# Eng. 100: Music Signal Processing
# DSP Lecture 4
# Lab 3: Signal Spectra

## Curiosity

- https://www.youtube.com/watch?v=rtR63-ecUNo (Oscilloscope art)
- http://musicmachinery.com/2010/05/21/the-swinger (musical style modulation)

## Announcements:

- Lab2 due Friday
- HW1 due Friday
- HW2 due next Friday
- Finish Project1 in Lab this week (at latest)
- Project 1 presentations in discussion sections next week.
- Start reading Lab 3 for next week — it is longer!

# Part 0: Lab 2 summary

# Lab 2 summary

What you learned (hopefully)

- Frequencies of musical tones in "The Victors":
  392, 440, 494, 523, 587, 659 Hz (rounded to nearest integer)
- Semi-log plot revealed missing frequencies:
  415, 466, 554, 622, 698, 740 Hz (mostly accidentals).
- 12 frequencies with common ratio about 1.06

The exact ratio is $2^{1/12} \approx 1.059463094359295$
Twelve such "half steps" in an "octave" leads to a frequency ratio of 2, *i.e.*, the frequency doubles each octave. More later...

Q0.1 What model relates piano key number to frequency?
A: exponential B: power C: None of these ??

# Outline

- What: The spectrum of a signal (first class)
  - Part 1. Why we need spectra
  - Part 2. Periodic signals $x(t) = c_0 + \sum_{k=1}^{K} c_k \cos\left(2\pi \frac{k}{T} t - \theta_k\right)$
  - Part 3. Band-limited signals, $K = \lfloor BT \rfloor$
- How: Methods for computing spectra (second class)
  - Part 4. Sampling rate $S > 2B$
  - Part 5. By hand by solving systems of equations
  - Part 6. Using general Fourier series solution
  - Part 7. Using fast Fourier transform (FFT), *e.g.*, in `Julia`
- Why: Using a signal's spectrum (third class)
  - to determine note frequencies: $f = \frac{k}{N} S$
  - to remove unwanted noise
  - to visualize frequency content (spectrogram)
  - Lab 3

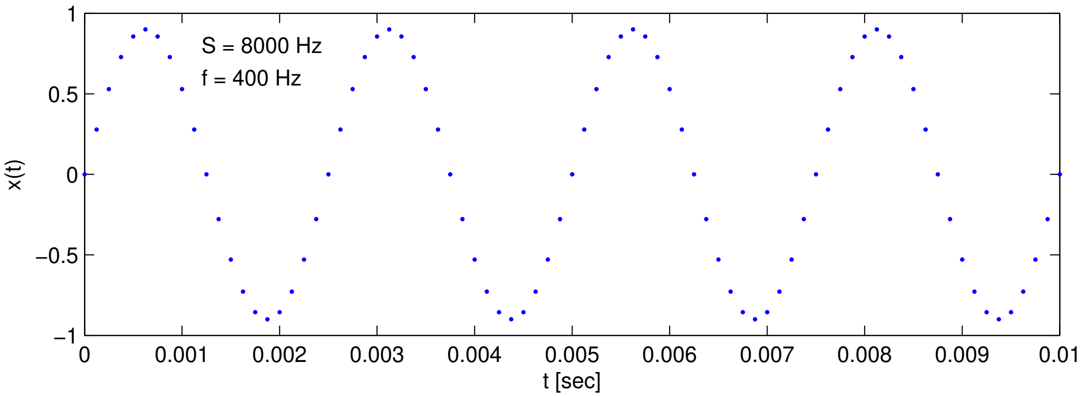
# Learning objectives

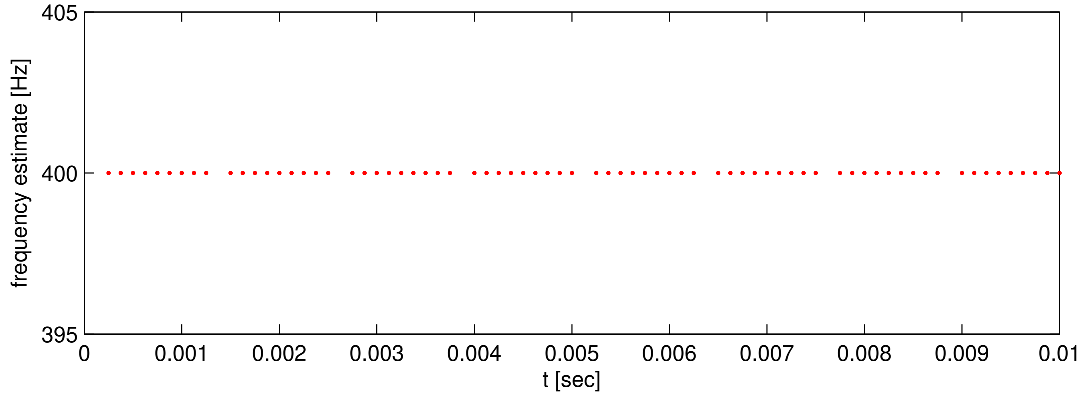
- Understand concepts of spectra and Fourier series for periodic signals
- Convert between spectra plot and equation form
- Understand band limited signal spectra
- Determine band limit given spectra plot or Fourier series equation
- Understand Nyquist-Shannon sampling requirement $S > 2B$

# Part 1. Why we need spectra

# Project 1 Transcriber for ideal sinusoid (works!)

Sinusoidal signal:




play

Method:

$$f = \frac{S}{2\pi} \arccos\left(\frac{x[n+1]+x[n-1]}{2x[n]}\right).$$

# Project 1: Transcriber limitations

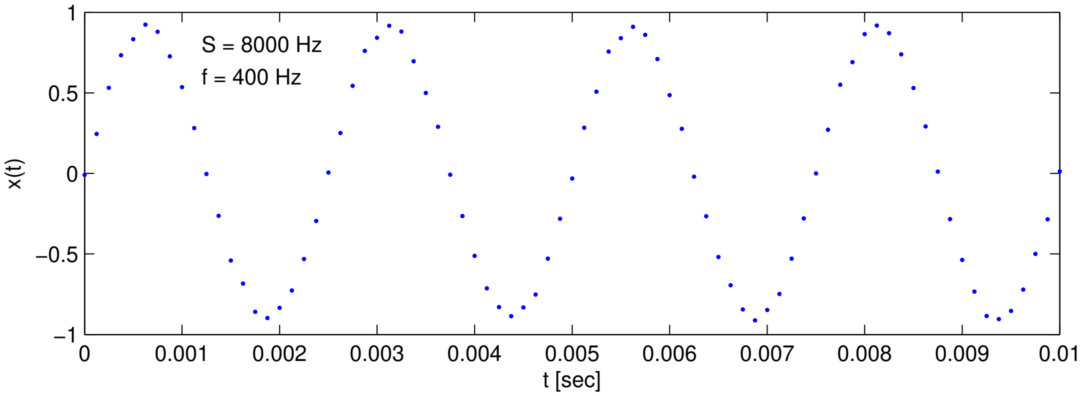
What are limitations of transcribers implemented in Project 1?

??

We need more sophisticated method(s)
for finding the (fundamental) frequency of a music signal.

# Project 1 Transcriber for noisy sinusoid (stinks!)

Sinusoidal signal with noise (*e.g.*, in any audio recording):

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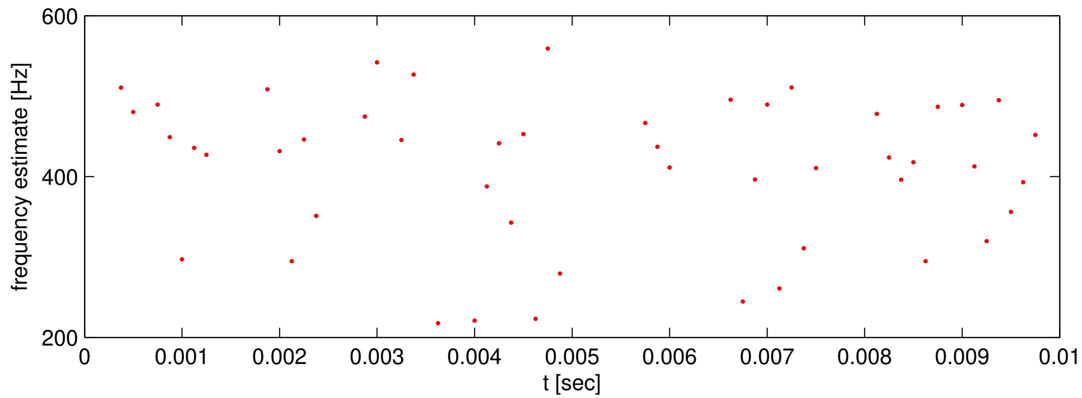
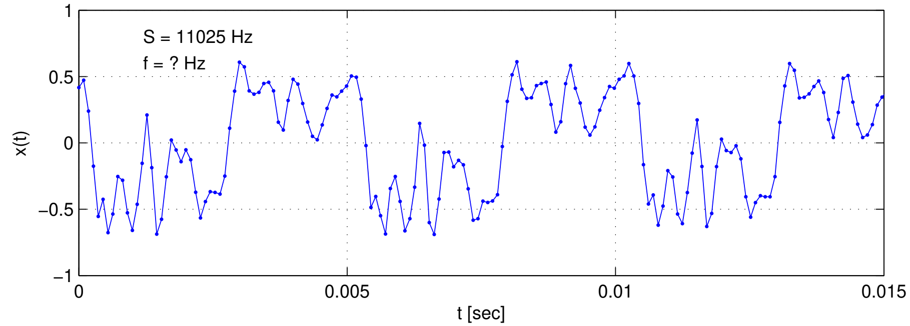


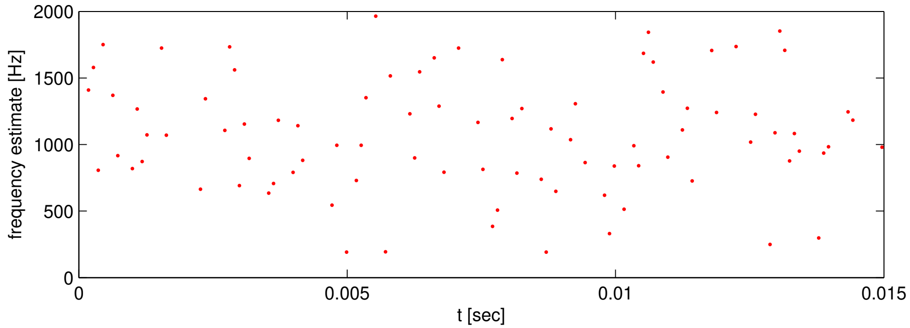

Method:

$$f = \frac{S}{2\pi} \arccos\left(\frac{x[n+1] + x[n-1]}{2x[n]}\right).$$

# Project 1 Transcriber for clarinet (stinks!)

Clarinet signal (roughly G below middle C): play



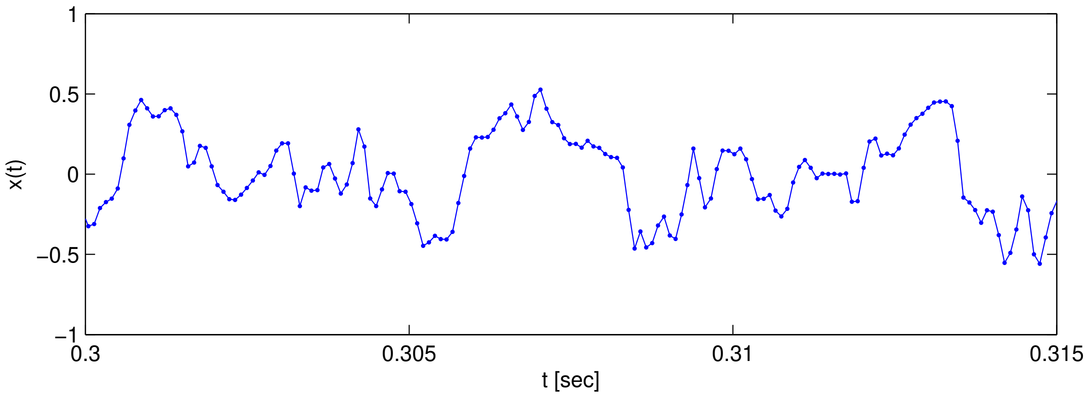




Method:

$$f = \frac{S}{2\pi} \arccos\left(\frac{x[n+1]+x[n-1]}{2x[n]}\right).$$

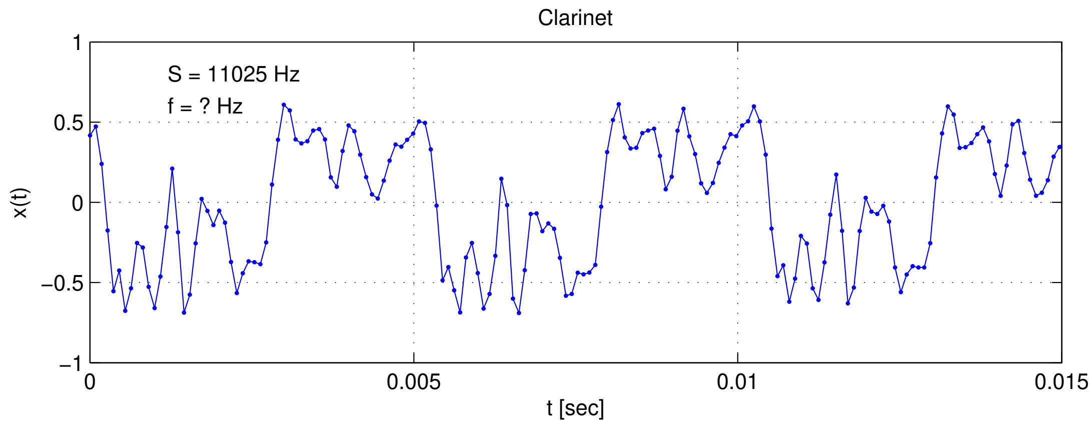
# Project 1 Transcriber for polyphony (stinks!)

Clarinet and guitar duet (even more challenging case): play




- Methods that use the "time domain" are very unlikely to work when two instruments play simultaneously.
- We need to use the "frequency domain"
  aka the spectrum of a signal. (Human ears work this way!)
- The concept of spectra is used widely in engineering.

# Engineering strategy: Divide and conquer




This signal $x(t)$ looks complicated.
(Bamboo reed vibrations are approximately a square wave.)

Engineering strategy:
- Make complicated things by combining simpler things.
- Use tools from mathematics (and physics) as needed.

Mathematics provides us with a perfect tool in this case:
Fourier series.

# Part 2. Spectra of periodic signals

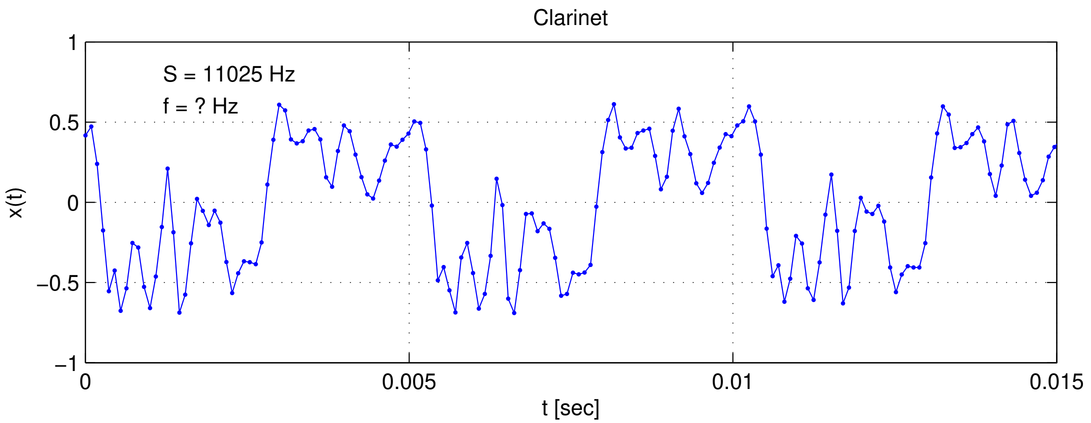
# Joseph Fourier laid the foundation for DSP

Joseph Fourier, 1768-1830

He died after falling down the stairs at his home.

- Fourier series theory developed circa 1807
  (Modern compared to our trigonometry method.)
- Motivating application: heat propagation in metal plates.

# Periodic signals are the key to music DSP




Key property of musical signals (over short time intervals):
periodicity.

A periodic signal (aka repeating signal) with period $=T$ satisfies

$$x(t) = x(t+T) = x(t+2T) = \cdots \text{ for all } t.$$

Example: $x(t) = \cos(2\pi 9t)$ is periodic with period $T = 1/9$ sec.
It is also periodic with period $T = 1/3$ sec.
The smallest period ($T = 1/9$ sec here) is the fundamental period.

# Exercise

Q0.2 What is the (approximate) period of the clarinet signal shown on the previous slide (in sec)?

A: 0.001 B: 0.005 C: 0.010 D: 0.015 E: 0.5 ??

# Fourier series of periodic signals

Amazing fact #1 (discovered by Joseph Fourier 200+ years ago):

Any real-world periodic signal with period $= T$ can be expanded (*i.e.*, expressed mathematically as a sum) as follows:

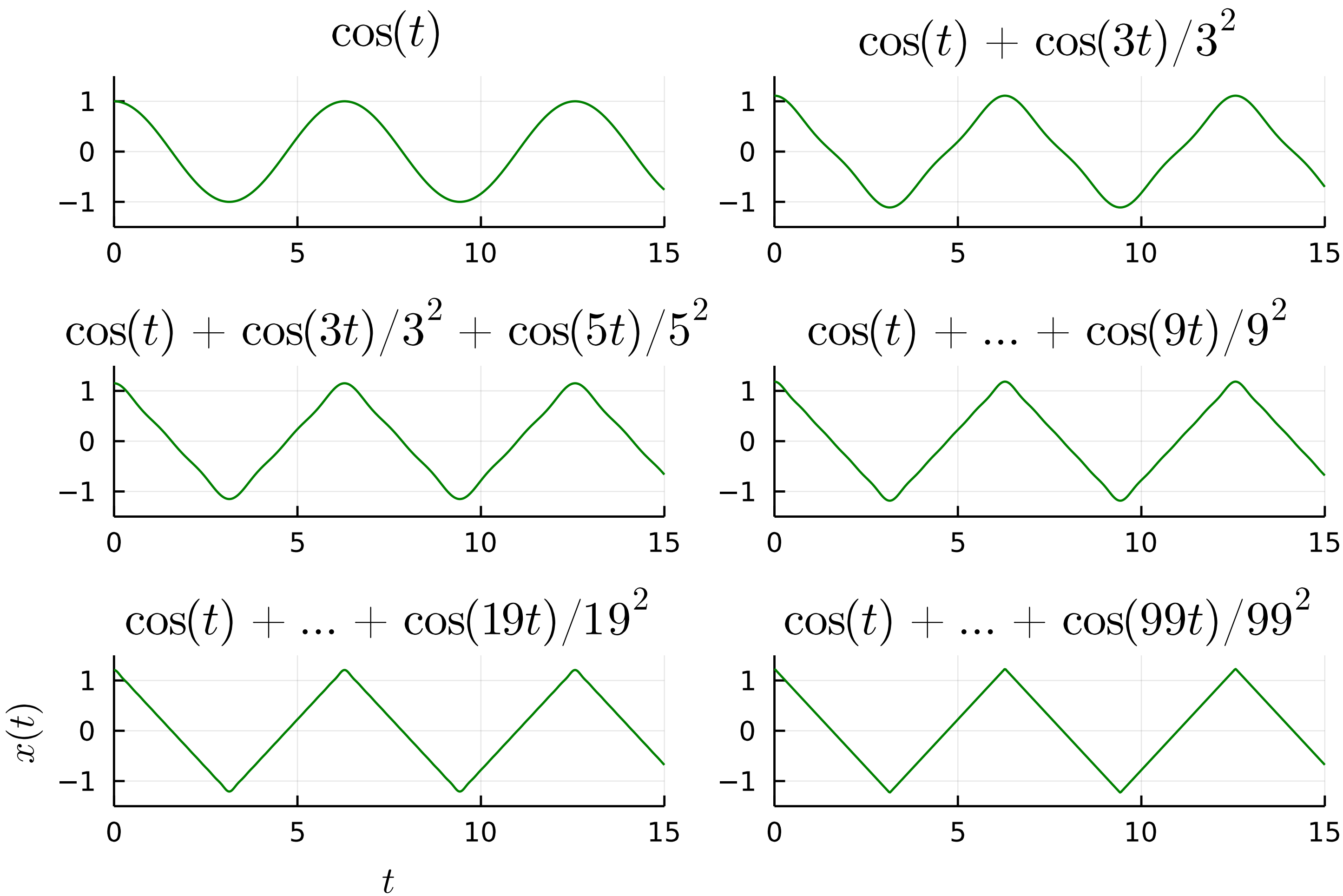
$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(2\pi \frac{k}{T} t - \theta_k\right)$$

$$= \underbrace{c_0}_{\substack{\text{DC term} \\ \text{DC value} \\ \text{DC constant}}} + \underbrace{c_1 \cos\left(2\pi \frac{1}{T} t - \theta_1\right)}_{\substack{\text{fundamental} \\ \text{period} = T \\ \text{frequency} = 1/T}} + \underbrace{c_2 \cos\left(2\pi \frac{2}{T} t - \theta_2\right)}_{\substack{\text{(first) harmonic} \\ \text{period} = T/2 \\ \text{frequency} = 2/T}} + \cdots$$

- $\{c_k\}$ called amplitudes
- $\{k/T\}$ called frequencies
- $\{\theta_k\}$ called phases

We can write even a "complicated" clarinet or guitar signal using such a "simple" sum of sinusoidal signals.

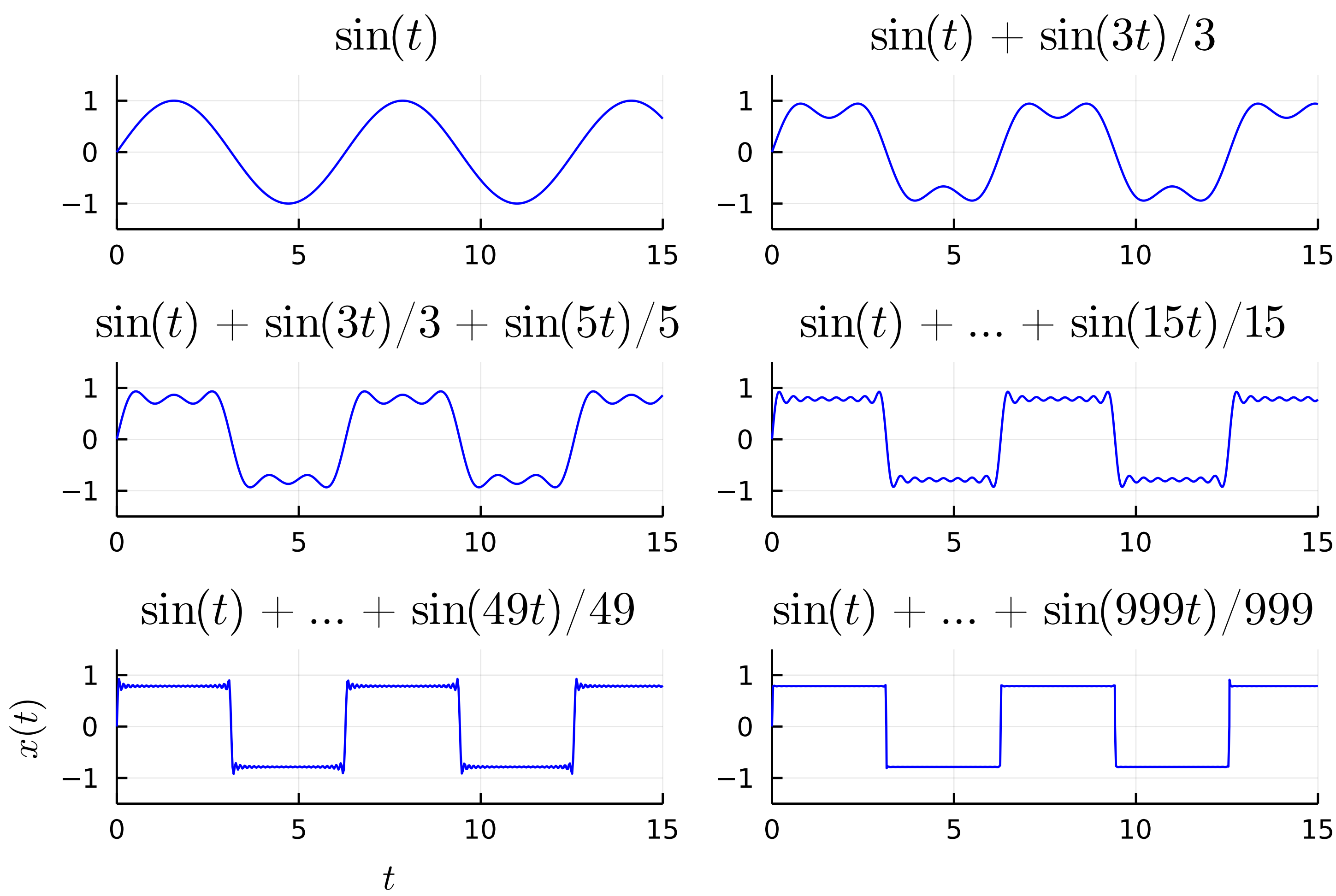
# Example: Triangle wave




More terms in sum $\Longrightarrow$ closer approximation to triangle wave.

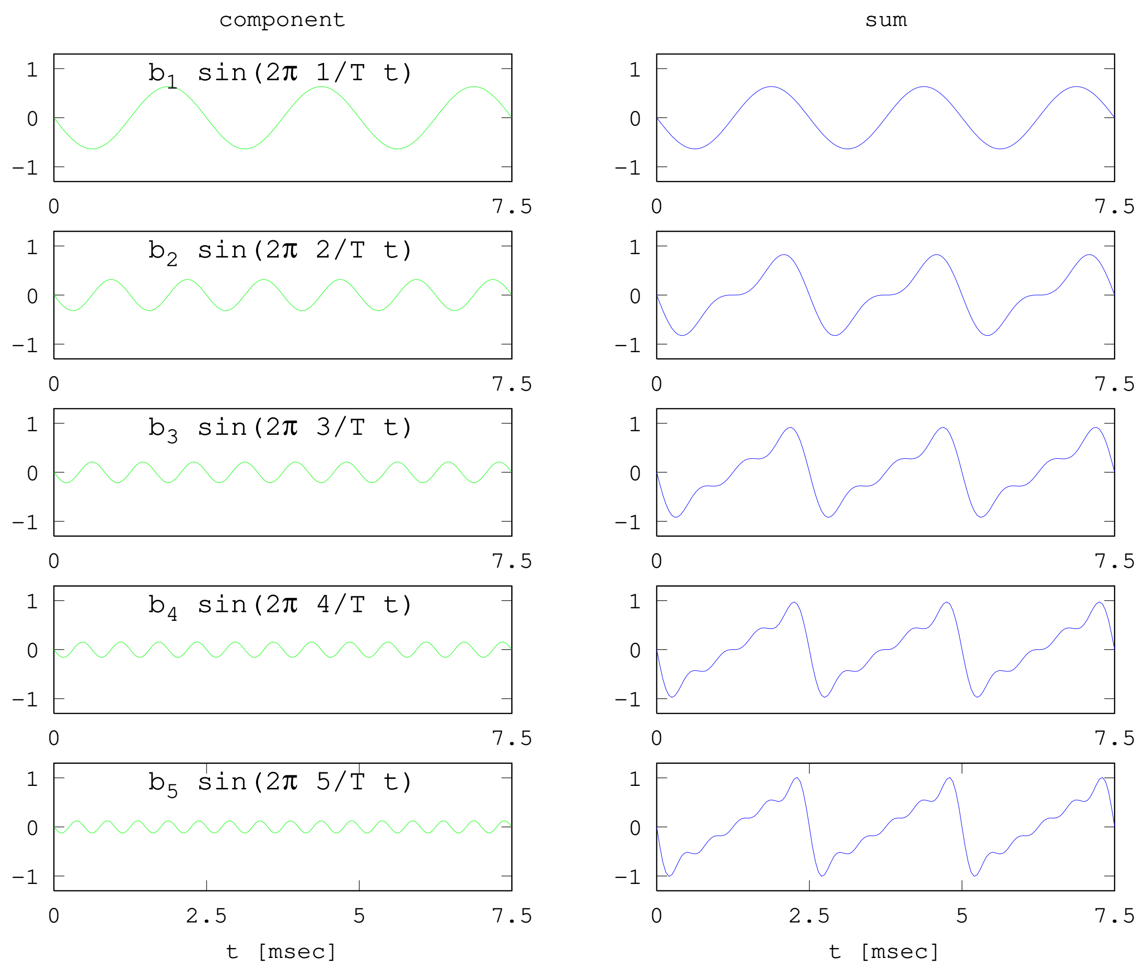
(Nice audio demo on wikipedia.)

# Example: Square wave




Sums of sinusoids can make "interesting" signals, like a square wave.
What is $T$ in this example? ??

# Example: Sawtooth wave




play

play

play

play

play

As we build this sawtooth wave, does the sound pitch sound change? ??
Fundamental frequency (in Hz)? ??

# The spectrum of a periodic signal

Every periodic signal can be written in the same form!

$$x(t) = c_0 + c_1 \cos\left(2\pi \frac{1}{T} t - \theta_1\right) + c_2 \cos\left(2\pi \frac{2}{T} t - \theta_2\right) + \cdots$$

So how do electric guitar and clarinet signals differ?

??

# The spectrum of a periodic signal

Every periodic signal can be written in the same form!

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So how do electric guitar and clarinet signals differ?

??

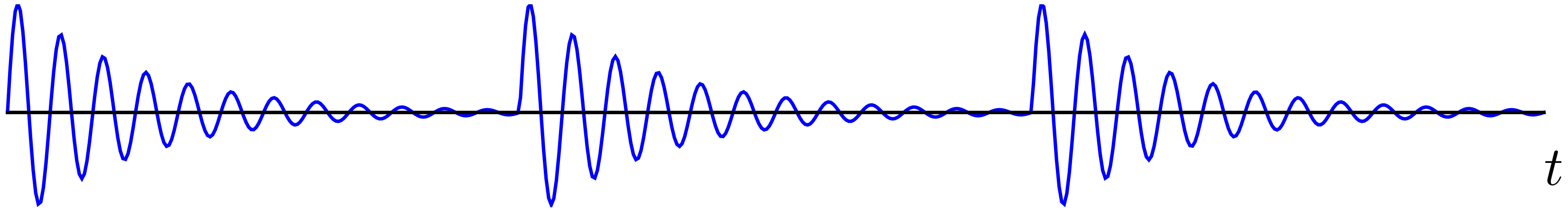
Definition. The spectrum of a signal $x(t)$ is just a stem plot of the amplitudes $\{c_k\}$ versus the frequencies $\{k/T\}$ in Hertz.

- The phases $\theta_k$ are unimportant for monophonic music.
- The DC term $c_0$ cannot be produced or heard either.
- Coefficients $\{c_k\}$ define timbre (TAM-ber) of sound aka "tone color"

# Spectra of periodic signals

In ENGR 100, we define spectra only of periodic signals. Why?

- Musical instruments produce approximately periodic signals.
- Definition and computation are much easier.
- Real-world non-periodic signals can be viewed as part of a periodic signal with a *very* long period.



# Example: AM Radio Signal

Two Michigan AM Radio stations are:
- WSDS, 1480 kHz, 3800W, Salem Township
- WABJ, 1490 kHz, 1000W, Adrian, MI

If WSDS broadcasts a 3000 Hz sinusoidal test tone
and WABJ broadcasts a 2000 Hz sinusoidal test tone, then
(you can learn in EECS 216) that an antenna in Saline
that can pick up both stations would receive this signal:

$$\begin{aligned} x(t) &= 40\cos(2\pi 1480000t) + 20\cos(2\pi 1483000t) + 20\cos(2\pi 1477000t) \\ &\quad + 10\cos(2\pi 1490000t) + 5\cos(2\pi 1492000t) + 5\cos(2\pi 1488000t)\,. \end{aligned}$$

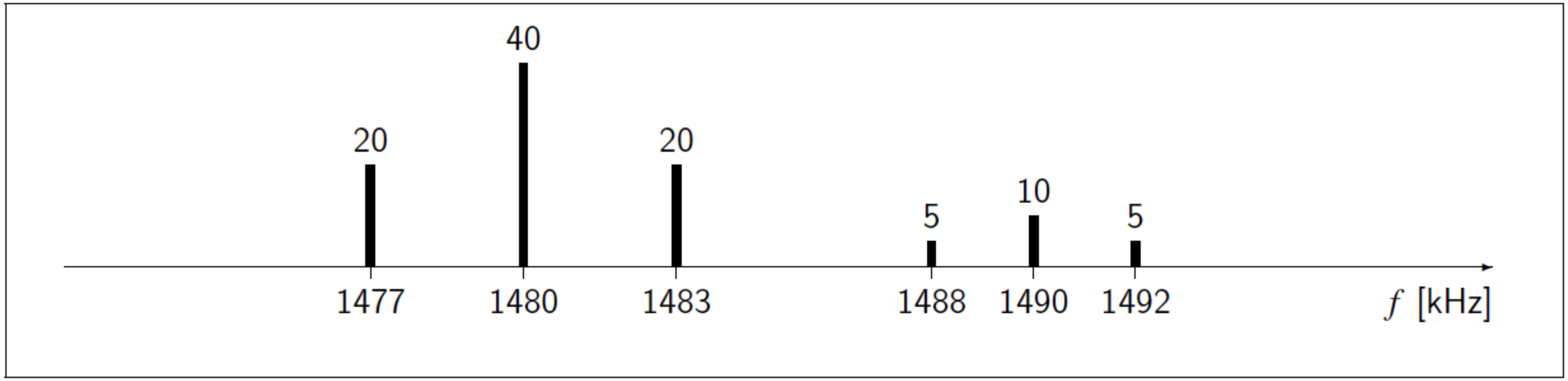
What is the spectrum of this signal?

# Example: AM Radio Signal Spectrum

AM radio signal (expressed as mathematical formula):

$$\begin{aligned} x(t) \;=\; & 40\cos(2\pi 1480000t) + 20\cos(2\pi 1483000t) + 20\cos(2\pi 1477000t) \\ & + 10\cos(2\pi 1490000t) + 5\cos(2\pi 1492000t) + 5\cos(2\pi 1488000t). \end{aligned}$$

Spectrum of this signal $x(t)$:




So the spectrum of a signal is a *graphical* representation.

Graphical representations are often beneficial.

Converting between these representations is a learning objective.

# Exercise

Find a *formula* for the signal that has the following spectrum.

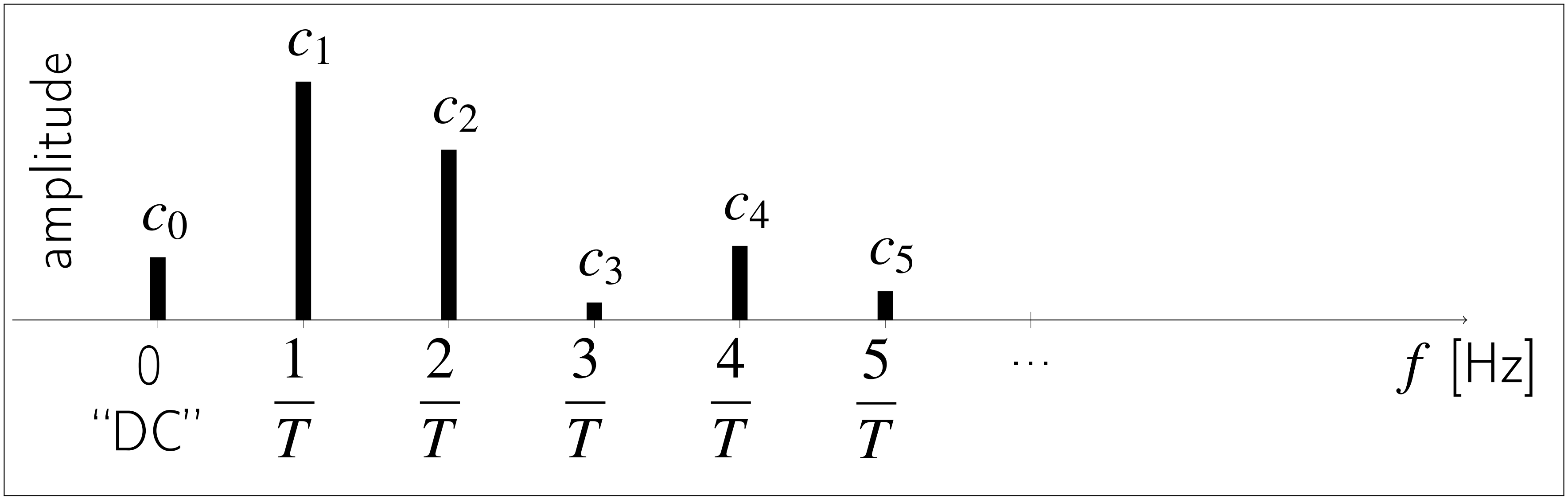

Could an audio signal (music) have this spectrum? play

??

(Working forwards and backwards...)

# Spectrum of a general periodic signal

A periodic signal with period $T$ has a spectrum that looks like:

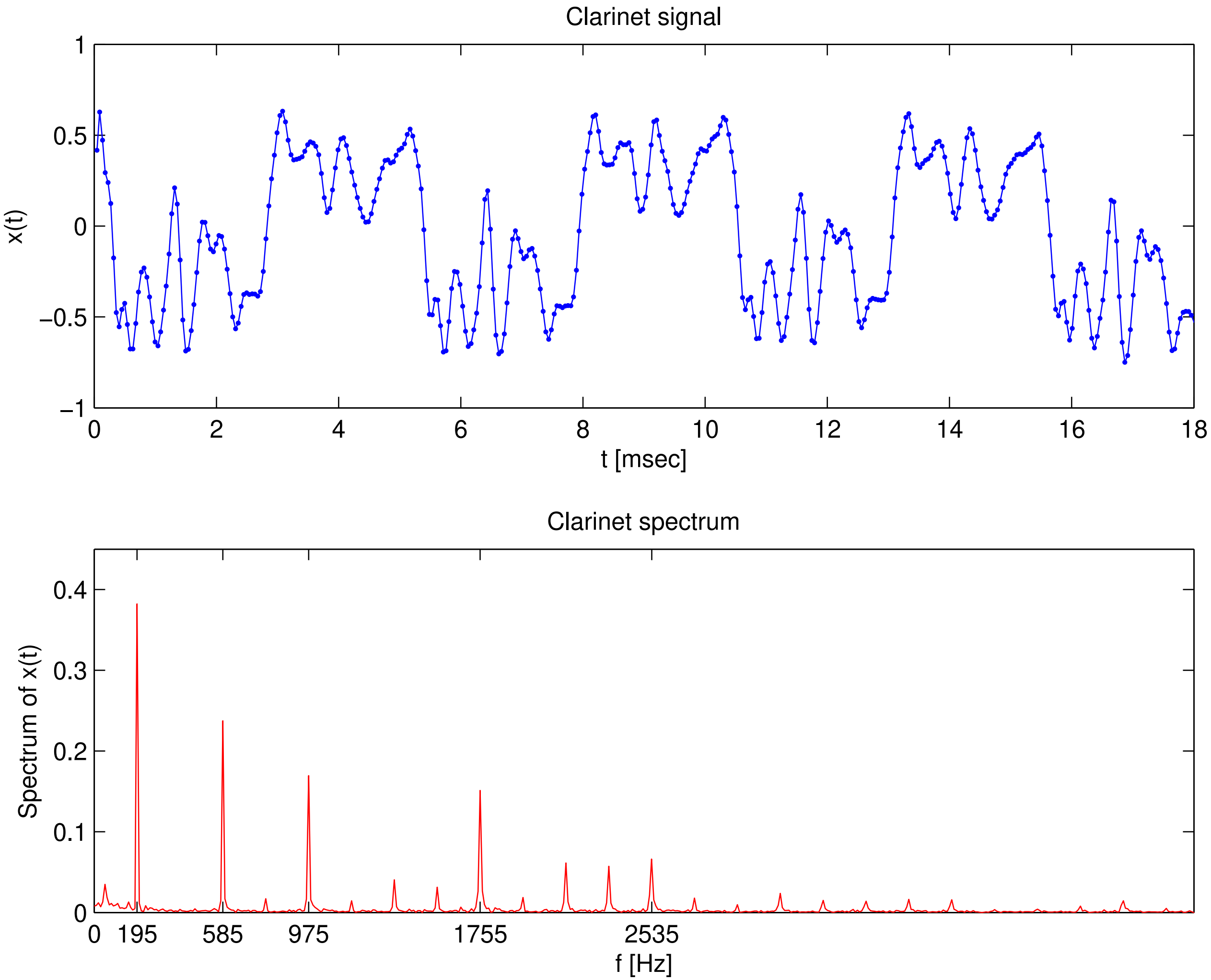



- The frequency components are $0,\ 1/T,\ 2/T,\ \ldots$
- The height of each line in the spectrum is an amplitude $c_k$

Ignoring phase: $x(t) = c_0 + c_1 \cos\left(2\pi \frac{1}{T} t\right) + c_2 \cos\left(2\pi \frac{2}{T} t\right) + \cdots$

What units are along the horizontal axis for a spectrum? ??

# Example: Clarinet spectrum

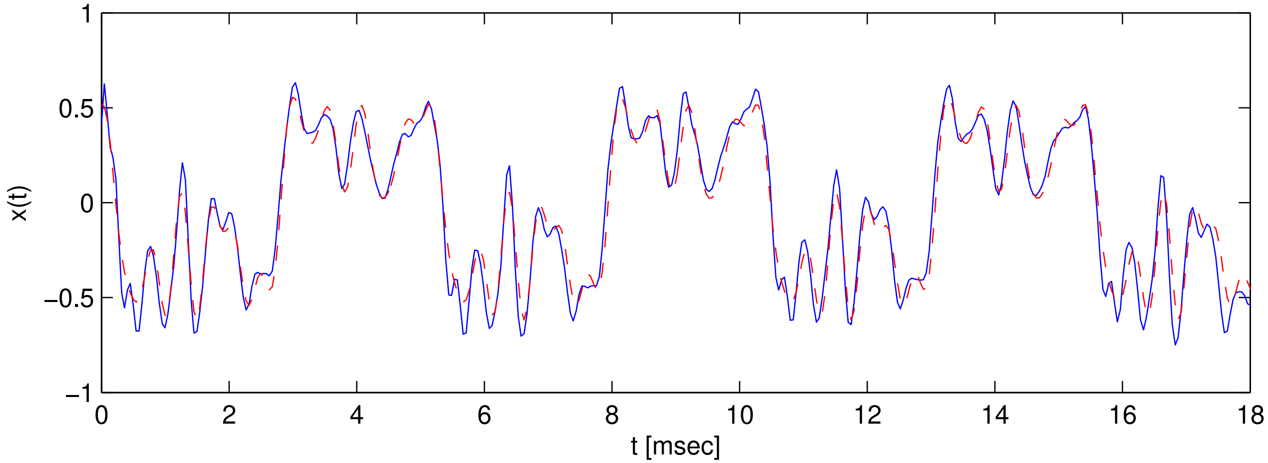


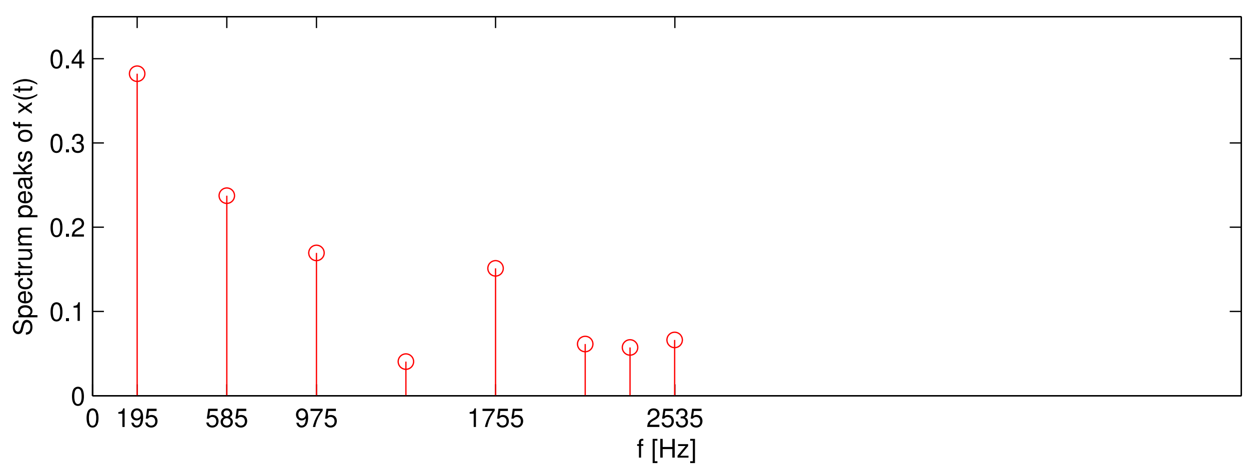

play

First significant peak: fundamental frequency $= 1/T \approx 195$ Hz

(Perfectly) periodic signals have (perfect) line spectra.

# Example: Clarinet synthesized




play

Synthesized using 8 largest peaks in spectrum.

Sounds more interesting than Project 1 synthesizer? Why?

# Example: Clarinet Fourier series

Expressing a complicated signal in terms of simple signals:

$$
\begin{aligned}
x(t) \approx\ & 0.382\cos(2\pi\ 195.0t + 1.35) + 0.237\cos(2\pi\ 584.9t + 0.48) + \\
& 0.169\cos(2\pi\ 974.8t + 0.30) + 0.151\cos(2\pi 1754.6t - 1.35) + \\
& 0.066\cos(2\pi 2534.5t - 1.41) + 0.061\cos(2\pi 2144.6t + 2.40) + \\
& 0.057\cos(2\pi 2339.5t + 0.40) + 0.041\cos(2\pi 1364.7t + 1.32)
\end{aligned}
$$

- Guitar signal would have different amplitudes and phases, even if playing the same note.
- MP3 audio coding exploits the "line" nature of music spectra.

- How did I make the spectrum plot on previous slide?
- How did I get all the numbers above?

# Part 3: Band-limited signals: towards computing a signal's spectrum

# Fourier Series: Trigonometric form

Fourier Series: Sinusoidal form:

$$x(t) = c_0 + c_1 \cos\left(2\pi \frac{1}{T} t - \theta_1\right) + c_2 \cos\left(2\pi \frac{2}{T} t - \theta_2\right) + \cdots$$

Fourier Series: Trigonometric form:

$$x(t) = a_0 + a_1 \cos\left(2\pi \frac{1}{T} t\right) + a_2 \cos\left(2\pi \frac{2}{T} t\right) + \cdots$$
$$+ b_1 \sin\left(2\pi \frac{1}{T} t\right) + b_2 \sin\left(2\pi \frac{2}{T} t\right) + \cdots$$

Coefficients in these two forms are related by:

$$a_0 = c_0$$
$$a_k = c_k \cos \theta_k$$
$$b_k = c_k \sin \theta_k$$
$$c_k = \sqrt{a_k^2 + b_k^2} \quad \text{(need this to plot spectra)}$$
$$\tan \theta_k = b_k / a_k$$

because (Lab 1): $\cos(t - \theta) = \cos(\theta)\cos(t) + \sin(\theta)\sin(t)$

We will focus on finding the $a_k$ and $b_k$ values for music signals.

# About those dots: $\cdots$

Example.
If $x(t)$ has period $= T = 0.01$ seconds then $x(t)$ has expansion:

$$
\begin{aligned}
x(t) = a_0 &+ a_1 \cos(2\pi 100 t) + a_2 \cos(2\pi 200 t) + a_3 \cos(2\pi 300 t) + \cdots \\
&+ b_1 \sin(2\pi 100 t) + b_2 \sin(2\pi 200 t) + b_3 \sin(2\pi 300 t) + \cdots
\end{aligned}
$$

| | fundamental = 1st harmonic | 2nd harmonic | 3rd harmonic |
|---|---|---|---|
| (DC) | (100 Hz) | (200 Hz) | (300 Hz) |

- Mathematical perspective: What does "$\cdots$" mean?

  ??

- Engineering perspective:
  Practical signals are, or can be made to be, band limited.
  - Physical limits
  - Perception limits
  - Anti-alias filters in A/D converters

# Band-limited signals have a maximum frequency

Definition. A signal is band limited to $B$ Hz if it has no frequency components *higher* than $B$ Hz.

Example.
If $x(t)$ has period $= T = 0.01$ seconds *and* is band-limited to 800 Hz then $x(t)$ has (finite!) Fourier series expansion:

$$\begin{aligned} x(t) = \; & a_0 + a_1 \cos(2\pi 100 t) + a_2 \cos(2\pi 200 t) + \cdots + a_8 \cos(2\pi 800 t) \\ & \quad + b_1 \sin(2\pi 100 t) + b_2 \sin(2\pi 200 t) + \cdots + b_8 \sin(2\pi 800 t) \end{aligned}$$

| | | | |
|---|---|---|---|
| | fundamental = 1st harmonic | 2nd harmonic | highest harmonic |
| (DC) | (100 Hz) | (200 Hz) | (800 Hz) |

Finite sum: $x(t) = a_0 + \sum_{k=1}^{8} (a_k \cos(2\pi 100 k t) + b_k \sin(2\pi 100 k t))$

This periodic, band-limited signal is "characterized completely" by the frequency (100 Hz) and just 17 other numbers:

$$\{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\}.$$

How do we find those values, called coefficients?

# Exercise

A periodic signal $x(t)$ has period $T = 0.02$ seconds and is known to be band-limited to 200 Hz.

How many (possibly nonzero) Fourier series coefficients does it have (in trigonometric form), including the DC coefficient?

??

Recall:

$$x(t) = a_0 + a_1 \cos\left(2\pi \frac{1}{T} t\right) + a_2 \cos\left(2\pi \frac{2}{T} t\right) + \cdots$$
$$+ b_1 \sin\left(2\pi \frac{1}{T} t\right) + b_2 \sin\left(2\pi \frac{2}{T} t\right) + \cdots$$

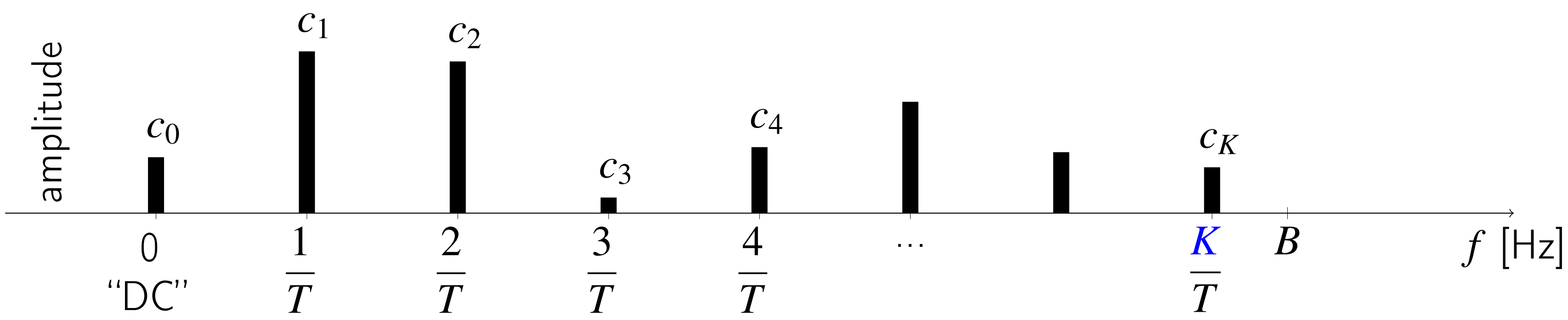
# How many coefficients?

In general, if a signal has period $T$ and is band-limited to $B$ Hz,
*how many* Fourier series coefficients $\{a_0, a_1, b_1, a_2, b_2, \ldots\}$ are needed?

??

# Spectrum of a band-limited periodic signal

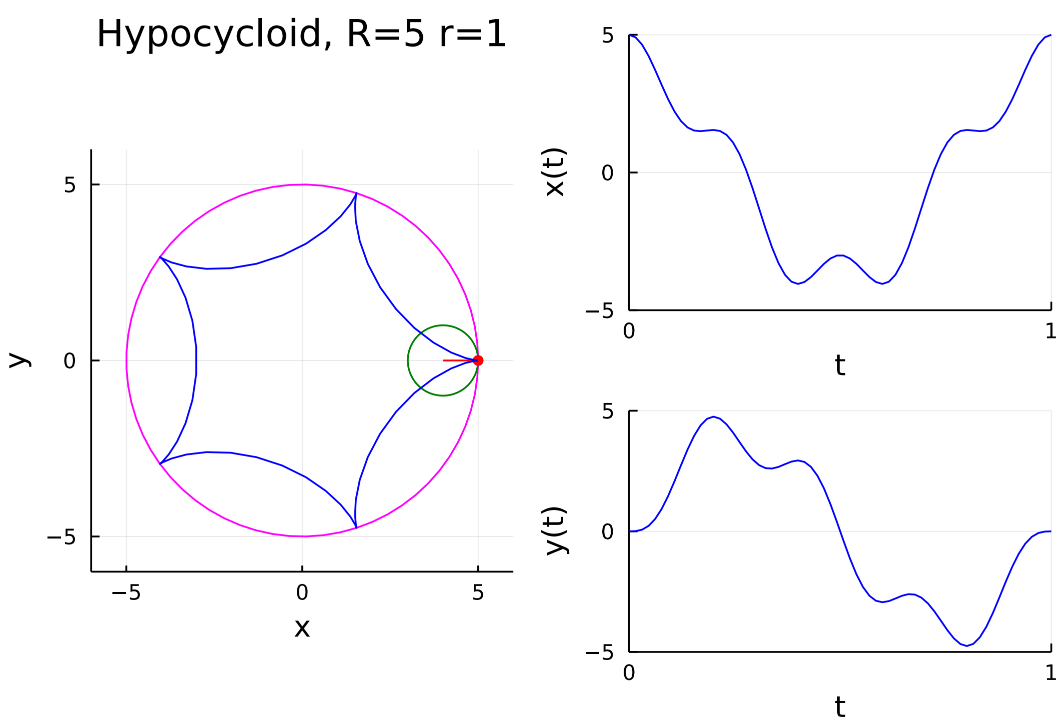
A periodic signal with period $T$, that is band-limited to $B$ Hz, has a spectrum that looks like:




- No frequency components above $B$ Hz
- No lines in spectrum past $B$ Hz
- $K = BT$ if it is an integer (# of sinusoids) (units of $BT$?)
- $K = \lfloor BT \rfloor$ more generally, $\lfloor x \rfloor$ = largest integer that is $\leq x$
  $\lfloor x \rfloor$ called `floor` function (below)
- Example: $T = 0.01$ s and $B = 360$ Hz $\Longrightarrow K = \lfloor 3.6 \rfloor = 3$

# Spectrum review: A non-music example

The following figure / demo illustrates a hypocycloid that is one special case of a spirograph.




demo: `fig_spirograph1.jl`

Formula:
$x(t) = (R-r)\cos(2\pi t) + r\cos\left(2\pi \frac{R-r}{r} t\right)$
$y(t) = (R-r)\sin(2\pi t) - r\sin\left(2\pi \frac{R-r}{r} t\right)$
Here, $R = 5$ (outer circle) and $r = 1$ (inner circle).

Recall: a signal is any time-varying quantity...

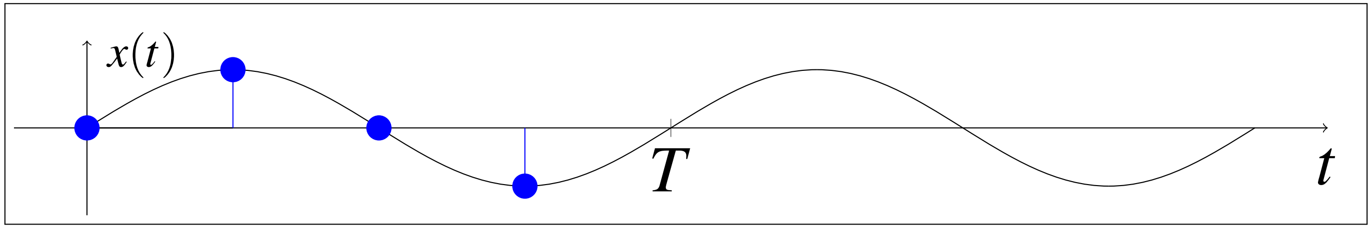
Exercise.
Sketch spectrum of $x(t)$ (or $y(t)$). ??
Is $x(t)$ band-limited? ??
What is the band-limit $B$? ??

# Computing a signal's spectrum

# Signal sampling

Idea. To determine $2BT+1$ Fourier coefficients of a signal with period $T$ that is band-limited to $B$ Hz, we try taking at least $N \geq 2BT+1$ samples of the signal over one period, *e.g.*, the interval $[0,T)$.

In other words, take $N > 2BT$ samples: $x[0], x[1], \ldots, x[N-1]$.




What will be the sampling interval? $\Delta = \dfrac{T}{N} < \dfrac{T}{2BT} = \dfrac{1}{2B}$.

The sampling rate is: $S = \dfrac{\#\text{ of samples}}{\text{time interval}} = \dfrac{N}{T} > \dfrac{2BT}{T} = 2B.$

Sample *faster* than twice the maximum frequency: $\boxed{S > 2B.}$

# 2B or not 2B

The formula $\boxed{S > 2B}$ is one of the most important in DSP!
It is the foundation for all digital audio and video and more.
CD players use a sampling rate of 44.1 kHz. Why? ??

Where did $T$ go?
The period $T$ need not affect the sampling rate!
We can choose $T$ arbitrarily large.

Amazing fact #2 (discovered by Claude Shannon 60+ years ago):

Nyquist-Shannon sampling theorem:

If we sample a band-limited signal $x(t)$ at a rate $S > 2B$, then
we can recover the signal from its samples $x[n] = x(n/S)$. (EECS 216)

Conversely:
sampling too slowly can cause bad effects called aliasing.
Example: wagon wheels in Western movies.

# Claude Shannon: Father of information theory

1916-2001
Born in Petosky, raised in Gaylord, MI.
UM EE Class of 1936.
Claude Shannon's statue is outside EECS.

http://www.computerhistory.org/collections/accession/102665758 circa 1980

*cf.* finite element models used by, *e.g.*, mechanical and aero engineers