The 1D Fourier Transform

Definition. The Fourier Transform (FT) relates a function to its frequency domain equivalent. The FT of a function *g(x)* is defined by the Fourier integral:

$$G(s) = F\{g(x)\} = \int_{-\infty}^{\infty} g(x) e^{-i2pxs} dx$$

for $x, s \in \Re$. There are a variety of existence criteria and the FT doesn't exist for all functions. For example, the function *g(x) = cos(1/x)* has an infinite number of oscillations as $x \to 0$ and the FT integral can't be evaluated. If the FT exists, then there is an inverse FT relationship:

$$g(x) = F^{-1}\{G(s)\} = \int_{-\infty}^{\infty} G(s) e^{i2pxs} ds$$

Uniqueness: Given the existence of the inverse FT, it follows that if the FT exists, it must be unique. That is, for a function forms a unique pair with its FT:

$$g(x) \leftrightarrow G(s)$$

Caveat. An exception to the uniqueness property is a class of functions called "massless" or "null" functions. An example is the continuous function $f(x) = \{ \begin{matrix} 1, x = 0 \\ 0, x \neq 0 \end{matrix}$. This function and others like it have the same Fourier transform as *f(x)* = 0: *F(s)* = 0. Thus, the uniqueness exists only for a function plus or minus arbitrary null functions. In practice, these functions are not realizable and thus, for the purposes of this class we will assume that the FT is unique.

Symmetry Definitions. We first decompose some function *g(x)* in to even and odd components, *e(x)* and *o(x)*, respectively, as follows:

$$e(x) = \tfrac{1}{2}[g(x) + g(-x)]$$

$$o(x) = \tfrac{1}{2}[g(x) - g(-x)]$$

thus,

$$g(x) = e(x) + o(x)$$

and

$$e(x) = e(-x) \text{ and } o(x) = -o(x)$$

A function, *g(x),* is Hermitian Symmetric (Conjugate Symmetric) if:

$$\text{Re}\{g(x)\} = e(x) \text{ and } \text{Im}\{g(x)\} = o(x)$$

thus,

$$g(x) = e(x) + io(x) = g^*(-x)$$

Symmetry Properties of the FT. There are several related properties:

1. If *g(x)* is real, then *G(s)* is Hermitian symmetric (e.g. *G(s) = G*(-s)*).
2. If *g(x)* is real and even, *G(s)* is real and even.
3. If *g(x)* is real and odd, *G(s)* is imaginary and odd.
4. If *g(x)* is real, *G(s)* can be specified entirely by non-negative frequencies ($s \geq 0$). That is, only ½ of the Fourier information is necessary to specify a real function.
5. If *g(x)* is imaginary, then *G(s)* is Anti-Hermitian symmetric (e.g. *G(s) = -G*(-s)*).

Proof of 1.

$$G(s) = \int g(x) e^{-i2psx} dx$$

$$= \int [e(x) + o(x)][\cos 2psx - i \sin 2psx] dx \qquad \text{(cos is even, sin is odd)}$$

$$= \int e(x) \cos 2psx dx + \int o(x) \cos 2psx dx - i \int e(x) \sin 2psx dx - i \int o(x) \sin 2psx dx$$

$$= \int e'(x) dx + \int o'(x) dx - i \int o''(x) dx - i \int e''(x) dx$$

$$= E(s) + 0 - i \cdot 0 - iO(s) \quad (\text{cos is even in } s, \text{sin is odd is } s, \int_{-\infty}^{\infty} odd(x) = 0)$$

$$= E(s) - iO(s)$$

$$= E(-s) + iO(-s)$$

$$= G^*(-s) \qquad \text{Q.E.D.}$$

Comment on negative frequencies. Consider a real-valued signal – imagine a voltage on a wire or the sound pressure against your eardrum – the Fourier transform of these is completely specified by the positive frequencies (e.g. *G(-s) = G*(s)*). We can argue that we have the concept of a frequency (oscillations/second), but it doesn't really make physical sense to talk about positive or negative frequencies. In this case, we could argue that the having positive and negative frequencies is merely a mathematical convenience. Are there cases where negative frequencies have meaning? Consider the bit in a drill – it can turn clockwise or counter clockwise and different rotational rates. Here positive and negative frequencies have physical meaning (the direction of rotation). As we shall see, the magnetic moment in MRI is a case where the sign indicates the direction of precession.

Convolution Definition. The convolution operator is defined as:

$$g(x) * h(x) = \int_{-\infty}^{\infty} g(x) h(x - x) dx$$

The convolution operator commutes:

$$g(x) * h(x) = \int_{-\infty}^{\infty} g(x) h(x - x) dx = \int_{-\infty}^{\infty} g(x - x) h(x) dx = h(x) * g(x)$$

The delta function, $d(x)$. The Dirac delta or impluse function is a mathematical construct that is infinitely high in amplitude, infinitely short in duration and has unity area:

$$d(x) = 0 \quad \text{everywhere except } x = 0 \quad \text{and} \quad \int d(x)dx = 1$$

Most properties of $d(x)$ can exist only in a limiting case (e.g. as a sequence of functions $g_n(x) \to d(x)$) or under an integral. Some important properties of $d(x)$:

$$\int d(x)g(x)dx = g(0), \text{ with } g(x) \text{ continuous at } x = 0$$

$$\int d(x-a)g(x)dx = g(a), \text{ with } g(x) \text{ continuous at } x = a$$

$$\int d(ax)g(x)dx = \frac{1}{|a|}g(0), \text{ with } g(x) \text{ continuous at } x = 0$$

$$F\{d(x)\} = 1$$

Delta function properties. First two are technically only defined under the integral, but we'll still talk about them.

Similarity (stretching)	$d(ax) = \frac{1}{\lvert a \rvert} d(x)$
Product/Sampling	$g(x)d(x-a) = g(a)d(x-a)$
Sifting	$\int g(x)d(x-a)dx = g(a)$
Convolution	$g(x) * d(x) = d(x) * g(x) = g(x)$ $g(x) * d(x-a) = d(x-a) * g(x) = g(x-a)$

Fourier Transform Theorems. There are many Fourier transform properties and theorems. This is a partial list. Assume that $F\{g(x)\} = G(s)$, $F\{h(x)\} = H(s)$ and that a and b are constants:

Linearity	$F\{ag(x) + bh(x)\} = aG(s) + bH(s)$
Similarity (stretching)	$F\{g(ax)\} = \frac{1}{\lvert a \rvert} G\left(\frac{s}{a}\right)$
Shift	$F\{g(x-a)\} = G(s)e^{-i2pas}$
Convolution	$F\{g(x) * h(x)\} = G(s)H(s)$
Product	$F\{g(x)h(x)\} = G(s) * H(s)$
Complex Modulation	$F\{g(x)e^{i2ps_0x}\} = G(s - s_0)$
Modulation	$F\{g(x)\cos(2ps_0x)\} = \frac{1}{2}[G(s-s_0) + G(s+s_0)]$ $F\{g(x)\sin(2ps_0x)\} = \frac{1}{2i}[G(s-s_0) - G(s+s_0)]$
Rayleigh's Power	$\int \lvert g(x) \rvert^2 dx = \int \lvert G(s) \rvert^2 ds$
Cross Power	$\int g(x)h^*(x)dx = \int G(s)H^*(s)ds$

Axis Reversal	$F\{g(-x)\} = G(-s)$
Complex Conjugation	$F\{g^*(x)\} = G^*(-s)$
Autocorrelation	$F\{g(x) * g^*(-x)\} = G(s)G^*(s) = \lvert G(s)\rvert^2$
Reverse Relationships	$F\{G(x)\} = g(-s)$ $F\{G^*(x)\} = g^*(s)$
Differentiation	$F\left\{\frac{d}{dx} g(x)\right\} = i2psG(s)$
Moments	$F\{xg(x)\} = \frac{i}{2p}\frac{d}{ds}G(s)$

Some common FT pairs:

$g(x)$	$G(s)$
1	$d(s)$
$d(x)$	1
$\cos(2ps_0x)$ $\sin(2ps_0x)$	$\frac{1}{2}[d(s-s_0)+d(s+s_0)]$ $\frac{1}{2i}[d(s-s_0)-d(s+s_0)]$
$\mathrm{rect}(x) = \{\begin{matrix} 1 & \lvert x\rvert < \frac{1}{2} \\ 0 & \lvert x\rvert \geq \frac{1}{2}\end{matrix}$	$\mathrm{sinc}(x) = \frac{\sin(px)}{px}$
$\mathrm{sinc}(x)$	$\mathrm{rect}(s)$
$\mathrm{triangle}(x) = \{\begin{matrix} 1-\lvert x\rvert & \lvert x\rvert < 1 \\ 0 & \lvert x\rvert \geq 1\end{matrix}$	$\mathrm{sinc}^2(s)$
e^{-px^2}	e^{-ps^2}
$\mathrm{sgn}(x) = \{\begin{matrix} 1 & x \geq 0 \\ -1 & x < 0\end{matrix}$	$\frac{1}{ips}$
$e^{-\lvert x\rvert}$	$\frac{2}{1+(2ps)^2}$
e^{-x}, for $x > 0$; 0, otherwise	$\frac{1-i2ps}{1+(2ps)^2}$
$\lvert x\rvert^{-1/2}$	$\lvert s\rvert^{-1/2}$
$J_0(2px)$	$\frac{\mathrm{rect}(s/2)}{p(1-s^2)^{1/2}}$
$\mathrm{comb}(x)$	$\mathrm{comb}(s)$

The comb function, comb(*x*). The sampling or "comb" function is a train of delta functions:

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} d(x-n)$$

The Fourier transform of comb(x) is:

$$F\{\mathrm{comb}(x)\} = \mathrm{comb}(s)$$

Proof.

$$F\{\text{comb}(x)\} = F\left\{\sum_{n=-\infty}^{\infty} d(x-n)\right\} = \sum_{n=-\infty}^{\infty} e^{i2pns} = F(s)$$

The RHS of the above expression can be viewed as the exponential Fourier series representation of a periodic function F(s) with period 1 and $a_n = 1$ for all n. The Fourier series expressions are:

$$F(s) = \sum_{n=-\infty}^{\infty} a_n e^{i2pns}, \text{ where } a_n = \int_{-1/2}^{1/2} F(s) e^{-i2pns} ds .$$

Now, let $G(s) = \text{rect}(s)F(s)$ (one period of $F(s)$) and thus $F(s) = \sum_{m=-\infty}^{\infty} G(s-m)$. Now observe that

$$a_n = \int_{-1/2}^{1/2} F(s) e^{-i2pns} ds = \int_{-\infty}^{\infty} G(s) e^{-i2pns} ds = F\{G(s)\}\big|_{x=n} = 1$$

One function that satisfies this relationship is $G(s) = d(s)$. Thus, one possible Fourier transform of comb(x) is:

$$F(s) = \sum_{m=-\infty}^{\infty} d(s-m) = \text{comb}(s)$$

By uniqueness of the Fourier transform, this is the unique Fourier transform of comb(x).

Sampling and replication by comb(x). The comb function can be used to sample or extract values of a continuous function $g(x)$. Sampling with period X can be done as:

$$g(x)\text{comb}(\tfrac{x}{X}) = \sum_{n=-\infty}^{\infty} g(x) d(\tfrac{x}{X} - n) = X \sum_{n=-\infty}^{\infty} g(x) d(x - nX) = X \sum_{n=-\infty}^{\infty} g(nX) d(x - nX) .$$

By the stretching and sifting properties of the delta function. A function $g(x)$ can be replicated with period X by convolving with a comb function:

$$g(x) * \text{comb}(\tfrac{x}{X}) = \sum_{n=-\infty}^{\infty} \left[g(x) * d(\tfrac{x}{X} - n)\right] = X \sum_{n=-\infty}^{\infty} \left[g(x) * d(x - nX)\right] = X \sum_{n=-\infty}^{\infty} g(x - nX)$$

By the stretching and convolution properties of the delta function.

Sampling Theory. When manipulating real objects in a computer, we must first sample the continuous domain object into a discretized version that the computer can handle. As described above, we can sample a function $g(x)$ at frequency $f_s = 1/X$ using the comb function:

$$g_s(x) = g(x)\text{comb}(\tfrac{x}{X}) = X \sum_{n=-\infty}^{\infty} g(nX) d(x - nX) .$$

The Fourier transform is:

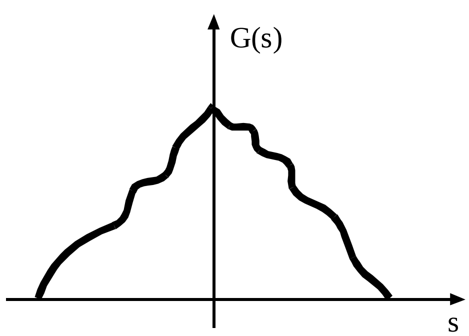
$$G_s(s) = G(s) * X\mathrm{comb}(Xs)$$

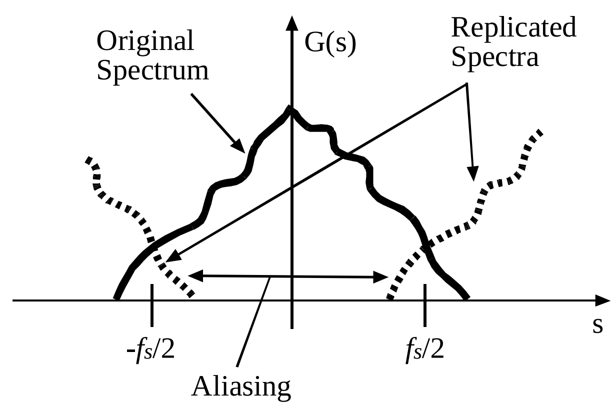
$$= G(s) * \sum_{m=-\infty}^{\infty} Xd(Xs - m)$$

$$= G(s) * \sum_{m=-\infty}^{\infty} d(s - mf_s)$$

$$= \sum_{m=-\infty}^{\infty} G(s - mf_s)$$

Thus, sampling in one domain leads to replication of the spectrum in the other domain. The spectrum is periodic with period f_s. Typically, only frequencies less than $f_s/2$ can be represented in the discrete domain signal. Any components that lie outside of this spectral region $(-f_s/2 \le s \le f_s/2)$ results in "aliasing" – the mis-assignment of spectral information.





The Whittaker-Shannon sampling theorem states that a band limited function with maximum frequency s_{max} can be fully represented by a discrete time equivalent provided the sampling frequency satisfies the Nyquist sampling criterion:

$$f_s = \frac{1}{X} \ge 2s_{\max}$$

If this is the case, then the original spectrum can be extracted (by filtering) and by uniqueness of the FT, the original signal can be reconstructed. To reconstruct the original signal, we apply a reconstruction filter $H(s) = \mathrm{rect}(s/f_s) = \mathrm{rect}(Xs)$:

$$\hat{G}(s) = G_s(s)H(s) = G_s(s)\mathrm{rect}(Xs)$$

$$= G(s), \text{if there is no aliasing}$$

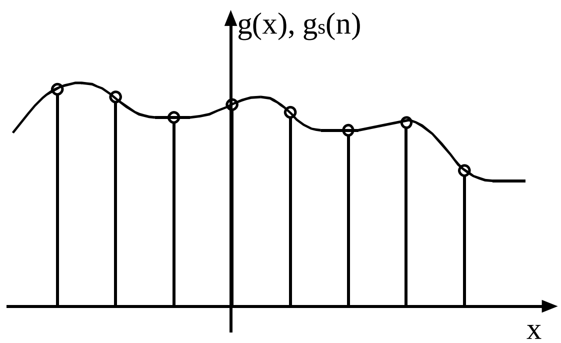
In the *x* domain, this results in "sinc" interpolation:

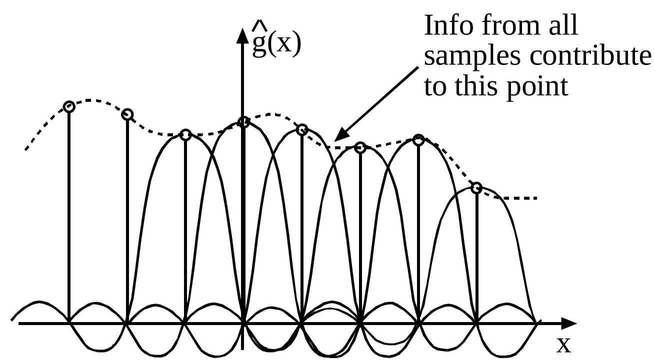
$$\hat{g}(x) = g_s(x) * \tfrac{1}{X}\,\mathrm{sinc}(\tfrac{x}{X})$$

$$= \left[\sum_{n=-\infty}^{\infty} g(nX)d(x-nX)\right] * \mathrm{sinc}(\tfrac{x}{X})$$

$$= \sum_{n=-\infty}^{\infty} \mathrm{sinc}(\tfrac{x-nX}{X})g(nX)$$

If the Nyquist criterion is met, then $\hat{g}(x) = g(x)$.





Units. If *x* has units of Q, then *s* will have units of “cycles/Q” or Q^{-1}. Please note that under our definition of the FT, this is not an angular frequency with units of radians/Q, but just plain Q^{-1}. Please also keep in mind that *x* is the index of variation – for example, we can have *g*(*x*) represent a velocity that varies as a function of spatial location *x*. The function *g*(*x*) has units cm/s, but *x* has units cm and *G*(*s*) has units of cm/s, but *s* has units of cm^{-1}.

Examples:

x	*s*
Time s (seconds)	Temporal Frequency s^{-1}, Hz, cycles/s
Distance cm	Spatial Frequency cm^{-1}, cycles/cm

The 2D Fourier Transform

Definition. The 2D Fourier Transform (FT) relates a function to its frequency domain equivalent. The FT of a function $g(x,y)$ is defined by the 2D Fourier integral:

$$G(u,v) = F\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-i2p(xu+vy)}dxdy$$

There is also an inverse FT relationship:

$$g(x,y) = F^{-1}\{G(u,v)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u.v)e^{i2p(xu+vy)}dudv$$

Uniqueness: Given the existence of the inverse FT, it follows that if the FT exists, it must be unique. That is, for a function forms a unique pair with its FT:

$$g(x,y) \leftrightarrow G(u,v)$$

2D FT in Polar Coordinates. We consider a special case where the functional form of $g(x,y)$ is separable in polar coordinates, that is, $g(r,q) = g_R(r)g_Q(q)$. Since $g_Q(q)$ is periodic in q, it has a Fourier series representation:

$$g_\Theta(q) = \sum_{n=-\infty}^{\infty} a_n e^{inq} \ .$$

It can be shown that

$$F_{2D}\left\{g_R(r)e^{inq}\right\} = (-i)^n e^{inf} \cdot \int_0^\infty 2pg_R(r)J_n(2prr)rdr$$

where the part under the integral in known as the Hankel transform of order n, and $J_n(\cdot)$, is the n^{th} order Bessel function of the first kind:

$$J_n(a) = \frac{1}{2p}\int_{-p}^{p} e^{i(a\sin j - nj)}dj \ .$$

(Derivation of the Hankel transform relationship relies on $e^{-i2p(xu+yv)} = e^{-i2prr\cos(q-f)}$.) Thus, the 2D FT in polar form is:

$$G(r,f) = F\left\{g_R(r)g_\Theta(q)\right\} = \sum_{n=-\infty}^{\infty} a_n(-i)^n e^{inf} \cdot \int_0^\infty 2pg_R(r)J_n(2prr)rdr$$

For the special case of circular symmetry of g, that is, $g(r,q) = g_R(r)$, then:

$$G(r,f) = G(r) = 2p\int_0^\infty g_R(r)J_0(2prr)rdr$$

which is also a circularly symmetric function. The inverse relationship is the same:

$$g_R(r) = 2p\int_0^\infty G(r)J_0(2prr)\,rdr$$

Some Symmetry Properties of the FT.

1. If $g(x,y)$ is real, then $G(u,v)$ is Hermitian Symmetric, that is, $G(u,v) = G^*(-u,-v)$.
2. If $G(u,v)$ is real, then $g(x,y)$ is Hermitian Symmetric, that is, $g(x,y) = g^*(-x,-y)$.
3. If $g(x,y)$ is real and even, then $G(u,v)$ is also real and even.
4. If $g(r,q) = g_R(r)$ (circularly symmetric), then $G(r,f) = G(r)$ (circularly symmetric).

The delta function, $d(x, y)$. The delta function in two is equal the to product of two 1D delta functions $d(x, y) = d(x)d(y)$. In a manner similar to the 1D delta function, the 2D delta function has the following definition:

$$d(x, y) = 0 \quad \text{everywhere except } (x, y) = (0,0) \text{ and } \int\int d(x, y)dxdy = 1$$

Most properties of $d(x, y)$ can be derived from the 1D delta function. There is also a polar coordinate version of the 2D delta function: $d(x, y) = d(r)/pr$.

Fourier Transform Theorems. Let *a* and *b* are non-zero constants and $F\{g(x,y)\} = G(u,v)$ and $F\{h(x,y)\} = H(u,v)$.

Linearity	$F\{ag(x, y) + bh(x, y)\} = aG(u, v) + bH(u, v)$
Magnification	$F\{g(ax, by)\} = \frac{1}{\|ab\|} G\left(\frac{u}{a}, \frac{v}{b}\right)$
Shift	$F\{g(x - a, y - b)\} = G(u, v)e^{-i2p(ua+vb)}$
Complex Modulation	$F\left\{g(x, y)e^{i2p(xa+yb)}\right\} = G(u - a, v - b)$
Convolution/Multiplication	$g(x, y) ** h(x, y) = \int\int g(x, h)h(x - x, y - h)dxdh$ $F\{g(x, y) ** h(x, y)\} = G(u, v)H(u, v)$ $F\{g(x, y)h(x, y)\} = G(u, v) ** H(u, v)$
Correlation	$g(x, y) \bullet\bullet h(x, y) = \int\int g(x, h)h^*(x + x, y + h)dxdh$ $F\{g(x, y) \bullet\bullet h(x, y)\} = G(u, v)H^*(u, v)$ $F\{g(x, y) \bullet\bullet g(x, y)\} = \|G(u, v)\|^2$
Separability	$g(x, y) = g_X(x)g_Y(y)$ $F\{g(x, y)\} = F_{1D,x}\{g_X(x)\}F_{1D,y}\{g_Y(y)\}$ $= G_X(u)G_Y(v)$
Power	$\iint \|g(x, y)\|^2 dxdy = \iint \|G(u, v)\|^2 dudv$ $\iint g(x, y)h^*(x, y)dxdy = \iint G(u, v)H^*(u, v)dudv$
Axis Reversal	$F\{g(-x,-y)\} = G(-u,-v)$
Conjugation	$F\{g^*(x, y)\} = G^*(-u,-v)$
Reverse Relationships	$F\{G(x, y)\} = g(-u,-v)$ $F\{G^*(x, y)\} = g(u, v)$
Derivative	$F\left\{\frac{\partial}{\partial x} g(x, y)\right\} = i2puG(u, v)$
DC Value	$G(0,0) = \iint g(x, y)dxdy$
Eigenfunction of Linear Space Invariant Systems	$F\left\{e^{i2p(ax+by)} ** h(x, y)\right\} = H(a, b)d(x - a, y - b)$ $e^{i2p(ax+by)} ** h(x, y) = H(a, b)e^{i2p(ax,by)}$

Some common 2D FT pairs:

1	$d(u,v)$
$d(x,y)$	1
$d(x-a,y-b)$	$e^{-i2p(ua+vb)}$
$e^{i2p(ax+by)}$	$d(u-a,v-b)$
$e^{-pr^2}=e^{-px^2}e^{-py^2}$	$e^{-pr^2}=e^{-pu^2}e^{-pv^2}$
$\cos(2px)=\cos(2px)\cdot 1$	$\frac{1}{2}\left[d(u-1)+d(u+1)\right]d(v)$
$\mathrm{rect}(y)=1\cdot\mathrm{rect}(y)$	$d(u)\mathrm{sinc}(v)$
$\mathrm{rect}(ax)\mathrm{rect}(by)$	$\frac{1}{\lvert ab\rvert}\mathrm{sinc}\left(\frac{u}{a}\right)\mathrm{sinc}\left(\frac{v}{b}\right)$
$\mathrm{circ}(r)=\{\begin{matrix}1, r\le 1\\ 0, r>1\end{matrix}$	$\frac{J_1(2pr)}{r}=\mathrm{jinc}(r))$
$1/r$	$1/r$
comb(x,y) = comb(x)comb(y)	comb(u,v) = comb(u)comb(v)

The comb function in 2D, comb(x,y). The 2D sampling or comb function is defined as comb(x,y)=comb(x)comb(y) and has the 2D FT F{comb(x,y)}= comb(u,v). Formally, the 2D comb function is defined as:

$$\mathrm{comb}(x,y)=\sum_{n,m=-\infty}^{\infty} d(x-n,y-m)$$

In a manner similar to the 1D case, we can prove that Fourier transform of the 2D comb function is also a 2D comb function as given in the above table.

Sampling Theory in 2D. In a manner similar to sampling in 1D, sampling in 2D can be modeled as multiplying a function times the 2D comb function. With sample spacing of X and Y, in the x and y directions, the sampled function is:

$$g_s(x,y)=g(x,y)\mathrm{comb}\left(\tfrac{x}{X},\tfrac{y}{Y}\right)=g(x,y)\sum_{n,m=-\infty}^{\infty} d\left(\tfrac{x}{X}-n,\tfrac{y}{Y}-m\right)$$

$$=XY\sum_{n,m=-\infty}^{\infty} d(x-nX,y-mY)g(x,y)$$

$$=XY\sum_{n,m=-\infty}^{\infty} d(x-nX,y-mY)g(nX,mY)$$

The discrete domain equivalent is $g_d(n,m)=g(nX,mY)=g_s(nX,mY)$. In the Fourier domain, the result is:

$$G_s(u,v) = G(u,v) ** XY\mathrm{comb}(Xu, Yv)$$

$$= G(u,v) ** \sum_{n,m=-\infty}^{\infty} d(u - \tfrac{n}{X}, v - \tfrac{m}{Y})$$

$$= \sum_{n,m=-\infty}^{\infty} G(u - \tfrac{n}{X}, v - \tfrac{m}{Y})$$

Thus, sampling in one domain leads to replication of the spectrum in the other domain. Spacing of the replicated spectra is (1/*X*,1/*Y*). The Whittaker-Shannon sampling theorem in 2D states that a band limited function with maximum frequencies $s_{max,x}$ and $s_{max,y}$ can be fully represented by a discrete time equivalent provided the sampling frequency satisfies the Nyquist sampling criterion:

$$\frac{1}{X} \geq 2s_{\mathrm{max},x} \text{ and } \frac{1}{Y} \geq 2s_{\mathrm{max},y}$$

Under these circumstances, there is no spectral overlap (or aliasing) the original spectrum and by uniqueness of the FT, the original signal can be reconstructed.

To reconstruct the original signal, we apply a reconstruction filter $H(u,v) = \mathrm{rect}(Xu)\mathrm{rect}(Yv)$.

$$\hat{G}(u,v) = G_s(u,v)H(u,v) = G_s(u,v)\mathrm{rect}(Xu)\mathrm{rect}(Yv)$$
$$= G(u,v), \text{if there is no aliasing}$$

In the (x,y) domain, this corresponds to "sinc" interpolation in 2D (sinc(x) = $\frac{\sin px}{px}$):

$$\hat{g}(x,y) = g_s(x,y) ** \tfrac{1}{XY}\mathrm{sinc}(\tfrac{x}{X})\mathrm{sinc}(\tfrac{y}{Y})$$

$$= \left[\sum_{n,m=-\infty}^{\infty} d(x - nX, y - mY)g(nX, mY)\right] ** \mathrm{sinc}(\tfrac{x}{X})\mathrm{sinc}(\tfrac{y}{Y})$$

$$= \sum_{n,m=-\infty}^{\infty} \mathrm{sinc}(\tfrac{x-nX}{X})\mathrm{sinc}(\tfrac{y-mY}{Y})g(nX, mY)$$

$$= \sum_{n,m=-\infty}^{\infty} \mathrm{sinc}(\tfrac{x-nX}{X})\mathrm{sinc}(\tfrac{y-mY}{Y})g_d(n,m)$$

The last line demonstrates how the original continuous signal can be retrieved from the discrete sampled version of *g*(*x*,*y*).

Relatives of the FT and Other FT Relationships

2D Discrete Space FT. Above, we saw that a sampled signal resulted in a periodic extensions in k-space. Accordingly, the signal is defined by a single period of that space, e.g. $G_s(u,v)\text{rect}(Xu)\text{rect}(Yv)$. The 2D discrete space Fourier transform is a normalized version of the 2D continuous domain Fourier transform of a sampled object.

$$G_d(w_X, w_Y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} g_d(n,m) e^{-i(w_X n + w_Y m)}$$

where is $g_d(n,m) = g(nX, mY)$. The inverse FT relationship is:

$$g_d(n,m) = \frac{1}{4p^2} \int_{-p}^{p} \int_{-p}^{p} G_d(w_X, w_Y) e^{i(w_X n + w_Y m)} dw_X \, dw_Y$$

we now recognize that

$$G_s(u,v) = XY\, G_d(w_X, w_Y)\, \Big|_{w_X = 2puX,\, w_Y = 2pvY}$$

One can consider $G_d(w_X, w_Y)$ either to be space limited to (-*p*, *p*) or periodic with period 2*p* in both directions.

2D FT of Period Signals. Suppose we define a periodic signal $\tilde{g}(x,y)$ that is period in *x* with period *X* and periodic in *y* with period *Y* as:

$$\tilde{g}(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} g_{XY}(x - nX, y - mY)$$

$$= g_{XY}(x,y) ** \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} d(x - nX, y - mY)$$

$$= g_{XY}(x,y) ** \frac{1}{XY} comb\left(\frac{x}{X}, \frac{y}{Y} \right)$$

where $g_{XY}(x,y)$ is zero outside of the domain $[0:X) \times [0:Y)$. The Fourier transform of this function is:

$$\tilde{G}(u,v) = G_{XY}(u,v) comb(Xu, Yv)$$

$$= G_{XY}(u,v) \left(\frac{1}{XY} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} d\left(u - \frac{k}{X}, v - \frac{l}{Y} \right) \right)$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{1}{XY} G_{XY}\left(\frac{k}{X}, \frac{l}{Y} \right) d\left(u - \frac{k}{X}, v - \frac{l}{Y} \right)$$

2D Fourier Series. The above expression makes the relationship between the 2D Fourier transform and the 2D Fourier series obvious. Taking the inverse 2D FT of the above we get:

$$\tilde{g}(x,y) = F^{-1}\{\tilde{G}(u,v)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{G}(u.v)e^{i2p(xu+vy)}dudv$$

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{1}{XY} G_{XY}\left(\frac{k}{X},\frac{l}{Y}\right)e^{i2p\left(\frac{kx}{X}+\frac{ly}{Y}\right)}$$

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_{k,l}e^{i2p\left(\frac{kx}{X}+\frac{ly}{Y}\right)}$$

which is the 2D Fourier series representation of a 2D periodic signal, where the Fourier series coefficients $c_{k,l}$ are:

$$c_{k,l} = \frac{1}{XY} G_{XY}\left(\frac{k}{X},\frac{l}{Y}\right)$$

$$= \frac{1}{XY} F\{g_{XY}(x,y)\}\Big|_{u=\frac{k}{X},u=\frac{l}{Y}}$$

$$= \frac{1}{XY}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_{XY}(x,y)e^{-i2p\left(\frac{xk}{X}+\frac{vl}{Y}\right)}dxdy$$

$$= \frac{1}{XY}\int_0^X\int_0^Y g(x,y)e^{-i2p\left(\frac{xk}{X}+\frac{vl}{Y}\right)}dxdy$$

Thus, a periodic signal can be represented by a discrete set of coefficients.

2D FT of Discrete and Periodic Signals. A discrete signal leads to period Fourier domain and the periodic signal leads to a discrete Fourier domain. So, the 2D FT of a discrete and periodic signal should be discrete and periodic. Let $\tilde{g}_d(n,m)$ be period with periods *N*, *M*. To find its 2D FT we take $G_d(w_X, w_Y)$ and evaluate it at $w_X = \frac{2pk}{N}, w_Y = \frac{2pl}{M}$ to yield the 2D Discrete FT:

$$\tilde{G}_d(k,l) = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \tilde{g}_d(n,m)e^{-i2p\left(\frac{nk}{N}+\frac{ml}{M}\right)}$$

for $k,l \in Z^2$. $\tilde{G}_d(k,l)$ is also periodic with periods *N*, *M*. The inverse FT is:

$$\tilde{g}_d(n,m) = \frac{1}{NM}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1} \tilde{G}_d(k,l)e^{i2p\left(\frac{nk}{N}+\frac{ml}{M}\right)}$$

2D Discrete FT of a finite series. In the above case, since both $\tilde{g}_d(n,m)$ and $\tilde{G}_d(k,l)$ are periodic with periods *N*, *M*. Each can be complete described by a finite 2D series (of sizes *N*x*M*). Thus, we can define the 2D DFT of a finite discrete series, $g_d(n,m)$, using an assumption of a periodic extension – that is, in determining its spectrum we assume that we are just looking at a single period of the function. Thus, the 2D DFT is:

$$G_d(k,l) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g_d(n,m) e^{-i2p\left(\frac{nk}{N}+\frac{ml}{M}\right)}$$

for $n,m \in (0:N-1)\times(0:M-1)$ and $k,l \in (0:N-1)\times(0:M-1)$. The inverse FT is:

$$g_d(n,m) = \frac{1}{NM} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} G_d(k,l) e^{i2p\left(\frac{nk}{N}+\frac{ml}{M}\right)}$$

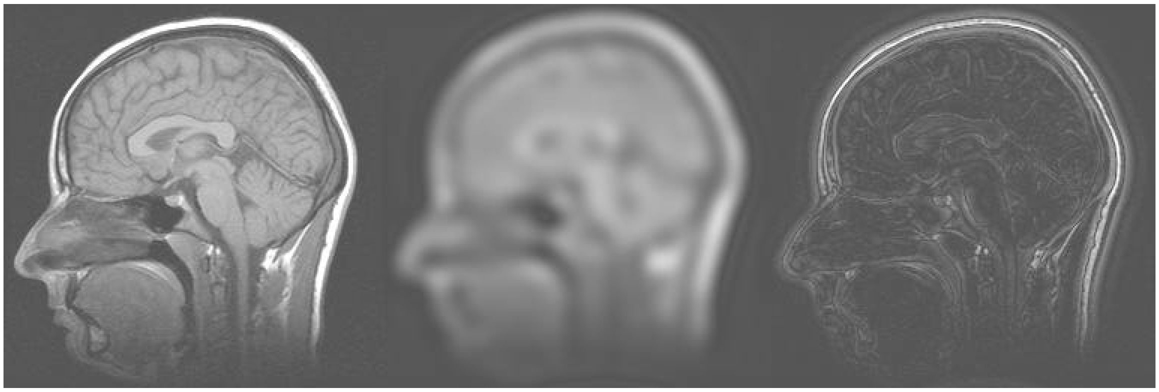
These functions are implemented by Matlab’s fft2 and ifft2 function.

These related to the continuous FT by the following relationshis:

$$G_d(k,l) = \frac{1}{XY} G_s\left(\frac{k}{NX}, \frac{l}{MY}\right)$$

(and recall the $G_s = G$ for a bandlimited *g*).

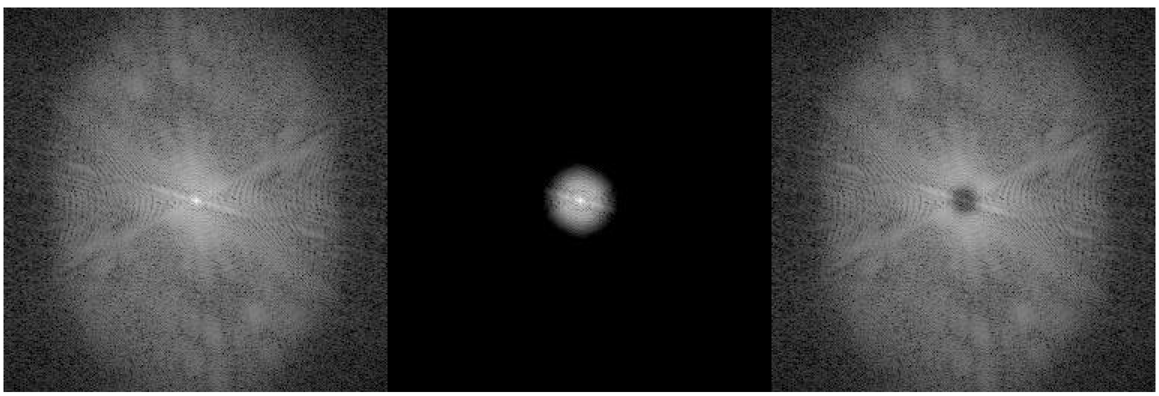
Examples of Fourier Transforms:



2D data in image domain

Low spatial freq data (image domain)

High spatial freq data (image domain)



2D data in Fourier domain

Low spatial freq data (Fourier domain)

High spatial freq data (Fourier domain)

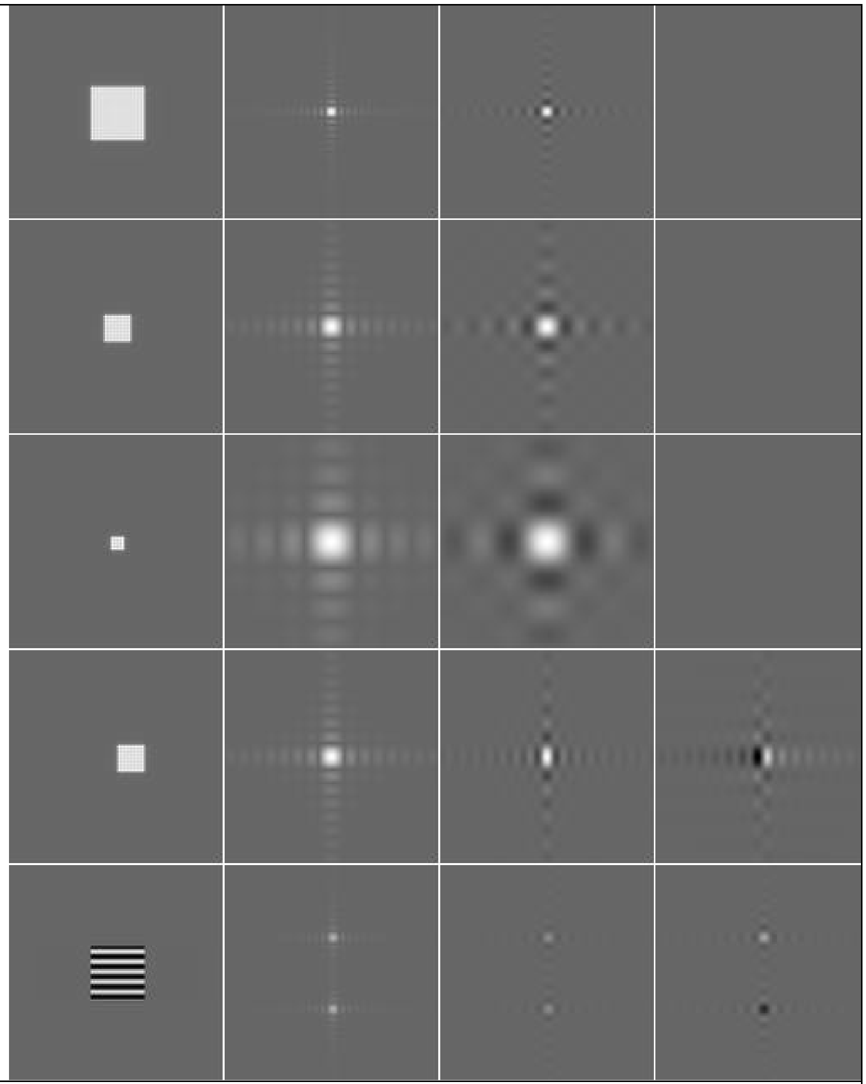
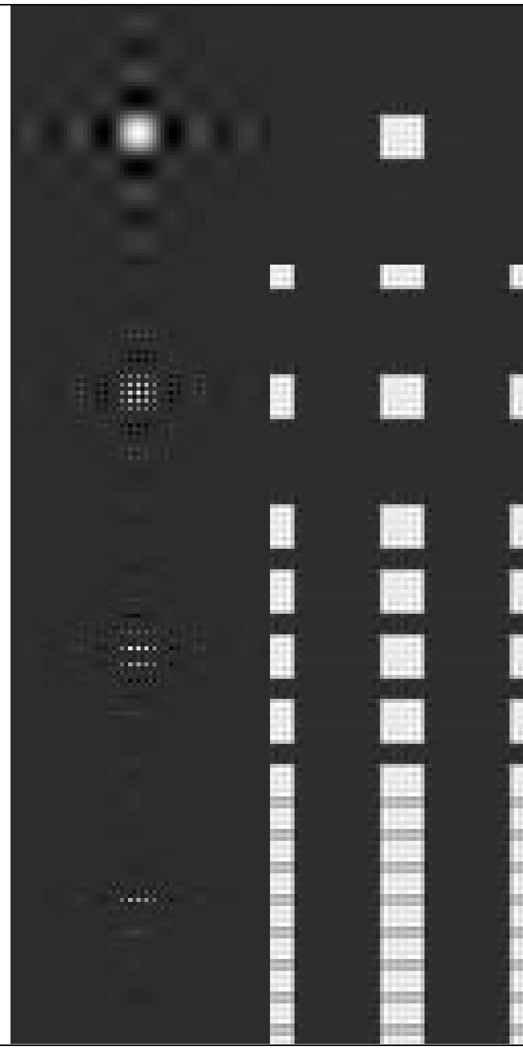


	Image	Abs(Fourier)	Real(Fourier)	Imag(Fourier)
g(x,y) = rect(x)rec(y) G(u,v) = sinc(u)sinc(v)				
scaling (magnification) property				
scaling (magnification) property				
shifting property				
modulation				

g(x,y) = sinc(x)sinc(y) G(u,v) = rect(u)rect(v) sampling pattern with $\Delta x = \Delta y$ In the Fourier transform we have the replication pattern with spacing $1/\Delta x = 1/\Delta y$ sampling pattern with $\Delta x < \Delta y$ In the Fourier transform we have the replication pattern with spacing $1/\Delta x > 1/\Delta y$ sampling pattern with $\Delta x << \Delta y$ This has aliasing in the y (v) direction	
	Image Data Fourier Data