Take-Home Final Exam

Last Possible Due Date: Dec. 21, 2004, 5 p.m.

Your solutions to the exam should be handed in to the instructor (BIRB 1088) or to Eve Gochis, the MRI lab administrator (BIRB 1072) no later than 5 pm on the due date.

Rules governing this exam:

One the first page of your solutions, write "I have neither given nor received aid on this exam," and sign your name below it. Any exam that does not have this honor code declaration will not be graded.

You may have this exam for no more than 72 hours. When you take and return the exam, you should mark the times on the sign-out sheet. Keeping the exam more than 72 hours will result in a penalty of 10 percentage points for every hour the exam is late.

You may use your books, your own notes, web resources, computers and calculators.

You may not seek assistance, share or borrow notes from any current or former students or from any other individual (instructors' notes available on the web are OK). While all problems are different from any previous years' exams, you may not look at solutions from those exams while taking this exam.

This exam is governed by the Engineering Honor Code that requires that you do not seek assistance on the exam and that you report any violations of the Honor Code. (For more info see http://honor.personal.engin.umich.edu/).

Other tips:

Please remember to put your name on your exam solutions.

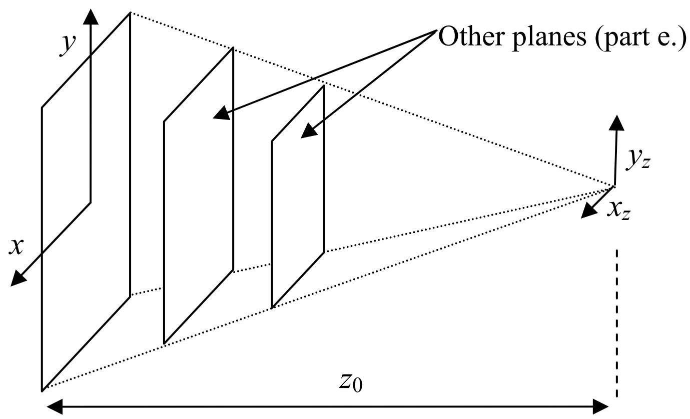
Read the questions carefully.

Show your work (it's hard to give partial credit without it). Define all constants, magnification factors, etc. that you use in your solutions.

If parts of the exam are not clear and you cannot find the instructor to ask for clarification, please write your assumptions and proceed with the question. If you cannot solve a part that is needed later in the solution, define a parameter to represent the answer of that part and continue.

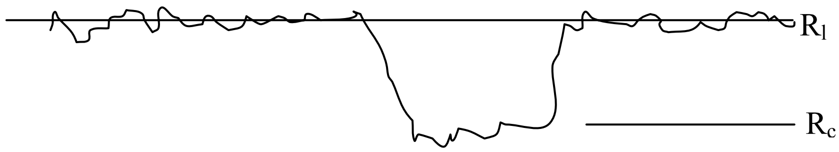
Write legibly please.

1. (20 pts.) Ultrasonics. You've recently been hired by a small ultrasonics company called Cookers-R-US, Inc. As your first assignment, you will evaluate the operational parameters for a therapeutic ultrasound device used to heat and destroy cancerous tissue. Consider the following system: a square transducer having edges length L is set to focus on an on-axis point at z_0. The pressure function amplitude at the transducer face is A, where A^2 is a power density (units: W/cm^2). The energy density (power density times the pulse duration) is the critical parameter in determining if tissue is destroyed.

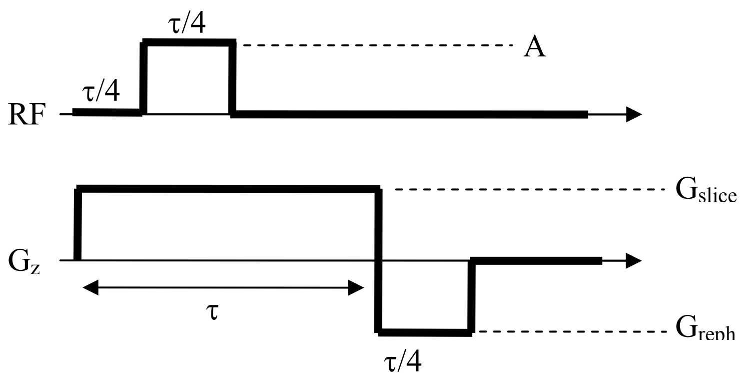


   a. Give an expression for the total power transmitted.
   b. We set the source function to $s(x,y) = A/\lambda\ \mathrm{rect}(x/L)\mathrm{rect}(y/L)$. Derive an expression for the steady state pressure pattern in the focal plane, $p(z,x_z,y_z)$. [The $1/\lambda$ factor is necessary to convert the source pressure densities into the point transmitter used for the in-class derivations.]
   c. Add attenuation to the expression derived in b. Assume the linear attenuation coefficient is α and let $r_{0z} \approx z_0$ for purposes of estimating the effects of attenuation.
   d. Let the power density be $|p(z,x_z,y_z)|^2$. Determine the peak power density in the focal plane (including attenuation).
   e. The operational plan calls for tissue destruction at the focal plane, but not in planes close to the transducer. We will use a rather crude approximation to estimate the power density in planes for $z < z_0$. We will assume that as the focused wave approaches the focal point, the total power at the transducer face will be spread across a smaller square with edge lengths proportional to distance from the focal point. Give an expression for power density as a function of z.
   f. Add attenuation to the expression derived in e. Keep in mind the attenuation affects the pressure amplitude and the power is related to pressure squared.
   g. Give an expression for the ratio of power density at plane z to power density at the focal point.
   h. If the power for other planes is too high relative to the focal plane, what can be done to the system to improved performance of the system (dimensions, transmit frequency, etc.)?
   i. Let $c = 1.5$ mm/μs, $f_0 = 1.5$ MHz, $\alpha = 0.15$ cm^{-1}, $z_0 = 5$ cm, $L = 2$ cm. Approximately how large is the focal point? How much higher is the power density at the focal point relative to a plane at z = 2.5 cm?

2. (5 pts.) Noise in Ultrasound. Assume we are imaging the liver with a mean volume reflectivity R_l and we detect a cyst with volume reflectivity R_c. Determine the contrast to noise ratio (CNR) of the cyst.



3. (20 pts.) Slice selection in MRI. Consider the following selective excitation sequence



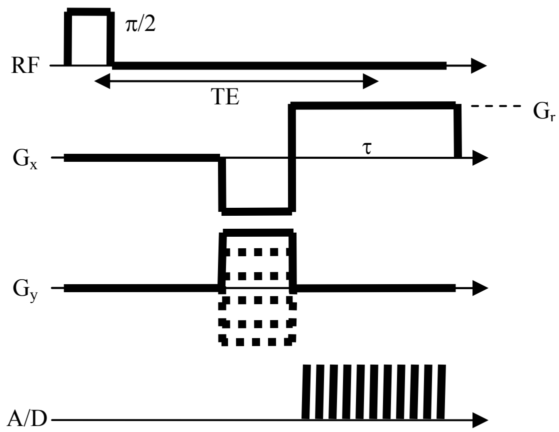
in which the RF pulse has the form $B_1(t) = A\,\mathrm{rect}(4(t - \frac{3}{8}\tau)/\tau)$.

a. Find an expression for A to produce a tip angle of 30 degrees at the center of the slice.
b. Find an expression for G_{reph} so that there is no residual phase across the slice at the end of this RF pulse.
c. Find an expression for the slice profile $p(z)$ at time $t = 5\tau/4$. You may use the small tip angle approximation.
d. Find an expression for G_{slice} to produce a slice thickness of Δz (please describe how you defined slice thickness for this part).
e. Write a few sentences describing the potential advantages and disadvantages of this RF pulse (e.g. vs. the sinc RF pulse described previously).

For the following parts, use the 30 degree RF pulse described in parts a-d and use an object with initial magnetization described by $m_0(x,y,z) = m_0\,\mathrm{rect}(x/X)\,\mathrm{rect}(y/Y)\,\mathrm{rect}(z/Z)$.

f. Find an expression for the received signal $s(t)$ at time $t = 5\tau/4$ for Z infinitely large. Assume the G_x and G_y are both zero. Do not leave in integral form.
g. Consider the received signal $s(t)$ at time $t = 5\tau/4$ for two different cases of Z: (1) $Z = Z_1 = 16\pi/(\gamma G_{slice}\tau)$ and (2) $Z = 2Z_1 = 32\pi/(\gamma G_{slice}\tau)$. Determine the relationship (<,> or =) for $s(t)$ between these two cases (you do not need to explicitly calculate $s(t)$ for this problem).

4. (25 pts.) MRI. In nearly all of our analysis regarding MRI, we've assumed that the underlying magnetic field is perfectly uniform, $B_z(x,y,z) = B_0$. In practice, the magnetic field is not uniform and this can create distortions in the resultant images. Here, we will examine some of these distortions. We will consider use the usual spin warp pulse sequence:



The object to be imaged is $m_0(x,y) = \mathrm{rect}(x/L)\,\mathrm{rect}(y/L)$. Neglect relaxation and ignore the z-dimension in all of these analyses. The standard signal equation is:

$$s_0(t) = \iint m_0(x,y)\exp\left(-i2\pi\left(k_x(t)x + k_y(t)y\right)\right)dxdy.$$

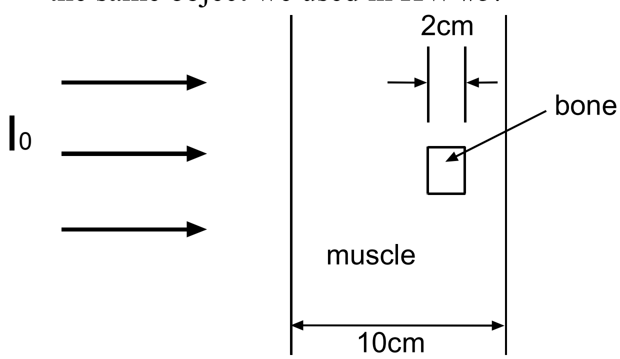
We will now assume the underlying magnetic field is not uniform: $B_z(x,y) = B_0 + \Delta B(x,y)$. In this case, we will use a modified received signal equation:

$$s_2(t) = \iint m_0(x,y)\exp\left(-i2\pi\left(k_x(t)x + k_y(t)y + \Delta f(x,y)t\right)\right)dxdy.$$

a. In the above equation, define $\Delta f(x,y)$ in terms of $\Delta B(x,y)$.
b. Give an expression for $k_x(t)$ during the period that the A/D's are turned on: [TE-τ/2, TE+τ/2]. Create an expression for t in terms of k_x.
c. Let $\Delta B(x,y) = \Delta B$ an undesired spatially uniform offset from B_0. Describe the $s_2(t)$ in terms of $s_0(t)$.
d. Determine the newly acquired k-space data, $M_2(k_x,k_y)$. The result of b. may be helpful here. You may assume an infinite support region in k-space, if you like.
e. Determine the reconstructed image, $m_2(x,y)$ (again, infinite support is ok).
f. Describe in general terms the effects of a field shift of ΔB on the reconstructed image.
g. Now, let $\Delta B(x,y) = G_b x$ an undesired linearly varying magnetic field. By combining the effects of G_x and G_b, we can consider that we have a new k-space trajectory $k_{x,net}(t)$. Show that we can write $k_{x,net}(t) = ak_x(t) + k_0$ for $t \in$ [TE-τ/2, TE+τ/2] and give values for a and k_0.
h. Determine the newly acquired k-space data, $M_2(k_x,k_y)$ (infinite support is ok).
i. Determine the reconstructed image, $m_2(x,y)$ (infinite support is ok).
j. Describe, in general terms, the effects of a background gradient, G_b, on the reconstructed image. Under what relationship between G_x and G_b will the image acquisition fail?
k. For these two cases (c. and g.), what should be done to G_x to minimize image distortions?

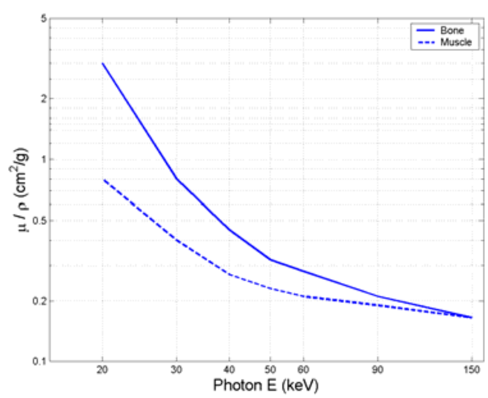
5. (20 pts.) Central section theorem. Use the central section theorem from CT to answer the following questions. In each case, please show how the theorem was used.
   a. Determine the Radon space, $g_\theta(R)$, of $f(x,y) = \exp\left(-\pi\left(x^2+y^2\right)\right)\cos(2\pi a y)$.
   b. Determine the inverse 1D FT of $G(\rho) = \mathrm{jinc}(\beta\rho)$.
   c. Determine the forward 2DFT of $f(x,y) = f(r) = \dfrac{\mathrm{circ}(r/a)}{\sqrt{a^2 - r^2}}$. Hint: $\int_{-b}^{b} \dfrac{1}{\sqrt{b^2 - x^2}}\,dx = \pi$.
   d. Determine the original object, $f(x,y)$, given $g_\theta(R) = \exp(-\pi(R - a\cos\theta - b\sin\theta)^2 / c^2)$.

6. (10 pts.) Dual energy x-ray. One challenge in x-ray imaging is the complication of overlying tissue and bone. This can be partially addressed by creating images at two different energies and taking a weighted combination of these to accentuate one component or another. In this problem, we will treat a simplified case where were we will consider simple subtraction of two images: $D(x,y) = |I_A(x,y) - I_B(x,y)|$ where I_A and I_B are two monoenergetic images of the same object we used in HW #5.



Mass Attenuation Coefficients (μ/ρ)

Photon E (keV)	Bone (cm^2/g)	Muscle (cm^2/g)
30	.80	.40
40	.45	.27
50	.32	.23
60	.28	.21
90	.21	.19
150	.17	.17



The densities (ρ) of muscle and bone are 1.0 and 1.75 g/cm^3, respectively.

   a. Letting I_0 be the same for both energies, determine which two energies (A, B) will maximize the contrast for bone in $D(x,y)$.
   b. If you could set I_0 separately for the two energies, describe how would you go about determining the "best" energies? Consider two cases: (1) Suppose the muscle layer is of variable and unknown thickness, and (2) Suppose the images are noisy and the quantity $(I_{0,A} + I_{0,B})$ must be held constant at I_{total}. Write a few sentences describing the issues and how you would approach the optimization.