tissues

	T1	T2	Rel. density
Distilled water	3 s	3 s	1.0
Cerebro Spinal Fluid	3 s	300 ms	1.0
Gray matter	1.2 s	60-80 ms	.98
White matter	800 ms	45 ms	.80
Fat	150 ms	35 ms	1.0



Steady State Magnetization for α pulses

The above description of signal intensity holds for 90 degree RF pulses. Occasionally, it is desirable to use a short TR (10 to 100 ms). This means that signal intensity would be very small for all tissues. In these cases, it is useful to use an RF pulse with a "tip angle" or "flip angle" less than 90 degrees. Here, we can examine what happens to the *z* magnetization before and after an α degree pulse:

$$m_z^+ = m_z^- \cos a$$

The z magnetization recovers according to T1 for a period of time TR:

$$m_z(TR) = m_0 + (m_z^+ - m_0)e^{-TR/T1}$$

Under steady state conditions, $m_z(TR) = m_z^-$, and thus:

$$m_z^+ = \left[m_z^+ e^{-TR/T1} + m_0(1 - e^{-TR/T1})\right]\cos a$$

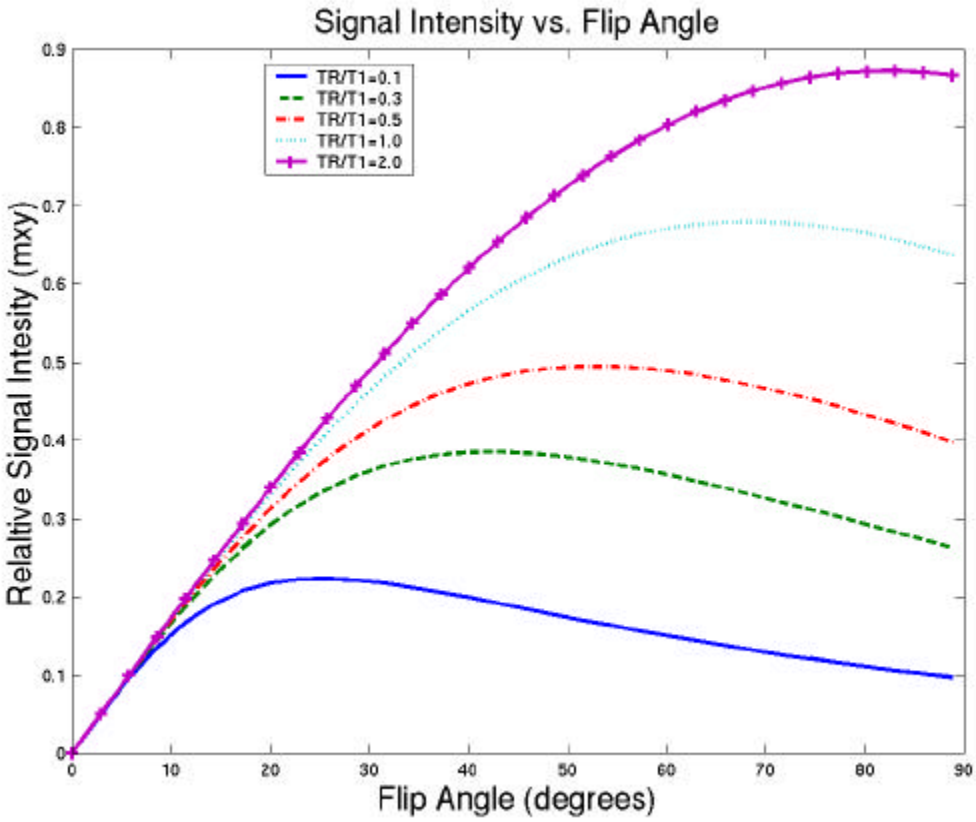
which can be solved to yield:

$$m_z^+ = m_0 \frac{1 - e^{-TR/T1}}{1 - \cos a \cdot e^{-TR/T1}} \cos a$$

$$m_z^- = m_0 \frac{1 - e^{-TR/T1}}{1 - \cos a \cdot e^{-TR/T1}}$$

The transverse component following an α degree pulse is:

$$m_{xy} = m_z^- \sin a = m_0 \frac{1 - e^{-TR/T1}}{1 - \cos a \cdot e^{-TR/T1}} \sin a$$



The above relationship can be differentiated to yield the optimal α (in terms of maximal signal):

$$a_{opt} = \arccos\left(e^{-TR/T1}\right)$$

This is known as the "Ernst Angle" (in recognition of Nobel Laureate, Richard Ernst).