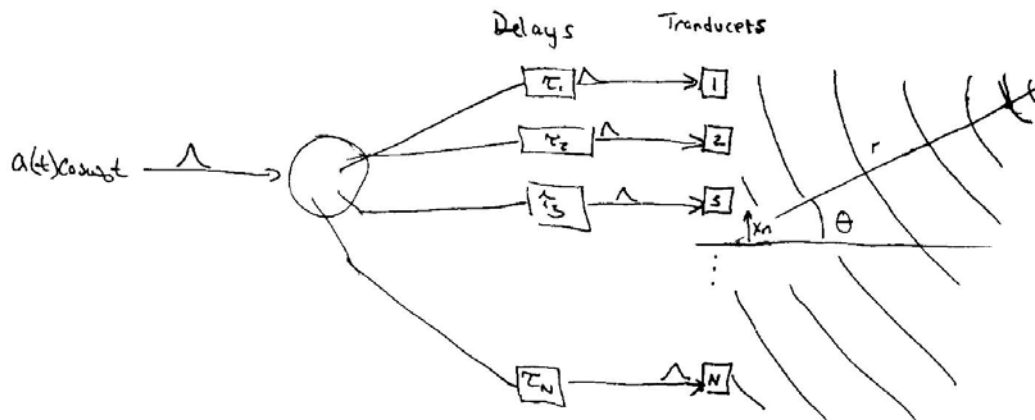


Ultrasound Notes 3: Array Systems

While early US systems used a single focus transducer and mechanical sweeping of the transducer to different angles, nearly all modern US systems are array systems where focussing is not preset and beam steering is done through time delays associated with each element of the array.

Transmit mode: Focussing and Beam Steering

Typical array system shown here in transmit mode:



where x_n is the x coordinate of the n^{th} element (0 is at the center of the array).

For focussing at $(z, x_z=0)$ set the delay to:

$$\tau'_n = \tau'(x_n, z, x_z = 0) = \frac{x_n^2}{2cz}$$

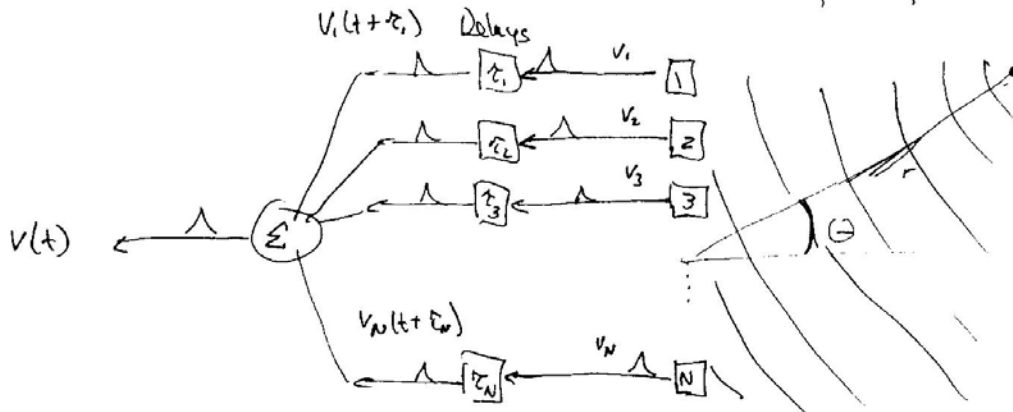
(Actually, this is a negative delay – to focus on-axis in the Fresnel zone, we require that the edge elements fire before the center element because they have farther to propagate.)

In polar coordinates, for focussing at (r_0, θ_0) :

$$\tau'_n = \tau'(x_n, r_0, \theta_0) = -\frac{x_n \sin \theta_0}{c} + \frac{x_n^2 \cos^2 \theta_0}{2cr_0}$$

A focal depth and direction (r_0, θ_0) is selected once when the pulse is transmitted. Once it leaves the transducer, it can no longer be changed.

Receive mode: Delay-Sum Beamforming



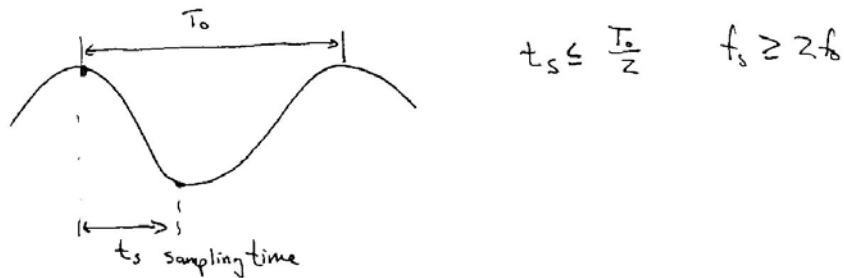
- Signals from each transducer are delayed and summed to produce a signal localized to the focal point, e.g. (r_0, θ_0) :

$$v(t) = \sum_{n=1}^N v_n(t + \tau'_n)$$

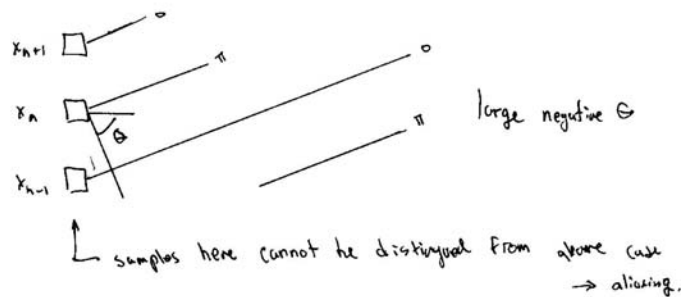
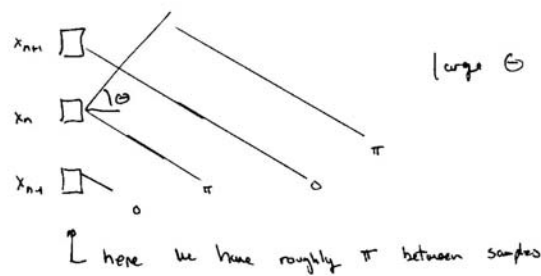
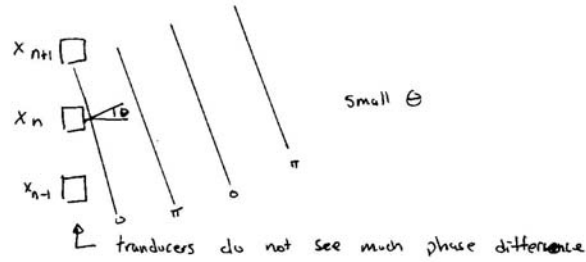
- Signal from other angles are suppressed by destructive interference of the wavefronts when summed together (recall the each pulse is made up of modulated waves at a carrier frequency, f_0).
- Unlike the transmit case, the received data can be combined in many different ways. In other words, we can separately focus for any point in time (depth plane, r). Also we can focus to any angle at by reprocessing the data. Typically, “dynamic” focussing is used to focus for each depth plane – here τ'_n changes for every depth $r = ct/2$.

Sampling in Space

At each point in time, the array is actually sampling the instantaneous pressure wave as a function of space. Aliasing is potentially a problem. A quick review of sampling:



Aliasing can be prevented if less than π has accrued between neighboring samples. For our case of waves impinging upon the detector array, sampling becomes a bigger issues if the source of the waves is coming from a large angle, θ :



The sampling requirement for this case can be written as the maximum difference in propagation delay between a point (r, θ) and two neighboring transducer elements cannot lead to phase accrual of greater than π :

$$|(\tau(x_n, r, \theta) - \tau(x_{n+1}, r, \theta)) \cdot \omega_0| \leq \pi \quad \forall r, \theta, n$$

assuming the element spacing is d (e.g. $x_{n+1} = x_n + d$), then:

$$\begin{aligned}
& \left| (\tau(x_n, r, \theta) - \tau(x_{n+1}, r, \theta)) \cdot \omega_0 \right| \\
&= \left| \left(r - x_n \sin \theta + \frac{x_n^2 \cos^2 \theta}{2r} - r + (x_n + d) \sin \theta - \frac{(x_n + d)^2 \cos^2 \theta}{2r} \right) \cdot \frac{\omega_0}{c} \right| \\
&= \left| \left(d \sin \theta - \frac{2x_n d \cos^2 \theta}{2r} - \frac{d^2 \cos^2 \theta}{2r} \right) \cdot \frac{\omega_0}{c} \right|
\end{aligned}$$

The worst case is for plane waves, that is $r \rightarrow \infty$:

$$\begin{aligned}
\left| (\tau(x_n, r, \theta) - \tau(x_{n+1}, r, \theta)) \cdot \omega_0 \right| &= \left| d \sin \theta \cdot \frac{\omega_0}{c} \right| \\
&= \left| d \sin \theta \cdot \frac{2\pi}{\lambda} \right| \leq \pi
\end{aligned}$$

and thus:

$$|d \sin \theta| \leq \frac{\lambda}{2}$$

and if we want to unambiguously distinguish between arrival directions (points of reflectivity) over a full π of angle, then:

$$d \leq \frac{\lambda}{2}$$

In this case, the number of transducer elements, N , should be:

$$N = \frac{2a}{d} \geq \frac{4a}{\lambda} = 2 \cdot (\text{Numerical Aperture})$$

The “Numerical Aperture” for an array is the size of the transducer in terms of the number of wavelengths (e.g. N.A. = $2a/\lambda$).

Comments:

- We’ve just talked about the array sampling a pressure wave propagating towards the transducer. It is important to realize that sampling also occurs during transmit as well
 - in order to unambiguously transmit a wave in a particular direction (over π angles), we also need to satisfy the above transducer spacing.
- Below under discussion of the point spread function, we will see the effects of insufficient sampling by the array.

Angular Sampling of the Object

For this analysis, we will consider the far-field (or focal plane) solution:

$$p(r, \theta) = \frac{\cos \theta \cdot e^{ikr}}{r} \mathfrak{F}\{s(x)\}\Big|_{u=\frac{\sin \theta}{\lambda}}$$

$$= K'' S\left(\frac{\sin \theta}{\lambda}\right)$$

As described above, we must transmit our beam at a particular angle (θ) and thus, this axis is discretized as well:

$$S(\cdot) \text{ is sampled at locations } \frac{\sin \theta_i}{\lambda}$$

This is a reverse sampling argument – we are sampling the FT of $s(x)$ and thus the samples of $S(u)$ must be positioned at least as close as half the maximum extent of $s(x)$.

Thus, if $s(x)$ goes from $[-a, a]$, then:

$$\Delta \left[\frac{\sin \theta}{\lambda} \right] \leq \frac{1}{2a}$$

$$\Delta \sin \theta \leq \frac{\lambda}{2a} = \frac{1}{\text{Numerical Aperture}}$$

This is the solution for the receive only or a transmit only case.

Recall that the actual beam function is:

$$B(r, \theta) = h_T(\omega, r, \theta) h_R(\omega, r, \theta)$$

$$= [p(r, \theta)]^2$$

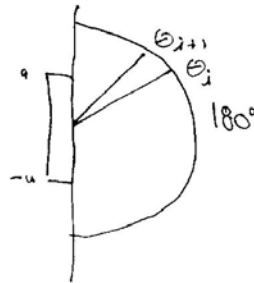
$$= [K'']^2 \left[S\left(\frac{\sin \theta}{\lambda}\right) \right]^2$$

and this is the function that is used to sample the object in the angular direction. Since we must now sample the square of the FT of the space limited aperture function, we must know the maximum extent of the its IFT, which is $s(x)^* s(x)$. If $s(x)$ goes from $[-a, a]$,

then $s(x) * s(x)$ is bounded by a function that extends of $[-2a, 2a]$. For example, if $s(x) = \text{rect}(x/2a)$, then $s(x) * s(x) = \text{triangle}(x/2a)$, which goes from $-2a$ to $2a$. Thus:

$$\Delta \sin \theta \leq \frac{\lambda}{4a} = \frac{1}{2 \cdot \text{N.A.}}$$

- How many beams to sample π ?



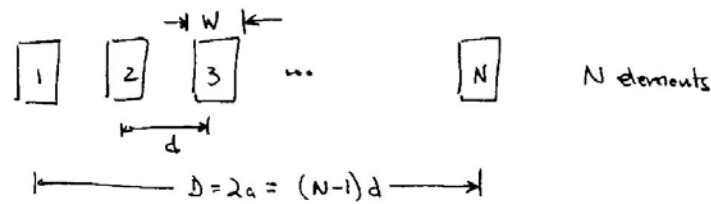
$$\frac{\max(\sin \theta) - \min(\sin \theta)}{\Delta(\sin \theta)} = \frac{1 - (-1)}{\lambda / 4a} = \frac{8a}{\lambda} = 4 \cdot \text{N.A.}$$

- Is sampling uniform in θ ? No, sampling is uniform in $\sin \theta$.



- Shouldn't we just sample more finely than we need to? No.
 - It takes extra time (you can't transmit the next pulse at a new angle until the last one is received).
 - It adds no new information.
- What happens if we don't sample finely enough? We might miss an object feature.

Point Spread Function for Discrete Transducers



For odd N :

$$s(x) = \sum_{-(N-1)/2}^{(N-1)/2} \text{rect}\left(\frac{x - nd}{w}\right)$$

$$= \left[\frac{1}{d} \text{comb}\left(\frac{x}{d}\right) \text{rect}\left(\frac{x}{D}\right) \right] * \text{rect}\left(\frac{x}{w}\right)$$

and the FT is:

$$S(u) = w \text{sinc}(wu) \cdot [\text{comb}(du) * D \text{sinc}(Du)]$$

$$= \frac{Dw}{d} \text{sinc}(wu) \cdot \sum_{n=-\infty}^{\infty} \text{sinc}\left(D\left(u - \frac{n}{d}\right)\right)$$

Substituting $u = \sin \theta / \lambda$, in the far field (or focal plane) for an on-axis beam, we get:

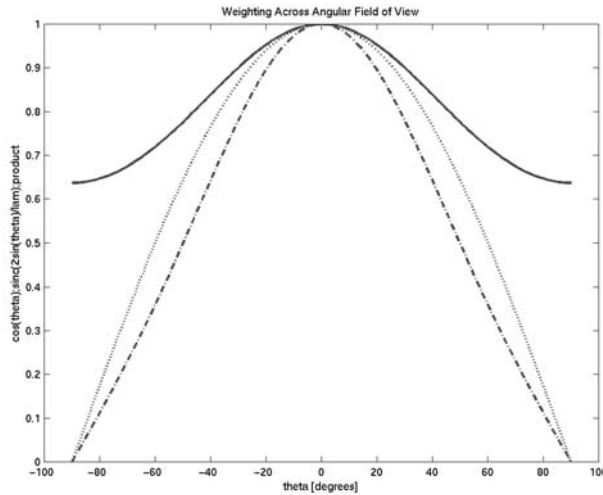
$$p(r, \theta) = K \cos \theta \text{sinc}\left(w \frac{\sin \theta}{\lambda}\right) \cdot \sum_{n=-\infty}^{\infty} \text{sinc}\left(2a \left(\frac{\sin \theta}{\lambda} - \frac{n}{d}\right)\right)$$

- the first sinc function provides additional weighting as a function of θ . This could reduce the angular “field of view” by restricting over what range of angles we can effectively transmit to/receive from.

Let’s look at these effects for a Nyquist sampled array ($d = \lambda/2$). First we look at the effect of the terms weighting the angular field of view:

$$\cos \theta \quad \text{and} \quad \text{sinc}\left(w \frac{\sin \theta}{\lambda}\right)$$

We know that $w < d$, so $w < \lambda/2$. The worst case (narrowest) is for $w = \lambda/2$. Here are plots of these functions (and the product of these functions – the lowest line on the plots):



Thus the weighting, while present, is not significantly different than the original $\cos \theta$ weighting. These weighting functions are independent of the numerical aperture.

Simulations

For the Nyquist sampled array ($d=\lambda/2$), we get:

$$p(r, \theta) = K \cos \theta \operatorname{sinc}\left(w \frac{\sin \theta}{\lambda}\right) \cdot \sum_{n=-\infty}^{\infty} \operatorname{sinc}((\text{N.A.}) \cdot (\sin \theta - 2n))$$

The argument of the sinc is zero only for $n=0$ – the main lobe (desired beam) and this occurs at $\theta=0$. For $d=\lambda$, we get:

$$p(r, \theta) = K \cos \theta \operatorname{sinc}\left(w \frac{\sin \theta}{\lambda}\right) \cdot \sum_{n=-\infty}^{\infty} \operatorname{sinc}((\text{N.A.}) \cdot (\sin \theta - n))$$

The argument of the sinc is zero for.

n	θ	Type
0	0	Main Lobe
1	$\pi/2$	Grating
-1	$-\pi/2$	Grating

For $d=2\lambda$, we get:

$$p(r, \theta) = K \cos \theta \operatorname{sinc}\left(w \frac{\sin \theta}{\lambda}\right) \cdot \sum_{n=-\infty}^{\infty} \operatorname{sinc}\left((\text{N.A.}) \cdot \left(\sin \theta - \frac{n}{2}\right)\right)$$

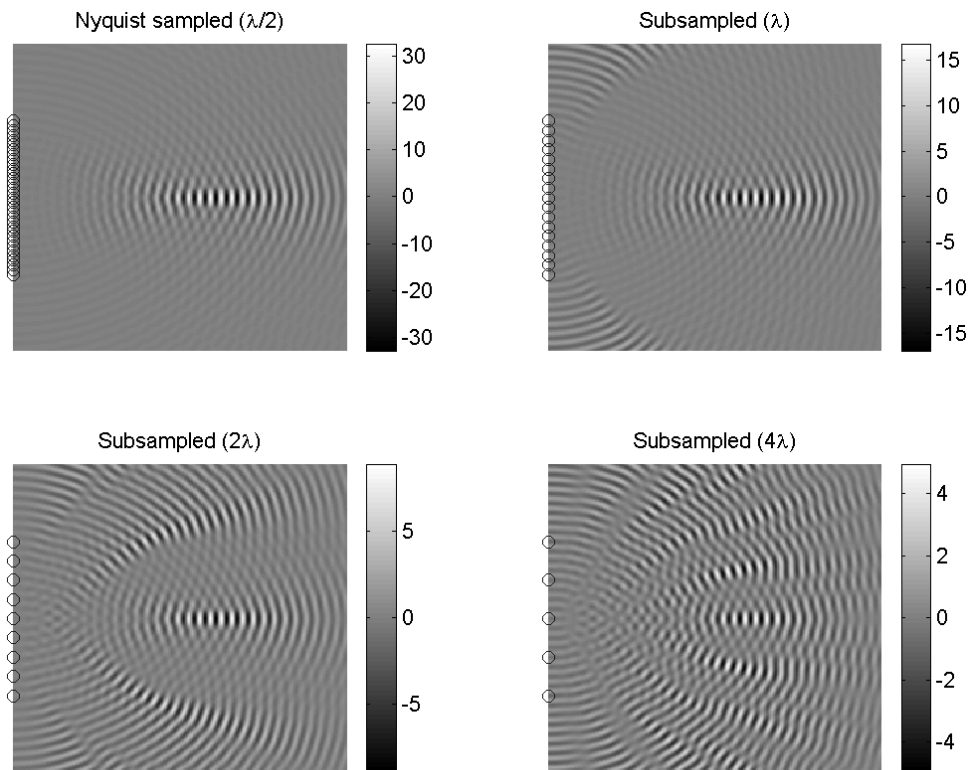
The argument of the sinc is zero for.

n	θ	Type
0	0	Main Lobe
1	$\pi/6$	Grating
-1	$-\pi/6$	Grating
2	$\pi/2$	Grating
-2	$-\pi/2$	Grating

Comments:

- Observe that the main lobe doesn't change in each case.
- The grating lobes are aliases resulting from subsampling of the spatial waves by the discrete transducer.
- In the sum of sinc functions, the $n=0$ term is the main beam
- The $n \neq 0$ terms are “grating” lobes – additional responses at angular locations other than the desired angle (in this case $\theta_0 = 0$).

Main and grating lobes in the following simulation of discrete transmitters:



Grating Lobes with Beam Steering

We can derive the above expressions for the case where the beam is steered to another location, θ_0 :

$$s(x) = \left[\frac{1}{d} \text{comb}\left(\frac{x}{d}\right) \text{rect}\left(\frac{x}{D}\right) e^{-ikx \sin \theta_0} \right] * \text{rect}\left(\frac{x}{w}\right)$$

which has as its transform:

$$S(u) = \frac{Dw}{d} \text{sinc}(wu) \cdot \sum_{n=-\infty}^{\infty} \text{sinc}\left(D\left(u - \frac{\sin \theta_0}{\lambda} - \frac{n}{d}\right)\right)$$

and thus:

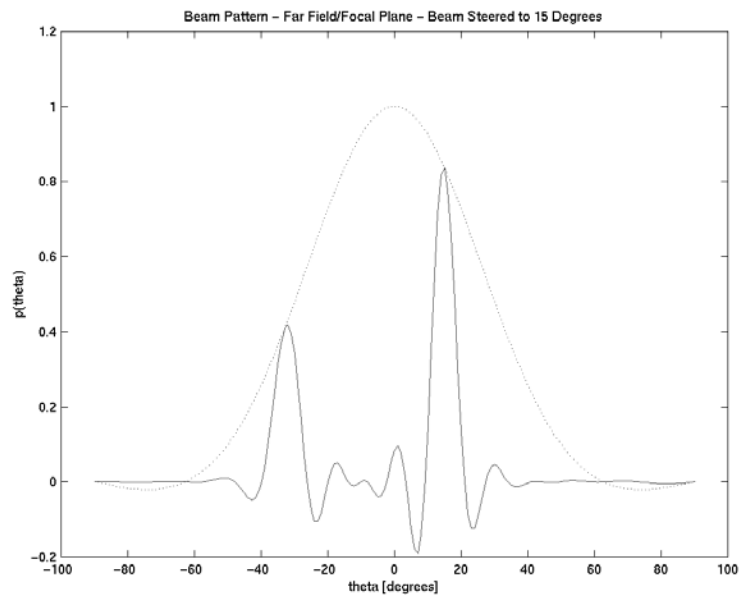
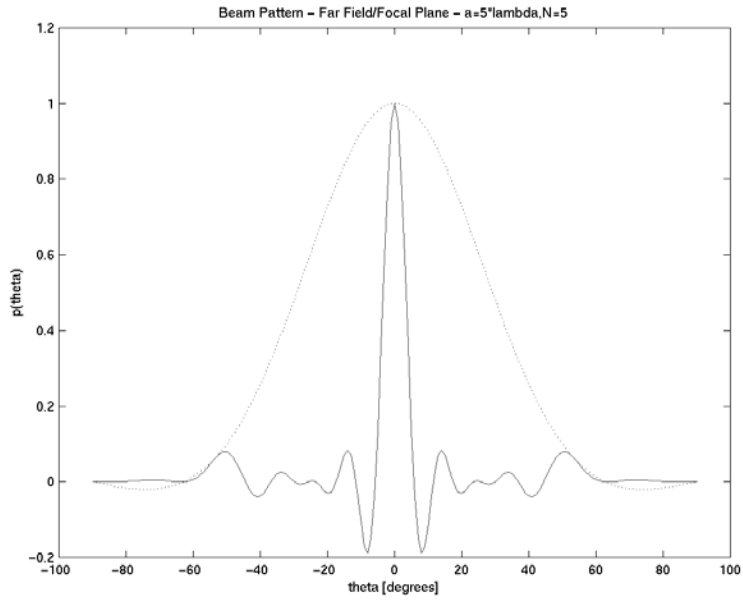
$$p(r, \theta) = K \cos \theta \text{sinc}\left(w \frac{\sin \theta}{\lambda}\right) \cdot \sum_{n=-\infty}^{\infty} \text{sinc}\left(2a\left(\frac{\sin \theta}{\lambda} - \frac{\sin \theta_0}{\lambda} - \frac{n}{d}\right)\right)$$

Finally, for a Nyquist sampled array ($d=\lambda/2$):

$$p(r, \theta) = K \cos \theta \text{sinc}\left(w \frac{\sin \theta}{\lambda}\right) \cdot \sum_{n=-\infty}^{\infty} \text{sinc}((\text{N.A.}) \cdot (\sin \theta - \sin \theta_0 - 2n))$$

The size of the grating lobes, however, is dramatically affected by the numerical aperture. First, let's determine if the main lobe of any of the grating lobes can appear in the field of view. For $-\pi < \theta, \theta_0 < \pi$, there are no cases for which $(\sin \theta - \sin \theta_0) \geq 2$. Thus, the argument of the sinc can never be zero and the main lobe of any of the grating lobes will not appear in the field of view. Side lobes, however, can appear. The value of the argument of the sinc is weighted by the numerical aperture and thus, we are pushed farther out. In general, these sidelobes produce additional ripple in the background.

Here are two cases of arrays for which Nyquist sampling has not been achieved.



Comments:

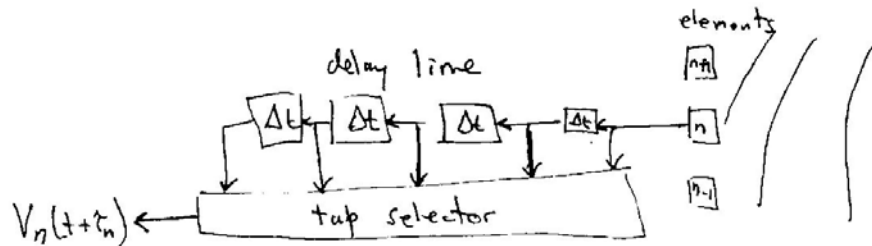
- Observe that the grating lobes are worse for off-axis beams.
- For Nyquist sampled arrays, the effects of grating lobes is minimal.
- The grating lobes affect both the transmit and receive patterns.

Processing for Delay-Sum Beamforming

- The US pulse is fired once → therefore, we must choose a particular steering and depth of focus and set up the delays at the time of transmission. This is easily done - for focussing at (r_0, θ_0) :

$$\tau'_n = \tau'(x_n, r_0, \theta_0) = -\frac{x_n \sin \theta_0}{c} + \frac{x_n^2 \cos^2 \theta_0}{2cr_0}$$

- The receive signal at time t corresponds to depth $r = tc/2$.
 - At each point t we need a new focal depth, r .
 - Usually, the receive angle is the same as the transmit angle, θ_0 .
 - We need a different set of delay for each depth (time sample).
- One common way of implementing the delays is with a tapped delay line. Each transducer has a delay line such as:



where $\tau_n \approx m\Delta t$

First, we look at focussing only:

$$\tau'_n = \frac{x_n^2}{2cr} = \frac{x_n^2}{c^2 t} \approx m\Delta t$$

thus the tap number is set by:

$$m(t) = \text{int} \left[\frac{x_n^2}{c^2 t \Delta t} \right]$$

The particular tap that has been selected is changed when:

$$m(t_k) - m(t_{k+1}) = 1.$$

- How small does Δt need to be? The phase error introduced by the coarseness of the delay line is:

$$\Delta\phi = 2\pi f_0 \Delta t$$

- Typically, a phase error less than 10 degrees is necessary for good quality beam forming:

$$\Delta t = \frac{2\pi}{36} \frac{1}{2\pi f_0} = \frac{1}{36 f_0}$$

- For example, if $f_0 = 5$ MHz, then $\Delta t \leq 6$ ns (sampling rate = 180 MHz).
 - This is a high sampling rate – devices would be expensive.
 - Synchrony and accuracy along a long delay line for many channels is difficult.
- How many taps would be necessary? Let's look at focusing and beam steering delays for a sample configuration with $r = 2$ cm, $a = 2$ cm:

$$\max \tau_n = \frac{a^2}{2cr} = \frac{2\text{cm} \cdot 2\text{cm}}{2 \cdot 0.15\text{cm}/\mu\text{s} \cdot 2\text{cm}} \approx 6.7 \mu\text{s} = 6700 \text{ ns}$$

This corresponds to about 10^3 taps.

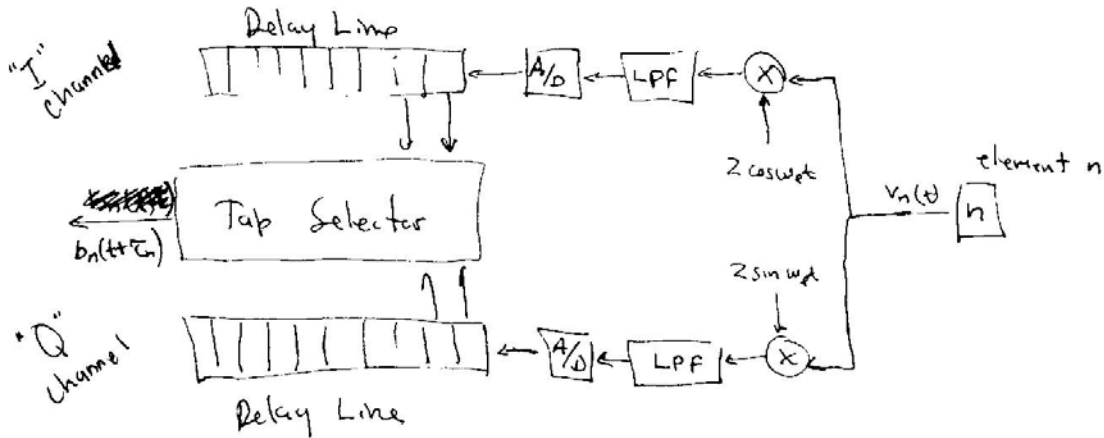
- For beam steering, the maximum delay (end-to-end of array is):

$$\frac{2a}{c} = \frac{2 \cdot 2\text{cm}}{0.15\text{cm}/\mu\text{s}} = 27 \mu\text{s}$$

This corresponds to about 5×10^3 taps.

Practical Beam Forming System

While the above could be built with today's technology, there is an eloquent solution to this hardware challenge that can lead to substantial cost savings. This solution involves the complex demodulation to an intermediate frequency, which is then sampled and followed by a software phase correction term.



Recall the actual received waveform is a real valued function:

$$v_n(t) = \Re\{a(t - \tau_n)e^{-i\omega_0(t - \tau_n)}\} = a(t - \tau_n) \cos \omega_0(t - \tau_n)$$

(with some scaling factors dropped). First, let's look at demodulation part of this receiver. There are two channels – the “I” or in-phase channel and “Q” or quadrature channel:

$$\begin{aligned} I &= \text{LPF}\{a(t - \tau_n) \cos \omega_0(t - \tau_n) 2 \cos \omega_l t\} \\ &= \text{LPF}\{a(t - \tau_n) [\cos((\omega_0 - \omega_l)t - \omega_0 \tau_n) + \cos(\omega_0 + \omega_l)t]\} \\ &= a(t - \tau_n) \cos(\Delta\omega t - \omega_0 \tau_n) \end{aligned}$$

where $\Delta\omega = \omega_0 - \omega_l \ll \omega_0$ (the difference between the transmit frequency and the local oscillator in the receiver). For the Q channel:

$$\begin{aligned} &= \text{LPF}\{a(t - \tau_n) \cos \omega_0(t - \tau_n) 2 \sin \omega_l t\} \\ &= \text{LPF}\{a(t - \tau_n) [\sin((\omega_0 - \omega_l)t - \omega_0 \tau_n) + \sin(\omega_0 + \omega_l)t]\} \\ &= a(t - \tau_n) \sin(\Delta\omega t - \omega_0 \tau_n) \end{aligned}$$

Now, we define the baseband signal for element n , $b_n(t)$:

$$b_n(t) = I \mp iQ = a(t - \tau_n) e^{\mp i\Delta\omega t} e^{\pm i\omega_0 \tau_n}$$

To simplify the following analysis, we will let $\omega_l = \omega_0$ or $\Delta\omega = 0$, or:

$$b_n(t) = a(t - \tau_n) e^{\pm i\omega_0 \tau_n}$$

The \pm is given to indicate that either the positive or negative frequency component can be used to reconstruct the baseband signal.

- Now, we need to sample $b_n(t)$, but this has phase variations that occur much more slowly. The sampling must occur only fast enough to capture:
 - Amplitude changes in $a(t)$
 - Changing phase delays for differing focal depths
 - Not $\omega_0 t$.
- Sampling at $f_s = (2-4)\times f_0$ is adequate for good beam forming. Thus we get a 10-fold reduction in the sampling rate and number of taps in the delay line.

When constructing the final baseband signal, we shift the demodulated signal around by discrete sample values - $[\tau_n']$. We also apply a phase term $\omega_l \tau_n'$, but this value needn't be discretized. The final baseband signal is:

$$\begin{aligned}
 b(t) &= \frac{1}{N} \sum_{n=1}^N b_n(t + [\tau_n']) e^{\mp i \omega_l (t + \tau_n')} \\
 &= \frac{1}{N} \sum_{n=1}^N a(t - \tau_n + [\tau_n']) e^{\pm i \omega_0 \tau_n} e^{\mp i \omega_l (t + \tau_n')} \\
 &= \frac{1}{N} \sum_{n=1}^N a(t - \tau_n + [\tau_n']) e^{\mp i \omega_0 (t - \tau_n + \tau_n')}
 \end{aligned}$$

Observe that the $\omega_l t$ term is common to all terms and can be pulled out of the summation. I've included the + and - terms here because you may not always know how your demodulation is done (e.g. in the US project). Now, keeping only the negative term, we make an approximation that $a(t)$ varies little over Δt and thus:

$$b(t) \approx \frac{1}{N} \sum_{n=1}^N a(t - \tau_n + \tau_n') e^{-i \omega_0 (t - \tau_n + \tau_n')}$$

Recall that:

$$\begin{aligned}
 \tau_n &= \frac{r}{c} - \frac{x_n \sin \theta_0}{c} + \frac{x_n^2 \cos^2 \theta_0}{2cr} \\
 \tau_n' &= -\frac{x_n \sin \theta_0}{c} + \frac{x_n^2 \cos^2 \theta_0}{2cr}, \text{ and } [\tau_n'] \text{ is discretized to } m\Delta t
 \end{aligned}$$

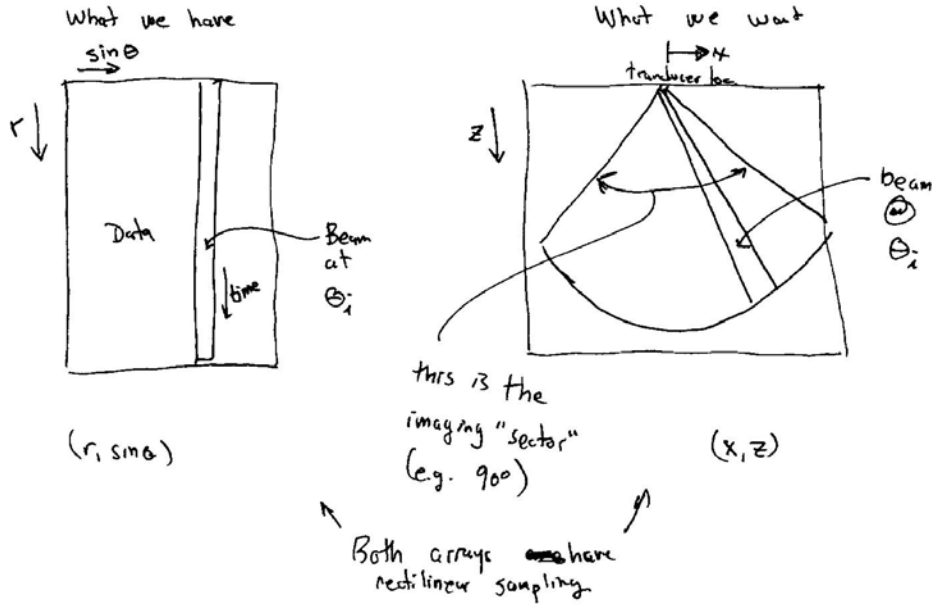
and thus:

$$b(t) = a(t - r/c) e^{-i \omega_0 (t - r/c)}$$

- Observe the additional phase term $e^{-i\omega_l(t+\tau_n)}$. This term is necessary to bring align the phases before summing. The $e^{-i\omega_l t}$ part is unnecessary because it does not depend on n and can be pulled outside the summation and we usually look at $|b(t)|$.
- The phase correction term is calculated only at the time of the samples (which are at $m\Delta t$) and thus, $\tau_n - \tau_n' \approx r/c$. This phase term is usually implemented in the digitally sampled data and thus can be performed with very high accuracy.
- The final received signal is the proportional to $b(t)$ and the reflectivity at a point at depth, $r = tc/2$.
- Commonly, data is presented as $|R(tc/2, \theta)b(t)|$ or $\log|R(tc/2, \theta)b(t)|$.
- Let's look again at errors – the error in phase due to the coarseness of samples is $\Delta\phi = \Delta\omega\Delta t$ and $\Delta\omega$ can be made arbitrarily small (even 0).
- The other source of error is scalar from poor sampling of $a(t)$, but with a sample rate of $f_s = (2-4)f_0$ and the pulse length $T=3/f_0$, we insure that 6-12 samples are made over the pulse shape. (The delays align the pulses prior to summation.)

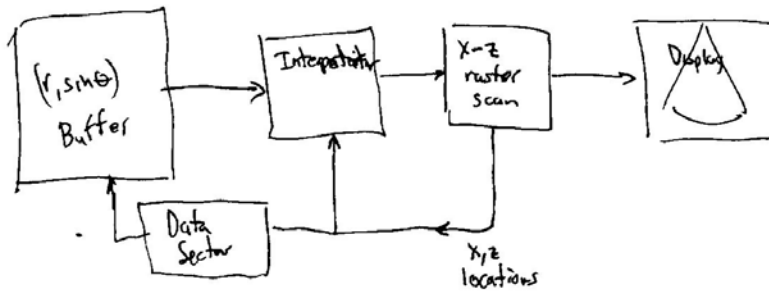
Scan Conversion

The final process in a US system is scan conversion to present the data as an image in a rectilinear coordinate space to the user. This involves conversion from our beam data, which exists in a domain that is equally spaced in $\sin\theta$ and in r , to physical space, which is a domain that is equally spaced in x and z :

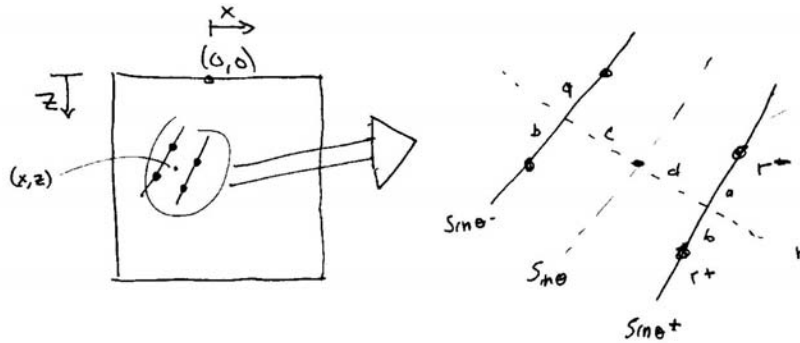


This process is often driven by needs of display memory, which scans its domain (x, z) and requests interpolates from the acquired data in $(\sin \theta, r)$ space. In this case, the output coordinates drives the interpolation process – in other words, it is “output-driven.” There are approaches to interpolation that are “input driven,” but we will not describe those here.

Display Hardware



One simple and common approach to interpolation is bilinear interpolation:



Here, we find the 4 nearest neighbors and perform linear interpolation in two directions. First, let's take our output coordinates and give its equivalent in the input space:

$$r = \sqrt{x^2 + z^2} \quad \text{and} \quad \sin \theta = \frac{x}{r}$$

The of four nearest neighbors are determined from:

$$r^+ = \Delta r \cdot \text{ceil}\left(\frac{r}{\Delta r}\right); r^- = \Delta r \cdot \text{floor}\left(\frac{r}{\Delta r}\right);$$

$$\sin \theta^+ = \Delta \sin \theta \cdot \text{ceil}\left(\frac{\sin \theta}{\Delta \sin \theta}\right); \sin \theta^- = \Delta \sin \theta \cdot \text{floor}\left(\frac{\sin \theta}{\Delta \sin \theta}\right)$$

and the data are:

$$P_1 = \text{data}(r^-, \sin \theta^-); P_2 = \text{data}(r^-, \sin \theta^+); P_3 = \text{data}(r^+, \sin \theta^-); P_4 = \text{data}(r^+, \sin \theta^+)$$

and distances of several segments are:

$$a = r - r^-; b = r^+ - r; c = \sin \theta - \sin \theta^-; d = \sin \theta^+ - \sin \theta$$

and finally:

$$P(x, z) = \frac{bdP_1 + bcP_2 + adP_3 + acP_4}{\Delta r \Delta \sin \theta}$$

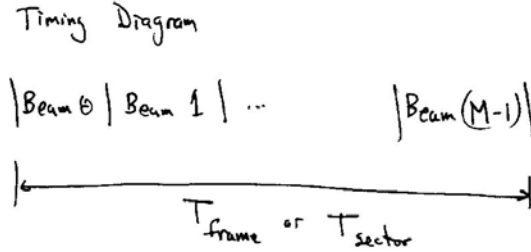
There are, of course, many other (and better) methods for interpolation in a 2D space.

Temporal Resolution

The temporal resolution of an US system is determined by the time required to image an entire sector. Typically only one pulse is traversing the body at any given time. Thus the minimum time between pulses is given by:

$$T = \frac{2r_{\max}}{c}$$

For example, if $r_{\max} = 20$ cm, then $T = 267$ μ s.



For M beams in a sector, the time for one frame will be $T_{\text{frame}} = TM$. For $r_{\max} = 20$ cm and $M = 128$ then $T_{\text{frame}} \approx 33$ ms. Thus the frame rate will be about 30 Hz.

Overall System Architecture

