Hybrid Zero Dynamics of N-Link Planar Biped Walkers: Equation Details

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I. Notation

The notation is as follows. The configuration coordinates are denoted by $q_1, \ldots, q_5$ and their velocities by $\dot{q}_1, \ldots, \dot{q}_5$. The link lengths, masses, inertias, and center of mass locations are denoted by L_*, M_*, I_*, and p_*^M, respectively.

II. Equations of Motion

The equations of motion during the swing phase is

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Bu$$

where

$$\begin{aligned}
D_{1,1}(q) &= I_f - 2p_t^M M_t L_f \cos(q_4) + I_T + M_t L_f^2 + I_t \\
D_{1,2}(q) &= -M_t L_f^2 + 2p_t^M M_t L_f \cos(q_4) - I_t - I_f \\
D_{1,3}(q) &= M_t L_f^2 - p_T^M M_T L_f \cos(q_1) \\
&\quad -2p_t^M M_t L_f \cos(q_4) - M_t L_f^2 \cos(q_1 - q_2) \\
&\quad -p_f^M M_f L_f \cos(q_1 - q_2) + I_T + I_f \\
&\quad +p_t^M M_t L_f \cos(-q_1 + q_2 + q_4) + I_t \\
D_{1,4}(q) &= p_t^M M_t L_f \cos(q_4) - I_t \\
D_{1,5}(q) &= M_t L_f^2 - M_t L_f^2 \cos(q_1 - q_2) \\
&\quad -2p_t^M M_t L_f \cos(q_4) - p_T^M M_T L_f \cos(q_1) \\
&\quad +p_f^M M_f L_t \cos(q_1 - q_2 + q_3) \\
&\quad -p_f^M M_f L_f \cos(q_1 - q_2) \\
&\quad -p_t^M M_t L_t \cos(-q_1 + q_2 - q_3 + q_4) \\
&\quad +M_t L_f L_t \cos(q_1 - q_2 + q_3) \\
&\quad +p_t^M M_t L_f \cos(-q_1 + q_2 + q_4) + I_f + I_t \\
&\quad +p_T^M M_T L_t \cos(q_1 + q_3) + I_T \\
D_{2,1}(q) &= -M_t L_f^2 + 2p_t^M M_t L_f \cos(q_4) - I_t - I_f \\
D_{2,2}(q) &= M_t L_f^2 - 2p_t^M M_t L_f \cos(q_4) + I_t + I_f \\
D_{2,3}(q) &= -M_t L_f^2 + 2p_t^M M_t L_f \cos(q_4) \\
&\quad +M_t L_f^2 \cos(q_1 - q_2) \\
&\quad +p_f^M M_f L_f \cos(q_1 - q_2) - I_f \\
&\quad -p_t^M M_t L_f \cos(-q_1 + q_2 + q_4) - I_t \\
D_{2,4}(q) &= -p_t^M M_t L_f \cos(q_4) + I_t \\
D_{2,5}(q) &= -M_t L_f^2 + M_t L_f^2 \cos(q_1 - q_2) \\
&\quad +2p_t^M M_t L_f \cos(q_4) \\
&\quad -p_f^M M_f L_t \cos(q_1 - q_2 + q_3) \\
&\quad +p_f^M M_f L_f \cos(q_1 - q_2) \\
&\quad +p_t^M M_t L_t \cos(-q_1 + q_2 - q_3 + q_4) \\
&\quad -M_t L_f L_t \cos(q_1 - q_2 + q_3) \\
&\quad -p_t^M M_t L_f \cos(-q_1 + q_2 + q_4) - I_f - I_t \\
D_{3,1}(q) &= M_t L_f^2 - p_T^M M_T L_f \cos(q_1) \\
&\quad -2p_t^M M_t L_f \cos(q_4) - M_t L_f^2 \cos(q_1 - q_2) \\
&\quad -p_f^M M_f L_f \cos(q_1 - q_2) + I_T + I_f \\
&\quad +p_t^M M_t L_f \cos(-q_1 + q_2 + q_4) + I_t \\
D_{3,2}(q) &= -M_t L_f^2 + 2p_t^M M_t L_f \cos(q_4) \\
&\quad +M_t L_f^2 \cos(q_1 - q_2) \\
&\quad +p_f^M M_f L_f \cos(q_1 - q_2) - I_f \\
&\quad -p_t^M M_t L_f \cos(-q_1 + q_2 + q_4) - I_t \\
D_{3,3}(q) &= 2M_t L_f^2 - 2p_T^M M_T L_f \cos(q_1) \\
&\quad -2p_t^M M_t L_f \cos(q_4) - 2M_t L_f^2 \cos(q_1 - q_2) \\
&\quad -2p_f^M M_f L_f \cos(q_1 - q_2) + I_T + 2I_f \\
&\quad +2M_f L_f^2 - 2p_f^M M_f L_f + M_T L_f^2 \\
&\quad +2p_t^M M_t L_f \cos(-q_1 + q_2 + q_4) + I_t \\
D_{3,4}(q) &= p_t^M M_t L_f \cos(q_4) \\
&\quad -p_t^M M_t L_f \cos(-q_1 + q_2 + q_4) - I_t \\
D_{3,5}(q) &= 2M_t L_f^2 - 2M_t L_f^2 \cos(q_1 - q_2) \\
&\quad -2p_t^M M_t L_f \cos(q_4) - 2p_T^M M_T L_f \cos(q_1) \\
&\quad +p_f^M M_f L_t \cos(q_1 - q_2 + q_3) \\
&\quad -2p_f^M M_f L_f \cos(q_1 - q_2) \\
&\quad -p_t^M M_t L_t \cos(-q_1 + q_2 - q_3 + q_4) \\
&\quad +M_t L_f L_t \cos(q_1 - q_2 + q_3) \\
&\quad +2p_t^M M_t L_f \cos(-q_1 + q_2 + q_4) \\
&\quad +2I_f + p_f^M M_f L_t \cos(q_3) \\
&\quad -M_T L_f L_t \cos(q_3) - M_t L_f L_t \cos(q_3) \\
&\quad -2M_f L_f L_t \cos(q_3) + M_T L_f^2 \\
&\quad -2p_f^M M_f L_f + 2M_f L_f^2 + I_t \\
&\quad +p_T^M M_T L_t \cos(q_1 + q_3) + I_T \\
D_{4,1}(q) &= p_t^M M_t L_f \cos(q_4) - I_t \\
D_{4,2}(q) &= -p_t^M M_t L_f \cos(q_4) + I_t \\
D_{4,3}(q) &= p_t^M M_t L_f \cos(q_4) \\
&\quad -p_t^M M_t L_f \cos(-q_1 + q_2 + q_4) - I_t \\
D_{4,4}(q) &= I_t \\
D_{4,5}(q) &= -I_t - p_t^M M_t L_f \cos(-q_1 + q_2 + q_4) \\
&\quad +p_t^M M_t L_f \cos(q_4)
\end{aligned}$$

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\begin{aligned}
&+p_t^M M_t L_t \cos\left(-q_1+q_2-q_3+q_4\right)\\
D_{5,1}(q) &= M_t L_f^2 - M_t L_f^2 \cos\left(q_1-q_2\right)\\
&-2p_t^M M_t L_f \cos\left(q_4\right) - p_T^M M_T L_f \cos\left(q_1\right)\\
&+p_f^M M_f L_t \cos\left(q_1-q_2+q_3\right)\\
&-p_f^M M_f L_f \cos\left(q_1-q_2\right)\\
&-p_t^M M_t L_t \cos\left(-q_1+q_2-q_3+q_4\right)\\
&+M_t L_f L_t \cos\left(q_1-q_2+q_3\right)\\
&+p_t^M M_t L_f \cos\left(-q_1+q_2+q_4\right) + I_f + I_t\\
&+p_T^M M_T L_t \cos\left(q_1+q_3\right) + I_T\\
D_{5,2}(q) &= -M_t L_f^2 + M_t L_f^2 \cos\left(q_1-q_2\right)\\
&+2p_t^M M_t L_f \cos\left(q_4\right)\\
&-p_f^M M_f L_t \cos\left(q_1-q_2+q_3\right)\\
&+p_f^M M_f L_f \cos\left(q_1-q_2\right)\\
&+p_t^M M_t L_t \cos\left(-q_1+q_2-q_3+q_4\right)\\
&-M_t L_f L_t \cos\left(q_1-q_2+q_3\right)\\
&-p_t^M M_t L_f \cos\left(-q_1+q_2+q_4\right) - I_f - I_t\\
D_{5,3}(q) &= 2M_t L_f^2 - 2M_t L_f^2 \cos\left(q_1-q_2\right)\\
&-2p_t^M M_t L_f \cos\left(q_4\right) - 2p_T^M M_T L_f \cos\left(q_1\right)\\
&+p_f^M M_f L_t \cos\left(q_1-q_2+q_3\right)\\
&-2p_f^M M_f L_f \cos\left(q_1-q_2\right)\\
&-p_t^M M_t L_t \cos\left(-q_1+q_2-q_3+q_4\right)\\
&+M_t L_f L_t \cos\left(q_1-q_2+q_3\right)\\
&+2p_t^M M_t L_f \cos\left(-q_1+q_2+q_4\right) + 2I_f\\
&+p_f^M M_f L_t \cos\left(q_3\right) - M_T L_f L_t \cos\left(q_3\right)\\
&-M_t L_f L_t \cos\left(q_3\right) - 2M_f L_f L_t \cos\left(q_3\right)\\
&+M_T L_f^2 - 2p_f^M M_f L_f + 2M_f L_f^2 + I_t\\
&+p_T^M M_T L_t \cos\left(q_1+q_3\right) + I_T\\
D_{5,4}(q) &= -I_t - p_t^M M_t L_f \cos\left(-q_1+q_2+q_4\right)\\
&+p_t^M M_t L_f \cos\left(q_4\right)\\
&+p_t^M M_t L_t \cos\left(-q_1+q_2-q_3+q_4\right)\\
D_{5,5}(q) &= 2p_f^M M_f L_t \cos\left(q_3\right) - 2p_f^M M_f L_f \cos\left(q_1-q_2\right)\\
&-2M_t L_f^2 \cos\left(q_1-q_2\right)\\
&+2p_T^M M_T L_t \cos\left(q_1+q_3\right)\\
&+2p_f^M M_f L_t \cos\left(q_1-q_2+q_3\right)\\
&-2p_T^M M_T L_f \cos\left(q_1\right) + 2I_t + M_T L_t^2\\
&-2M_T L_f L_t \cos\left(q_3\right) - 2M_t L_f L_t \cos\left(q_3\right)\\
&-4M_f L_f L_t \cos\left(q_3\right) + 2I_f + I_T\\
&+2M_t L_f L_t \cos\left(q_1-q_2+q_3\right) + 2M_t L_f^2\\
&-2p_f^M M_f L_f + 2M_t L_t^2 + 2M_f L_t^2 + 2M_f L_f^2\\
&-2p_t^M M_t L_t + M_T L_f^2\\
&+2p_t^M M_t L_f \cos\left(-q_1+q_2+q_4\right)\\
&-2p_t^M M_t L_f \cos\left(q_4\right)\\
&-2p_t^M M_t L_t \cos\left(-q_1+q_2-q_3+q_4\right),
\end{aligned}
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\begin{aligned}
C_{1,1}(q) &= M_t p_t^M L_f \sin\left(q_4\right)\dot{q}_4\\
C_{1,2}(q) &= -M_t p_t^M L_f \sin\left(q_4\right)\dot{q}_4\\
C_{1,3}(q) &= -L_f\left(\dot{q}_3 p_T^M M_T \sin\left(q_1\right)\right.\\
&\quad +L_f\dot{q}_3 M_t \sin\left(q_1-q_2\right)\\
&\quad +\dot{q}_3 p_f^M M_f \sin\left(q_1-q_2\right)\\
&\quad +\dot{q}_3 p_t^M M_t \sin\left(-q_1+q_2+q_4\right)\\
&\quad -p_t^M M_t \sin\left(q_4\right)\dot{q}_4\\
&\quad +\dot{q}_5 p_T^M M_T \sin\left(q_1\right) + L_f\dot{q}_5 M_t \sin\left(q_1-q_2\right)\\
&\quad +\dot{q}_5 p_f^M M_f \sin\left(q_1-q_2\right)\\
&\quad \left.+\dot{q}_5 p_t^M M_t \sin\left(-q_1+q_2+q_4\right)\right)\\
C_{1,4}(q) &= p_t^M M_t L_f \sin\left(q_4\right)\left(\dot{q}_1-\dot{q}_2+\dot{q}_3-\dot{q}_4+\dot{q}_5\right)\\
C_{1,5}(q) &= -\dot{q}_3 p_T^M M_T L_f \sin\left(q_1\right) - L_f^2\dot{q}_3 M_t \sin\left(q_1-q_2\right)\\
&\quad -\dot{q}_3 p_f^M M_f L_f \sin\left(q_1-q_2\right)\\
&\quad -\dot{q}_3 p_t^M M_t L_f \sin\left(-q_1+q_2+q_4\right)\\
&\quad +p_t^M M_t L_f \sin\left(q_4\right)\dot{q}_4\\
&\quad +\dot{q}_5 M_t L_f L_t \sin\left(q_1-q_2+q_3\right)\\
&\quad -\dot{q}_5 p_t^M M_t L_f \sin\left(-q_1+q_2+q_4\right)\\
&\quad -\dot{q}_5 p_f^M M_f L_f \sin\left(q_1-q_2\right)\\
&\quad -L_f^2\dot{q}_5 M_t \sin\left(q_1-q_2\right)\\
&\quad -\dot{q}_5 p_T^M M_T L_f \sin\left(q_1\right)\\
&\quad +\dot{q}_5 p_f^M M_f L_t \sin\left(q_1-q_2+q_3\right)\\
&\quad +\dot{q}_5 p_T^M M_T L_t \sin\left(q_1+q_3\right)\\
&\quad +\dot{q}_5 p_t^M M_t L_t \sin\left(-q_1+q_2-q_3+q_4\right)\\
C_{2,1}(q) &= -p_t^M M_t L_f \sin\left(q_4\right)\dot{q}_4\\
C_{2,2}(q) &= p_t^M M_t L_f \sin\left(q_4\right)\dot{q}_4\\
C_{2,3}(q) &= L_f\left(L_f\dot{q}_3 M_t \sin\left(q_1-q_2\right)\right.\\
&\quad +\dot{q}_3 p_f^M M_f \sin\left(q_1-q_2\right)\\
&\quad +\dot{q}_3 p_t^M M_t \sin\left(-q_1+q_2+q_4\right)\\
&\quad -p_t^M M_t \sin\left(q_4\right)\dot{q}_4 + L_f\dot{q}_5 M_t \sin\left(q_1-q_2\right)\\
&\quad +\dot{q}_5 p_f^M M_f \sin\left(q_1-q_2\right)\\
&\quad \left.+\dot{q}_5 p_t^M M_t \sin\left(-q_1+q_2+q_4\right)\right)\\
C_{2,4}(q) &= -p_t^M M_t L_f \sin\left(q_4\right)\left(\dot{q}_1-\dot{q}_2+\dot{q}_3-\dot{q}_4+\dot{q}_5\right)\\
C_{2,5}(q) &= L_f^2\dot{q}_3 M_t \sin\left(q_1-q_2\right)\\
&\quad +\dot{q}_3 p_f^M M_f L_f \sin\left(q_1-q_2\right)\\
&\quad +L_f\dot{q}_3 p_t^M M_t \sin\left(-q_1+q_2+q_4\right)\\
&\quad -p_t^M M_t L_f \sin\left(q_4\right)\dot{q}_4\\
&\quad +L_f^2\dot{q}_5 M_t \sin\left(q_1-q_2\right)\\
&\quad -\dot{q}_5 p_f^M M_f L_t \sin\left(q_1-q_2+q_3\right)\\
&\quad +\dot{q}_5 p_f^M M_f L_f \sin\left(q_1-q_2\right)\\
&\quad -\dot{q}_5 p_t^M M_t L_t \sin\left(-q_1+q_2-q_3+q_4\right)\\
&\quad -\dot{q}_5 M_t L_f L_t \sin\left(q_1-q_2+q_3\right)\\
&\quad +L_f\dot{q}_5 p_t^M M_t \sin\left(-q_1+q_2+q_4\right)\\
C_{3,1}(q) &= L_f\left(\dot{q}_1 p_T^M M_T \sin\left(q_1\right) + L_f\dot{q}_1 M_t \sin\left(q_1-q_2\right)\right.
\end{aligned}
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\begin{aligned}
&+\dot{q}_1 p_f^M M_f \sin(q_1 - q_2)\\
&+\dot{q}_1 p_t^M M_t \sin(-q_1 + q_2 + q_4)\\
&-L_f \dot{q}_2 M_t \sin(q_1 - q_2)\\
&-\dot{q}_2 p_f^M M_f \sin(q_1 - q_2)\\
&-\dot{q}_2 p_t^M M_t \sin(-q_1 + q_2 + q_4)\\
&+\dot{q}_3 p_T^M M_T \sin(q_1) + L_f \dot{q}_3 M_t \sin(q_1 - q_2)\\
&+\dot{q}_3 p_f^M M_f \sin(q_1 - q_2)\\
&+\dot{q}_3 p_t^M M_t \sin(-q_1 + q_2 + q_4)\\
&+p_t^M M_t \sin(q_4)\,\dot{q}_4\\
&-p_t^M M_t \dot{q}_4 \sin(-q_1 + q_2 + q_4)\\
&+\dot{q}_5 p_T^M M_T \sin(q_1) + L_f \dot{q}_5 M_t \sin(q_1 - q_2)\\
&+\dot{q}_5 p_f^M M_f \sin(q_1 - q_2)\\
&+\dot{q}_5 p_t^M M_t \sin(-q_1 + q_2 + q_4)\big)
\end{aligned}
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\begin{aligned}
C_{3,2}(q) \;=\; & -L_f \big(L_f \dot{q}_1 M_t \sin(q_1 - q_2)\\
&+\dot{q}_1 p_f^M M_f \sin(q_1 - q_2)\\
&+\dot{q}_1 p_t^M M_t \sin(-q_1 + q_2 + q_4)\\
&-L_f \dot{q}_2 M_t \sin(q_1 - q_2)\\
&-\dot{q}_2 p_f^M M_f \sin(q_1 - q_2)\\
&-\dot{q}_2 p_t^M M_t \sin(-q_1 + q_2 + q_4)\\
&+L_f \dot{q}_3 M_t \sin(q_1 - q_2)\\
&+\dot{q}_3 p_f^M M_f \sin(q_1 - q_2)\\
&+\dot{q}_3 p_t^M M_t \sin(-q_1 + q_2 + q_4)\\
&+p_t^M M_t \sin(q_4)\,\dot{q}_4\\
&-p_t^M M_t \dot{q}_4 \sin(-q_1 + q_2 + q_4)\\
&+L_f \dot{q}_5 M_t \sin(q_1 - q_2)\\
&+\dot{q}_5 p_f^M M_f \sin(q_1 - q_2)\\
&+\dot{q}_5 p_t^M M_t \sin(-q_1 + q_2 + q_4)\big)\\
C_{3,3}(q) \;=\; & L_f \big(\dot{q}_1 p_T^M M_T \sin(q_1) + L_f \dot{q}_1 M_t \sin(q_1 - q_2)\\
&+\dot{q}_1 p_f^M M_f \sin(q_1 - q_2)\\
&+\dot{q}_1 p_t^M M_t \sin(-q_1 + q_2 + q_4)\\
&-L_f \dot{q}_2 M_t \sin(q_1 - q_2)\\
&-\dot{q}_2 p_f^M M_f \sin(q_1 - q_2)\\
&-\dot{q}_2 p_t^M M_t \sin(-q_1 + q_2 + q_4)\\
&+p_t^M M_t \sin(q_4)\,\dot{q}_4\\
&-p_t^M M_t \dot{q}_4 \sin(-q_1 + q_2 + q_4)\big)\\
C_{3,4}(q) \;=\; & p_t^M M_t L_f (\dot{q}_1 - \dot{q}_2 + \dot{q}_3 - \dot{q}_4 + \dot{q}_5)\\
&\times (\sin(q_4) - \sin(-q_1 + q_2 + q_4))\\
C_{3,5}(q) \;=\; & \dot{q}_1 p_T^M M_T L_f \sin(q_1) + \dot{q}_1 M_t L_f^2 \sin(q_1 - q_2)\\
&+\dot{q}_1 p_f^M M_f L_f \sin(q_1 - q_2)\\
&+p_t^M M_t L_f \dot{q}_1 \sin(-q_1 + q_2 + q_4)\\
&-\dot{q}_2 M_t L_f^2 \sin(q_1 - q_2)\\
&-\dot{q}_2 p_f^M M_f L_f \sin(q_1 - q_2)\\
&-p_t^M M_t L_f \dot{q}_2 \sin(-q_1 + q_2 + q_4)\\
&+L_f p_t^M M_t \sin(q_4)\,\dot{q}_4
\end{aligned}
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\begin{aligned}
&-p_t^M M_t L_f \dot{q}_4 \sin(-q_1 + q_2 + q_4)\\
&+\dot{q}_5 M_t L_f L_t \sin(q_1 - q_2 + q_3)\\
&-2\dot{q}_5 M_f L_f L_t \sin(q_3) - \dot{q}_5 M_t L_f L_t \sin(q_3)\\
&-\dot{q}_5 M_T L_f L_t \sin(q_3)\\
&+\dot{q}_5 p_f^M M_f L_t \sin(q_1 - q_2 + q_3)\\
&+\dot{q}_5 p_T^M M_T L_t \sin(q_1 + q_3)\\
&+\dot{q}_5 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4)\\
&+\dot{q}_5 p_f^M M_f L_t \sin(q_3)
\end{aligned}
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\begin{aligned}
C_{4,1}(q) \;=\; & -p_t^M M_t L_f \sin(q_4)\,(\dot{q}_1 - \dot{q}_2 + \dot{q}_3 + \dot{q}_5)\\
C_{4,2}(q) \;=\; & p_t^M M_t L_f \sin(q_4)\,(\dot{q}_1 - \dot{q}_2 + \dot{q}_3 + \dot{q}_5)\\
C_{4,3}(q) \;=\; & -p_t^M M_t L_f \big(\sin(q_4)\,\dot{q}_1 - \sin(q_4)\,\dot{q}_2\\
&+\sin(q_4)\,\dot{q}_3 - \dot{q}_3 \sin(-q_1 + q_2 + q_4)\\
&-\dot{q}_5 \sin(-q_1 + q_2 + q_4) + \sin(q_4)\,\dot{q}_5\big)\\
C_{4,4}(q) \;=\; & 0\\
C_{4,5}(q) \;=\; & -p_t^M M_t \big(L_f \sin(q_4)\,\dot{q}_1 - L_f \sin(q_4)\,\dot{q}_2\\
&+L_f \sin(q_4)\,\dot{q}_3 - L_f \dot{q}_3 \sin(-q_1 + q_2 + q_4)\\
&-L_f \dot{q}_5 \sin(-q_1 + q_2 + q_4) + L_f \sin(q_4)\,\dot{q}_5\\
&+\dot{q}_5 L_t \sin(-q_1 + q_2 - q_3 + q_4)\big)
\end{aligned}
$$

$$
\begin{aligned}
C_{5,1}(q) \;=\; & \dot{q}_3 M_t L_f^2 \sin(q_1 - q_2)\\
&-\dot{q}_1 p_T^M M_T L_t \sin(q_1 + q_3)\\
&-\dot{q}_5 p_T^M M_T L_t \sin(q_1 + q_3)\\
&+\dot{q}_3 p_T^M M_T L_f \sin(q_1)\\
&-\dot{q}_3 p_T^M M_T L_t \sin(q_1 + q_3)\\
&-\dot{q}_5 M_t L_f L_t \sin(q_1 - q_2 + q_3)\\
&+\dot{q}_5 p_T^M M_T L_f \sin(q_1) + \dot{q}_1 p_T^M M_T L_f \sin(q_1)\\
&-\dot{q}_2 p_t^M M_t L_f \sin(-q_1 + q_2 + q_4)\\
&-\dot{q}_3 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4)\\
&+\dot{q}_2 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4)\\
&-\dot{q}_4 p_t^M M_t L_f \sin(-q_1 + q_2 + q_4)\\
&+\dot{q}_4 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4)\\
&+\dot{q}_1 p_t^M M_t L_f \sin(-q_1 + q_2 + q_4)\\
&-p_t^M M_t \dot{q}_5 L_t \sin(-q_1 + q_2 - q_3 + q_4)\\
&+p_t^M M_t L_f \dot{q}_5 \sin(-q_1 + q_2 + q_4)\\
&+p_t^M M_t L_f \dot{q}_3 \sin(-q_1 + q_2 + q_4)\\
&-\dot{q}_1 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4)\\
&+p_t^M M_t L_f \sin(q_4)\,\dot{q}_4\\
&+\dot{q}_1 p_f^M M_f L_f \sin(q_1 - q_2)\\
&+\dot{q}_2 p_f^M M_f L_t \sin(q_1 - q_2 + q_3)\\
&-\dot{q}_3 p_f^M M_f L_t \sin(q_1 - q_2 + q_3)\\
&-\dot{q}_2 p_f^M M_f L_f \sin(q_1 - q_2)\\
&+\dot{q}_5 p_f^M M_f L_f \sin(q_1 - q_2)\\
&-\dot{q}_1 p_f^M M_f L_t \sin(q_1 - q_2 + q_3)\\
&+\dot{q}_3 p_f^M M_f L_f \sin(q_1 - q_2)\\
&-\dot{q}_5 p_f^M M_f L_t \sin(q_1 - q_2 + q_3)
\end{aligned}
$$

$$
\begin{aligned}
& -\dot{q}_3 M_t L_f L_t \sin(q_1 - q_2 + q_3) \\
& +\dot{q}_1 M_t L_f^2 \sin(q_1 - q_2) \\
& -\dot{q}_1 M_t L_f L_t \sin(q_1 - q_2 + q_3) \\
& +\dot{q}_5 M_t L_f^2 \sin(q_1 - q_2) \\
& -\dot{q}_2 M_t L_f^2 \sin(q_1 - q_2) \\
& +\dot{q}_2 M_t L_f L_t \sin(q_1 - q_2 + q_3)
\end{aligned}
$$

$$
\begin{aligned}
C_{5,2}(q) \;=\; & -\dot{q}_3 M_t L_f^2 \sin(q_1 - q_2) \\
& -\dot{q}_1 p_t^M M_t L_f \sin(-q_1 + q_2 + q_4) \\
& -\dot{q}_2 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& -p_t^M M_t L_f \sin(q_4)\, \dot{q}_4 \\
& -\dot{q}_3 p_t^M M_t L_f \sin(-q_1 + q_2 + q_4) \\
& +\dot{q}_4 p_t^M M_t L_f \sin(-q_1 + q_2 + q_4) \\
& +\dot{q}_5 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& +\dot{q}_1 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& -\dot{q}_5 p_t^M M_t L_f \sin(-q_1 + q_2 + q_4) \\
& +\dot{q}_2 p_t^M M_t L_f \sin(-q_1 + q_2 + q_4) \\
& +\dot{q}_3 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& -\dot{q}_4 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& +\dot{q}_5 M_t L_f L_t \sin(q_1 - q_2 + q_3) \\
& -\dot{q}_5 p_f^M M_f L_f \sin(q_1 - q_2) \\
& -\dot{q}_2 p_f^M M_f L_t \sin(q_1 - q_2 + q_3) \\
& +\dot{q}_5 p_f^M M_f L_t \sin(q_1 - q_2 + q_3) \\
& +\dot{q}_1 p_f^M M_f L_t \sin(q_1 - q_2 + q_3) \\
& -\dot{q}_1 p_f^M M_f L_f \sin(q_1 - q_2) \\
& +\dot{q}_3 p_f^M M_f L_t \sin(q_1 - q_2 + q_3) \\
& -\dot{q}_3 p_f^M M_f L_f \sin(q_1 - q_2) \\
& +\dot{q}_2 p_f^M M_f L_f \sin(q_1 - q_2) \\
& +\dot{q}_3 M_t L_f L_t \sin(q_1 - q_2 + q_3) \\
& -\dot{q}_1 M_t L_f^2 \sin(q_1 - q_2) \\
& +\dot{q}_1 M_t L_f L_t \sin(q_1 - q_2 + q_3) \\
& -\dot{q}_5 M_t L_f^2 \sin(q_1 - q_2) \\
& +\dot{q}_2 M_t L_f^2 \sin(q_1 - q_2) \\
& -\dot{q}_2 M_t L_f L_t \sin(q_1 - q_2 + q_3)
\end{aligned}
$$

$$
\begin{aligned}
C_{5,3}(q) \;=\; & -\dot{q}_3 p_f^M M_f L_t \sin(q_1 - q_2 + q_3) \\
& +\dot{q}_1 p_f^M M_f L_f \sin(q_1 - q_2) \\
& -\dot{q}_5 p_f^M M_f L_t \sin(q_1 - q_2 + q_3) \\
& -\dot{q}_1 p_f^M M_f L_t \sin(q_1 - q_2 + q_3) \\
& -\dot{q}_3 p_f^M M_f L_t \sin(q_3) \\
& -\dot{q}_2 p_f^M M_f L_f \sin(q_1 - q_2) \\
& -\dot{q}_5 p_f^M M_f L_t \sin(q_3) \\
& +\dot{q}_2 p_f^M M_f L_t \sin(q_1 - q_2 + q_3) \\
& -\dot{q}_5 p_T^M M_T L_t \sin(q_1 + q_3) \\
& -\dot{q}_3 p_T^M M_T L_t \sin(q_1 + q_3) \\
& +\dot{q}_1 p_T^M M_T L_f \sin(q_1) \\
& -\dot{q}_1 p_T^M M_T L_t \sin(q_1 + q_3) \\
& +2\dot{q}_5 M_f L_f L_t \sin(q_3) \\
& +2\dot{q}_3 M_f L_f L_t \sin(q_3) + \dot{q}_3 M_t L_f L_t \sin(q_3) \\
& +\dot{q}_5 M_t L_f L_t \sin(q_3) + \dot{q}_3 M_T L_f L_t \sin(q_3) \\
& +\dot{q}_5 M_T L_f L_t \sin(q_3) \\
& -\dot{q}_5 M_t L_f L_t \sin(q_1 - q_2 + q_3) \\
& +\dot{q}_2 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& -\dot{q}_2 p_t^M M_t L_f \sin(-q_1 + q_2 + q_4) \\
& -\dot{q}_1 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& +\dot{q}_4 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& +p_t^M M_t L_f \sin(q_4)\, \dot{q}_4 \\
& +\dot{q}_1 p_t^M M_t L_f \sin(-q_1 + q_2 + q_4) \\
& -\dot{q}_4 p_t^M M_t L_f \sin(-q_1 + q_2 + q_4) \\
& -\dot{q}_5 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& -\dot{q}_3 p_t^M M_t L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& -\dot{q}_3 M_t L_f L_t \sin(q_1 - q_2 + q_3) \\
& +\dot{q}_1 M_t L_f^2 \sin(q_1 - q_2) \\
& -\dot{q}_1 M_t L_f L_t \sin(q_1 - q_2 + q_3) \\
& -\dot{q}_2 M_t L_f^2 \sin(q_1 - q_2) \\
& +\dot{q}_2 M_t L_f L_t \sin(q_1 - q_2 + q_3)
\end{aligned}
$$

$$
\begin{aligned}
C_{5,4}(q) \;=\; & p_t^M M_t \left(\dot{q}_1 - \dot{q}_2 + \dot{q}_3 - \dot{q}_4 + \dot{q}_5\right) \\
& \times \left(L_f \sin(q_4) + L_t \sin(-q_1 + q_2 - q_3 + q_4)\right. \\
& \left. -L_f \sin(-q_1 + q_2 + q_4)\right)
\end{aligned}
$$

$$
\begin{aligned}
C_{5,5}(q) \;=\; & p_t^M M_t \dot{q}_4 L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& +2\dot{q}_3 M_f L_f L_t \sin(q_3) \\
& +p_t^M M_t \dot{q}_2 L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& +p_t^M M_t \dot{q}_1 L_f \sin(-q_1 + q_2 + q_4) \\
& -p_t^M M_t \dot{q}_2 L_f \sin(-q_1 + q_2 + q_4) \\
& +p_t^M M_t \dot{q}_4 L_f \sin(q_4) \\
& -p_t^M M_t \dot{q}_3 L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& -p_t^M M_t \dot{q}_4 L_f \sin(-q_1 + q_2 + q_4) \\
& -p_t^M M_t \dot{q}_1 L_t \sin(-q_1 + q_2 - q_3 + q_4) \\
& +\dot{q}_3 M_t L_f L_t \sin(q_3) + \dot{q}_3 M_T L_f L_t \sin(q_3) \\
& -\dot{q}_1 M_t L_f L_t \sin(q_1 - q_2 + q_3) \\
& +\dot{q}_2 M_t L_f L_t \sin(q_1 - q_2 + q_3) \\
& -\dot{q}_3 M_t L_f L_t \sin(q_1 - q_2 + q_3) \\
& +\dot{q}_1 M_t L_f^2 \sin(q_1 - q_2) \\
& -\dot{q}_2 M_t L_f^2 \sin(q_1 - q_2) \\
& -\dot{q}_1 p_T^M M_T L_t \sin(q_1 + q_3) \\
& -\dot{q}_3 p_T^M M_T L_t \sin(q_1 + q_3) \\
& +\dot{q}_1 p_T^M M_T L_f \sin(q_1) \\
& -\dot{q}_3 p_f^M M_f L_t \sin(q_1 - q_2 + q_3) \\
& +\dot{q}_1 p_f^M M_f L_f \sin(q_1 - q_2) \\
& +\dot{q}_2 p_f^M M_f L_t \sin(q_1 - q_2 + q_3)
\end{aligned}
$$

$$
\begin{aligned}
&-\dot{q}_3 p_f^M M_f L_t \sin(q_3)\\
&-\dot{q}_2 p_f^M M_f L_f \sin(q_1-q_2)\\
&-\dot{q}_1 p_f^M M_f L_t \sin(q_1-q_2+q_3),
\end{aligned}
$$

$$
\begin{aligned}
G_1(q) &= -g\big(\sin(q_1+q_3+q_5)\,p_T^M M_T\\
&\quad +p_f^M M_f \sin(q_1-q_2+q_3+q_5)\\
&\quad +L_f \sin(q_1-q_2+q_3+q_5)\,M_t\\
&\quad +p_t^M M_t \sin(-q_1+q_2-q_3+q_4-q_5)\big)\\
G_2(q) &= g\big(p_f^M M_f \sin(q_1-q_2+q_3+q_5)\\
&\quad +L_f \sin(q_1-q_2+q_3+q_5)\,M_t\\
&\quad +p_t^M M_t \sin(-q_1+q_2-q_3+q_4-q_5)\big)\\
G_3(q) &= g\,(L_f \sin(q_3+q_5)\,M_T\\
&\quad -\sin(q_1+q_3+q_5)\,p_T^M M_T\\
&\quad +2L_f \sin(q_3+q_5)\,M_f - p_f^M M_f \sin(q_3+q_5)\\
&\quad -p_f^M M_f \sin(q_1-q_2+q_3+q_5)\\
&\quad +L_f \sin(q_3+q_5)\,M_t\\
&\quad -L_f \sin(q_1-q_2+q_3+q_5)\,M_t\\
&\quad -p_t^M M_t \sin(-q_1+q_2-q_3+q_4-q_5)\big)\\
G_4(q) &= g p_t^M M_t \sin(-q_1+q_2-q_3+q_4-q_5)\\
G_5(q) &= g\,(L_f \sin(q_3+q_5)\,M_T - L_t \sin(q_5)\,M_T\\
&\quad -\sin(q_1+q_3+q_5)\,p_T^M M_T\\
&\quad +2L_f \sin(q_3+q_5)\,M_f - 2L_t \sin(q_5)\,M_f\\
&\quad -p_f^M M_f \sin(q_3+q_5)\\
&\quad -p_f^M M_f \sin(q_1-q_2+q_3+q_5)\\
&\quad -2L_t \sin(q_5)\,M_t + p_t^M M_t \sin(q_5)\\
&\quad +L_f \sin(q_3+q_5)\,M_t\\
&\quad -L_f \sin(q_1-q_2+q_3+q_5)\,M_t\\
&\quad -p_t^M M_t \sin(-q_1+q_2-q_3+q_4-q_5)\big),
\end{aligned}
$$

and

$$
B = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

III. Kinetic and Potential Energies

The total kinetic energy of the robot is

$$
\begin{aligned}
K(q,\dot{q}) &= -p_f^M M_f \dot{q}_5^2 L_f - p_f^M M_f L_f \dot{q}_3^2 - p_t^M M_t \dot{q}_5^2 L_t\\
&\quad -2p_T^M M_T L_f \dot{q}_3 \dot{q}_5 \cos(q_1)\\
&\quad -p_T^M M_T L_f \dot{q}_3^2 \cos(q_1)\\
&\quad -p_T^M M_T L_f \dot{q}_3 \dot{q}_1 \cos(q_1)\\
&\quad +p_T^M M_T \dot{q}_5^2 L_t \cos(q_1+q_3)\\
&\quad -p_T^M M_T \dot{q}_5^2 L_f \cos(q_1)\\
&\quad -p_T^M M_T \dot{q}_5 L_f \dot{q}_1 \cos(q_1)\\
&\quad +p_T^M M_T \dot{q}_5 L_t \dot{q}_1 \cos(q_1+q_3)\\
&\quad +p_T^M M_T \dot{q}_5 L_t \dot{q}_3 \cos(q_1+q_3)\\
&\quad -2p_t^M M_t \dot{q}_5 L_f \dot{q}_1 \cos(q_4)\\
&\quad +2p_t^M M_t L_f \dot{q}_3 \dot{q}_2 \cos(q_4)\\
&\quad +2p_t^M M_t L_f \dot{q}_1 \dot{q}_2 \cos(q_4)\\
&\quad +p_t^M M_t L_f \dot{q}_1 \dot{q}_4 \cos(q_4)\\
&\quad +p_t^M M_t L_f \dot{q}_3 \dot{q}_4 \cos(q_4)\\
&\quad +p_t^M M_t L_f \dot{q}_3 \dot{q}_1 \cos(-q_1+q_2+q_4)\\
&\quad -p_t^M M_t L_f \dot{q}_3 \dot{q}_4 \cos(-q_1+q_2+q_4)\\
&\quad -p_t^M M_t L_f \dot{q}_3^2 \cos(q_4) - p_t^M M_t L_f \dot{q}_2^2 \cos(q_4)\\
&\quad -p_t^M M_t L_f \dot{q}_3 \dot{q}_2 \cos(-q_1+q_2+q_4)\\
&\quad -p_t^M M_t \dot{q}_5 L_f \dot{q}_4 \cos(-q_1+q_2+q_4)\\
&\quad -p_t^M M_t \dot{q}_5^2 L_f \cos(q_4)\\
&\quad -p_t^M M_t \dot{q}_5 L_t \dot{q}_1 \cos(-q_1+q_2-q_3+q_4)\\
&\quad -p_t^M M_t L_f \dot{q}_1^2 \cos(q_4)\\
&\quad -p_t^M M_t \dot{q}_5^2 L_t \cos(-q_1+q_2-q_3+q_4)\\
&\quad +p_t^M M_t \dot{q}_5^2 L_f \cos(-q_1+q_2+q_4)\\
&\quad +p_t^M M_t L_f \dot{q}_3^2 \cos(-q_1+q_2+q_4)\\
&\quad +2p_t^M M_t L_f \dot{q}_3 \dot{q}_5 \cos(-q_1+q_2+q_4)\\
&\quad +p_t^M M_t \dot{q}_5 L_t \dot{q}_2 \cos(-q_1+q_2-q_3+q_4)\\
&\quad -p_t^M M_t \dot{q}_5 L_t \dot{q}_3 \cos(-q_1+q_2-q_3+q_4)\\
&\quad +p_t^M M_t \dot{q}_5 L_t \dot{q}_4 \cos(-q_1+q_2-q_3+q_4)\\
&\quad -p_f^M M_f L_f \dot{q}_3^2 \cos(q_1-q_2)\\
&\quad -2p_f^M M_f L_f \dot{q}_3 \dot{q}_5 \cos(q_1-q_2)\\
&\quad -p_f^M M_f \dot{q}_5 L_t \dot{q}_2 \cos(q_1-q_2+q_3)\\
&\quad +p_f^M M_f \dot{q}_5 L_t \dot{q}_3 \cos(q_1-q_2+q_3)\\
&\quad -p_t^M M_t L_f \dot{q}_2 \dot{q}_4 \cos(q_4)\\
&\quad +p_t^M M_t \dot{q}_5 L_f \dot{q}_4 \cos(q_4)\\
&\quad -p_t^M M_t L_f \dot{q}_2 \dot{q}_5 \cos(-q_1+q_2+q_4)\\
&\quad +p_t^M M_t \dot{q}_5 L_f \dot{q}_1 \cos(-q_1+q_2+q_4)\\
&\quad -2p_t^M M_t L_f \dot{q}_3 \dot{q}_1 \cos(q_4)\\
&\quad -2p_t^M M_t L_f \dot{q}_3 \dot{q}_5 \cos(q_4)\\
&\quad +2p_t^M M_t L_f \dot{q}_2 \dot{q}_5 \cos(q_4)\\
&\quad +p_f^M M_f \dot{q}_5 L_f \dot{q}_2 \cos(q_1-q_2)\\
&\quad +p_f^M M_f L_f \dot{q}_3 \dot{q}_2 \cos(q_1-q_2)\\
&\quad +p_f^M M_f \dot{q}_5 L_t \dot{q}_3 \cos(q_3)\\
&\quad -p_f^M M_f L_f \dot{q}_3 \dot{q}_1 \cos(q_1-q_2)\\
&\quad -p_f^M M_f \dot{q}_5 L_f \dot{q}_1 \cos(q_1-q_2)\\
&\quad +p_f^M M_f \dot{q}_5^2 L_t \cos(q_3) - 2p_f^M M_f L_f \dot{q}_3 \dot{q}_5\\
&\quad -p_f^M M_f \dot{q}_5^2 L_f \cos(q_1-q_2)\\
&\quad +p_f^M M_f \dot{q}_5 L_t \dot{q}_1 \cos(q_1-q_2+q_3)\\
&\quad +p_f^M M_f \dot{q}_5^2 L_t \cos(q_1-q_2+q_3)\\
&\quad -M_t L_f^2 \dot{q}_2 \dot{q}_3 + M_f \dot{q}_5^2 L_t^2 + M_f L_f^2 \dot{q}_3^2\\
&\quad +I_T \dot{q}_1 \dot{q}_5 + M_t L_t^2 \dot{q}_5^2 + I_T \dot{q}_3 \dot{q}_5 + M_t \dot{q}_3^2 L_f^2\\
&\quad +\frac{1}{2} M_T \dot{q}_5^2 L_f^2 + \frac{1}{2} M_T \dot{q}_5^2 L_t^2 + \frac{1}{2} M_T L_f^2 \dot{q}_3^2
\end{aligned}
$$

$$
\begin{aligned}
&+2I_f\dot{q}_3\dot{q}_5+\frac{1}{2}M_tL_f^2\dot{q}_1^2+\frac{1}{2}M_tL_f^2\dot{q}_2^2\\
&-M_T\dot{q}_5^2L_fL_t\cos(q_3)+M_tL_f^2\dot{q}_1\dot{q}_3\\
&-M_tL_f^2\dot{q}_1\dot{q}_2+M_tL_f^2\dot{q}_1\dot{q}_5-M_tL_f^2\dot{q}_2\dot{q}_5\\
&+M_f\dot{q}_5^2L_f^2+M_TL_f^2\dot{q}_3\dot{q}_5\\
&-2M_tL_f^2\dot{q}_3\dot{q}_5\cos(q_1-q_2)\\
&-2M_f\dot{q}_5^2L_fL_t\cos(q_3)\\
&-2M_fL_f\dot{q}_3\dot{q}_5L_t\cos(q_3)\\
&-M_t\dot{q}_5^2L_tL_f\cos(q_3)-M_t\dot{q}_3L_f\dot{q}_5L_t\cos(q_3)\\
&-M_tL_f^2\dot{q}_1\dot{q}_3\cos(q_1-q_2)\\
&-M_t\dot{q}_3^2L_f^2\cos(q_1-q_2)+2M_tL_f^2\dot{q}_3\dot{q}_5\\
&-I_t\dot{q}_4\dot{q}_5-I_t\dot{q}_2\dot{q}_5-I_t\dot{q}_2\dot{q}_3+I_t\dot{q}_1\dot{q}_3+I_t\dot{q}_1\dot{q}_5\\
&-I_t\dot{q}_1\dot{q}_2+I_T\dot{q}_1\dot{q}_3-M_t\dot{q}_5^2L_f^2\cos(q_1-q_2)\\
&+M_tL_f^2\dot{q}_2\dot{q}_3\cos(q_1-q_2)\\
&+M_tL_f\dot{q}_1\dot{q}_5L_t\cos(q_1-q_2+q_3)\\
&-M_tL_f^2\dot{q}_1\dot{q}_5\cos(q_1-q_2)\\
&+M_tL_f^2\dot{q}_2\dot{q}_5\cos(q_1-q_2)\\
&-M_TL_f\dot{q}_3\dot{q}_5L_t\cos(q_3)+I_f\dot{q}_5^2+I_f\dot{q}_3^2\\
&+I_t\dot{q}_5^2+\frac{1}{2}I_T\dot{q}_3^2+\frac{1}{2}I_T\dot{q}_1^2+\frac{1}{2}I_T\dot{q}_5^2+\frac{1}{2}I_t\dot{q}_3^2\\
&+\frac{1}{2}I_t\dot{q}_1^2+\frac{1}{2}I_t\dot{q}_4^2+\frac{1}{2}I_t\dot{q}_2^2+\frac{1}{2}I_f\dot{q}_2^2+\frac{1}{2}I_f\dot{q}_1^2\\
&-I_f\dot{q}_2\dot{q}_3+I_f\dot{q}_1\dot{q}_5+M_t\dot{q}_5^2L_f^2-I_f\dot{q}_1\dot{q}_2\\
&+I_f\dot{q}_1\dot{q}_3-I_f\dot{q}_2\dot{q}_5+I_t\dot{q}_3\dot{q}_5-I_t\dot{q}_1\dot{q}_4\\
&+I_t\dot{q}_2\dot{q}_4-I_t\dot{q}_3\dot{q}_4+2M_fL_f^2\dot{q}_3\dot{q}_5\\
&-M_tL_f\dot{q}_2\dot{q}_5L_t\cos(q_1-q_2+q_3)\\
&+M_t\dot{q}_5^2L_tL_f\cos(q_1-q_2+q_3)\\
&+M_t\dot{q}_3L_f\dot{q}_5L_t\cos(q_1-q_2+q_3).
\end{aligned}
$$

The total potential energy of the robot is

$$
\begin{aligned}
V(q) = &-g\left(L_f\cos(q_3+q_5)M_T-L_t\cos(q_5)M_T\right.\\
&-\cos(q_1+q_3+q_5)p_T^MM_T\\
&+2L_f\cos(q_3+q_5)M_f-2L_t\cos(q_5)M_f\\
&-p_f^MM_f\cos(q_3+q_5)\\
&-p_f^MM_f\cos(q_1-q_2+q_3+q_5)\\
&-2L_t\cos(q_5)M_t+p_t^MM_t\cos(q_5)\\
&+L_f\cos(q_3+q_5)M_t\\
&-L_f\cos(q_1-q_2+q_3+q_5)M_t\\
&\left.+p_t^MM_t\cos(-q_1+q_2-q_3+q_4-q_5)\right).
\end{aligned}
$$

IV. Impact Equations

The matrices required to calculate the impact model are

$$
\begin{aligned}
D_{e,1,1}(q_e) &= I_t+I_f+I_T+M_tL_f^2-2p_t^MM_tL_f\cos(q_4)\\
D_{e,1,2}(q_e) &= 2p_t^MM_tL_f\cos(q_4)-I_f-I_t-M_tL_f^2\\
D_{e,1,3}(q_e) &= I_t+I_f+I_T+M_tL_f^2-2p_t^MM_tL_f\cos(q_4)\\
D_{e,1,4}(q_e) &= p_t^MM_tL_f\cos(q_4)-I_t\\
D_{e,1,5}(q_e) &= I_t+I_f+I_T+M_tL_f^2-2p_t^MM_tL_f\cos(q_4)\\
D_{e,1,6}(q_e) &= M_tL_f\cos(q_1-q_2+q_3+q_5)\\
&\quad -p_t^MM_t\cos(q_1-q_2+q_3-q_4+q_5)\\
&\quad +p_f^MM_f\cos(q_1-q_2+q_3+q_5)\\
&\quad +\cos(q_1+q_3+q_5)p_T^MM_T\\
D_{e,1,7}(q_e) &= -M_tL_f\sin(q_1-q_2+q_3+q_5)\\
&\quad +p_t^MM_t\sin(q_1-q_2+q_3-q_4+q_5)\\
&\quad -\sin(q_1+q_3+q_5)p_T^MM_T\\
&\quad -p_f^MM_f\sin(q_1-q_2+q_3+q_5)\\
D_{e,2,1}(q_e) &= 2p_t^MM_tL_f\cos(q_4)-I_f-I_t-M_tL_f^2\\
D_{e,2,2}(q_e) &= -2p_t^MM_tL_f\cos(q_4)+M_tL_f^2+I_t+I_f\\
D_{e,2,3}(q_e) &= 2p_t^MM_tL_f\cos(q_4)-I_f-I_t-M_tL_f^2\\
D_{e,2,4}(q_e) &= I_t-p_t^MM_tL_f\cos(q_4)\\
D_{e,2,5}(q_e) &= 2p_t^MM_tL_f\cos(q_4)-I_f-I_t-M_tL_f^2\\
D_{e,2,6}(q_e) &= p_t^MM_t\cos(q_1-q_2+q_3-q_4+q_5)\\
&\quad -M_tL_f\cos(q_1-q_2+q_3+q_5)\\
&\quad -p_f^MM_f\cos(q_1-q_2+q_3+q_5)\\
D_{e,2,7}(q_e) &= -p_t^MM_t\sin(q_1-q_2+q_3-q_4+q_5)\\
&\quad +M_tL_f\sin(q_1-q_2+q_3+q_5)\\
&\quad +p_f^MM_f\sin(q_1-q_2+q_3+q_5)\\
D_{e,3,1}(q_e) &= I_t+I_f+I_T+M_tL_f^2-2p_t^MM_tL_f\cos(q_4)\\
D_{e,3,2}(q_e) &= 2p_t^MM_tL_f\cos(q_4)-I_f-I_t-M_tL_f^2\\
D_{e,3,3}(q_e) &= -2p_t^MM_tL_f\cos(q_4)+2I_f+I_T+I_t\\
&\quad +2M_tL_f^2\\
D_{e,3,4}(q_e) &= p_t^MM_tL_f\cos(q_4)-I_t\\
D_{e,3,5}(q_e) &= -p_t^MM_tL_f\cos(q_3)-2p_t^MM_tL_f\cos(q_4)\\
&\quad +I_T+2I_f+I_t+2M_tL_f^2\\
D_{e,3,6}(q_e) &= M_tL_f\cos(q_3+q_5)\\
&\quad +M_tL_f\cos(q_1-q_2+q_3+q_5)\\
&\quad -p_t^MM_t\cos(q_1-q_2+q_3-q_4+q_5)\\
&\quad +p_f^MM_f\cos(q_1-q_2+q_3+q_5)\\
&\quad +\cos(q_1+q_3+q_5)p_T^MM_T\\
&\quad +p_f^MM_f\cos(q_3+q_5)\\
D_{e,3,7}(q_e) &= -M_tL_f\sin(q_3+q_5)\\
&\quad -M_tL_f\sin(q_1-q_2+q_3+q_5)\\
&\quad +p_t^MM_t\sin(q_1-q_2+q_3-q_4+q_5)\\
&\quad -\sin(q_1+q_3+q_5)p_T^MM_T\\
&\quad -p_f^MM_f\sin(q_1-q_2+q_3+q_5)\\
&\quad -p_f^MM_f\sin(q_3+q_5)\\
D_{e,4,1}(q_e) &= p_t^MM_tL_f\cos(q_4)-I_t\\
D_{e,4,2}(q_e) &= I_t-p_t^MM_tL_f\cos(q_4)\\
D_{e,4,3}(q_e) &= p_t^MM_tL_f\cos(q_4)-I_t\\
D_{e,4,4}(q_e) &= I_t\\
D_{e,4,5}(q_e) &= p_t^MM_tL_f\cos(q_4)-I_t\\
D_{e,4,6}(q_e) &= p_t^MM_t\cos(q_1-q_2+q_3-q_4+q_5)
\end{aligned}
$$

$$
\begin{aligned}
D_{e,4,7}(q_e) &= -p_t^M M_t \sin(q_1 - q_2 + q_3 - q_4 + q_5)\\
D_{e,5,1}(q_e) &= I_t + I_f + I_T + M_t L_f^2 - 2p_t^M M_t L_f \cos(q_4)\\
D_{e,5,2}(q_e) &= 2p_t^M M_t L_f \cos(q_4) - I_f - I_t - M_t L_f^2\\
D_{e,5,3}(q_e) &= -p_t^M M_t L_f \cos(q_3) - 2p_t^M M_t L_f \cos(q_4)\\
&\quad + I_T + 2I_f + I_t + 2M_t L_f^2\\
D_{e,5,4}(q_e) &= p_t^M M_t L_f \cos(q_4) - I_t\\
D_{e,5,5}(q_e) &= -2p_t^M M_t L_f \cos(q_3) - 2p_t^M M_t L_f \cos(q_4)\\
&\quad + I_T + 2I_f + 2I_t + 2M_t L_f^2\\
D_{e,5,6}(q_e) &= M_t L_f \cos(q_3 + q_5)\\
&\quad + M_t L_f \cos(q_1 - q_2 + q_3 + q_5)\\
&\quad - p_t^M M_t \cos(q_1 - q_2 + q_3 - q_4 + q_5)\\
&\quad + p_f^M M_f \cos(q_1 - q_2 + q_3 + q_5)\\
&\quad + \cos(q_1 + q_3 + q_5)\, p_T^M M_T\\
&\quad + p_f^M M_f \cos(q_3 + q_5) - p_t^M M_t \cos(q_5)\\
D_{e,5,7}(q_e) &= -M_t L_f \sin(q_3 + q_5)\\
&\quad - M_t L_f \sin(q_1 - q_2 + q_3 + q_5)\\
&\quad + p_t^M M_t \sin(q_1 - q_2 + q_3 - q_4 + q_5)\\
&\quad - \sin(q_1 + q_3 + q_5)\, p_T^M M_T\\
&\quad - p_f^M M_f \sin(q_1 - q_2 + q_3 + q_5)\\
&\quad - p_f^M M_f \sin(q_3 + q_5) + p_t^M M_t \sin(q_5)\\
D_{e,6,1}(q_e) &= M_t L_f \cos(q_1 - q_2 + q_3 + q_5)\\
&\quad - p_t^M M_t \cos(q_1 - q_2 + q_3 - q_4 + q_5)\\
&\quad + p_f^M M_f \cos(q_1 - q_2 + q_3 + q_5)\\
&\quad + \cos(q_1 + q_3 + q_5)\, p_T^M M_T\\
D_{e,6,2}(q_e) &= p_t^M M_t \cos(q_1 - q_2 + q_3 - q_4 + q_5)\\
&\quad - M_t L_f \cos(q_1 - q_2 + q_3 + q_5)\\
&\quad - p_f^M M_f \cos(q_1 - q_2 + q_3 + q_5)\\
D_{e,6,3}(q_e) &= M_t L_f \cos(q_3 + q_5)\\
&\quad + M_t L_f \cos(q_1 - q_2 + q_3 + q_5)\\
&\quad - p_t^M M_t \cos(q_1 - q_2 + q_3 - q_4 + q_5)\\
&\quad + p_f^M M_f \cos(q_1 - q_2 + q_3 + q_5)\\
&\quad + \cos(q_1 + q_3 + q_5)\, p_T^M M_T\\
&\quad + p_f^M M_f \cos(q_3 + q_5)\\
D_{e,6,4}(q_e) &= p_t^M M_t \cos(q_1 - q_2 + q_3 - q_4 + q_5)\\
D_{e,6,5}(q_e) &= M_t L_f \cos(q_3 + q_5)\\
&\quad + M_t L_f \cos(q_1 - q_2 + q_3 + q_5)\\
&\quad - p_t^M M_t \cos(q_1 - q_2 + q_3 - q_4 + q_5)\\
&\quad + p_f^M M_f \cos(q_1 - q_2 + q_3 + q_5)\\
&\quad + \cos(q_1 + q_3 + q_5)\, p_T^M M_T\\
&\quad + p_f^M M_f \cos(q_3 + q_5) - p_t^M M_t \cos(q_5)\\
D_{e,6,6}(q_e) &= 2M_f + 2M_t + M_T\\
D_{e,6,7}(q_e) &= 0\\
D_{e,7,1}(q_e) &= -M_t L_f \sin(q_1 - q_2 + q_3 + q_5)\\
&\quad + p_t^M M_t \sin(q_1 - q_2 + q_3 - q_4 + q_5)\\
&\quad - \sin(q_1 + q_3 + q_5)\, p_T^M M_T\\
&\quad - p_f^M M_f \sin(q_1 - q_2 + q_3 + q_5)\\
D_{e,7,2}(q_e) &= -p_t^M M_t \sin(q_1 - q_2 + q_3 - q_4 + q_5)\\
&\quad + M_t L_f \sin(q_1 - q_2 + q_3 + q_5)\\
&\quad + p_f^M M_f \sin(q_1 - q_2 + q_3 + q_5)\\
D_{e,7,3}(q_e) &= -M_t L_f \sin(q_3 + q_5)\\
&\quad - M_t L_f \sin(q_1 - q_2 + q_3 + q_5)\\
&\quad + p_t^M M_t \sin(q_1 - q_2 + q_3 - q_4 + q_5)\\
&\quad - \sin(q_1 + q_3 + q_5)\, p_T^M M_T\\
&\quad - p_f^M M_f \sin(q_1 - q_2 + q_3 + q_5)\\
&\quad - p_f^M M_f \sin(q_3 + q_5)\\
D_{e,7,4}(q_e) &= -p_t^M M_t \sin(q_1 - q_2 + q_3 - q_4 + q_5)\\
D_{e,7,5}(q_e) &= -M_t L_f \sin(q_3 + q_5)\\
&\quad - M_t L_f \sin(q_1 - q_2 + q_3 + q_5)\\
&\quad + p_t^M M_t \sin(q_1 - q_2 + q_3 - q_4 + q_5)\\
&\quad - \sin(q_1 + q_3 + q_5)\, p_T^M M_T\\
&\quad - p_f^M M_f \sin(q_1 - q_2 + q_3 + q_5)\\
&\quad - p_f^M M_f \sin(q_3 + q_5) + p_t^M M_t \sin(q_5)\\
D_{e,7,6}(q_e) &= 0\\
D_{e,7,7}(q_e) &= 2M_f + 2M_t + M_T
\end{aligned}
$$

and

$$
R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0\\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}.
$$

The function $\Upsilon(q)$ is

$$
\Upsilon(q) = \begin{bmatrix} q \\ -L_f \sin(q_3 + q_5) + L_t \sin(q_5) \\ -L_f \cos(q_3 + q_5) + L_t \cos(q_5) \end{bmatrix}.
$$

V. Swing phase zero dynamics

For the following choice of output and $\theta(q)$

$$
\begin{aligned}
y = h(q) &:= Aq - b(\theta(q))\\
\theta(q) &= cq
\end{aligned}
$$

where

$$
b(\theta(q)) := \begin{bmatrix} \bar{b}_1(\theta(q)) \\ \bar{b}_2(\theta(q)) \\ \vdots \\ \bar{b}_{N-1}(\theta(q)) \end{bmatrix},
$$

$$
\bar{\theta}(q) := \frac{\theta(q) - \theta^+}{\theta^- - \theta^+},
$$

and

$$
\bar{b}_i(\theta(q)) := b_i(\bar{\theta}(q))
$$

there associated swing phase zero dynamics are

$$
\begin{aligned}
\dot{\xi}_1 &= \kappa_1(\xi_1)\xi_2\\
\dot{\xi}_2 &= \kappa_2(\xi_1).
\end{aligned}
$$

The functions κ_1 and κ_2 are

$$
\begin{aligned}
\kappa_1(\xi_1) \;=\; & -2\Bigg[-\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}M_TL_f^2+4M_tL_f^2r_1(\xi_1)\\
&+\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}M_TL_t^2-4M_tL_fL_tr_2(\xi_1)\\
&+4M_TL_fL_t\cos\left(\bar{b}_3(\xi_1)\right)\\
&+4M_tL_fL_t\cos\left(\bar{b}_3(\xi_1)\right)\\
&+8M_fL_fL_t\cos\left(\bar{b}_3(\xi_1)\right)-2\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}M_tL_f^2\\
&-2\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}M_fL_f^2+2\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}M_fL_t^2\\
&+2\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}M_tL_t^2+2\frac{\partial\bar{b}_2(\xi_1)}{\partial\xi_1}M_tL_f^2\\
&+2\frac{\partial\bar{b}_4(\xi_1)}{\partial\xi_1}I_t-2\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}I_f-2M_TL_f^2\\
&-2M_TL_t^2+2\frac{\partial\bar{b}_2(\xi_1)}{\partial\xi_1}I_f+2\frac{\partial\bar{b}_2(\xi_1)}{\partial\xi_1}I_t\\
&-4M_tL_f^2-\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}I_T-4M_fL_f^2-4M_fL_t^2\\
&-4M_tL_t^2+2\frac{\partial\bar{b}_2(\xi_1)}{\partial\xi_1}M_tL_fL_tr_2(\xi_1)\\
&-4I_f-4I_t-2I_T-2\frac{\partial\bar{b}_2(\xi_1)}{\partial\xi_1}M_tL_f^2r_1(\xi_1)\\
&+2\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}M_tL_f^2r_1(\xi_1)\\
&-2\frac{\partial\bar{b}_2(\xi_1)}{\partial\xi_1}p_f^MM_fL_fr_1(\xi_1)\\
&+2\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}p_f^MM_fL_fr_1(\xi_1)\\
&+4p_f^MM_fL_f+4p_f^MM_fL_fr_1(\xi_1)\\
&-4p_f^MM_fL_t\cos\left(\bar{b}_3(\xi_1)\right)\\
&-4p_f^MM_fL_tr_2(\xi_1)\\
&+2\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}p_f^MM_fL_f\\
&+2\frac{\partial\bar{b}_2(\xi_1)}{\partial\xi_1}p_f^MM_fL_tr_2(\xi_1)\\
&-2\frac{\partial\bar{b}_2(\xi_1)}{\partial\xi_1}p_t^MM_tL_tr_5(\xi_1)\\
&+\Bigg[2\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}M_TL_f\cos\left(\bar{b}_1(\xi_1)\right)\\
&+\left(-2M_TL_tr_4(\xi_1)\right.\\
&\left.+2M_TL_f\cos\left(\bar{b}_1(\xi_1)\right)\right)\frac{\partial\bar{b}_1(\xi_1)}{\partial\xi_1}\\
&+4M_TL_f\cos\left(\bar{b}_1(\xi_1)\right)\\
&-4M_TL_tr_4(\xi_1)\Bigg]p_T^M\\
&-4\frac{\partial\bar{b}_2(\xi_1)}{\partial\xi_1}p_t^MM_tL_f\cos\left(\bar{b}_4(\xi_1)\right)\\
&-2\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}p_t^MM_tL_fr_3(\xi_1)\\
&+2\frac{\partial\bar{b}_4(\xi_1)}{\partial\xi_1}p_t^MM_tL_fr_3(\xi_1)\\
&+2\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}p_t^MM_tL_f\cos\left(\bar{b}_4(\xi_1)\right)\\
&-2\frac{\partial\bar{b}_4(\xi_1)}{\partial\xi_1}p_t^MM_tL_f\cos\left(\bar{b}_4(\xi_1)\right)\\
&+2\frac{\partial\bar{b}_2(\xi_1)}{\partial\xi_1}p_t^MM_tL_fr_3(\xi_1)\\
&-2\frac{\partial\bar{b}_4(\xi_1)}{\partial\xi_1}p_t^MM_tL_tr_5(\xi_1)\\
&+4p_t^MM_tL_f\cos\left(\bar{b}_4(\xi_1)\right)\\
&-4p_t^MM_tL_fr_3(\xi_1)+4p_t^MM_tL_tr_5(\xi_1)\\
&-2\frac{\partial\bar{b}_3(\xi_1)}{\partial\xi_1}p_t^MM_tL_t+4p_t^MM_tL_t\\
&+\Big[4p_t^MM_tL_f\cos\left(\bar{b}_4(\xi_1)\right)-2I_T-2I_f\\
&+2p_t^MM_tL_tr_5(\xi_1)-2I_t-2M_tL_f^2\\
&-2M_tL_fL_tr_2(\xi_1)-2p_f^MM_fL_tr_2(\xi_1)\\
&-2p_t^MM_tL_fr_3(\xi_1)+2M_tL_f^2r_1(\xi_1)\\
&+2p_f^MM_fL_fr_1(\xi_1)\Big]\frac{\partial\bar{b}_1(\xi_1)}{\partial\xi_1}\Bigg]^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\kappa_2(\xi_1) \;=\; & -g\left[L_fr_6(\xi_1)M_T-L_tr_7(\xi_1)M_T\right.\\
&-\sin\left(\bar{b}_1(\xi_1)+\frac{1}{2}\bar{b}_3(\xi_1)+\xi_1\right)p_T^MM_T\\
&+2L_fr_6(\xi_1)M_f-2L_tr_7(\xi_1)M_f\\
&-p_f^MM_fr_6(\xi_1)-p_f^MM_fr_8(\xi_1)\\
&-2L_tr_7(\xi_1)M_t+p_t^MM_tr_7(\xi_1)\\
&+L_fr_6(\xi_1)M_t-L_fr_8(\xi_1)M_t\\
&+p_t^MM_t\sin\Bigg(\bar{b}_1(\xi_1)-\bar{b}_2(\xi_1)\\
&\left.+\frac{1}{2}\bar{b}_3(\xi_1)-\bar{b}_4(\xi_1)+\xi_1\Bigg)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
r_1(\xi_1) &= \cos\left(\bar{b}_1(\xi_1)-\bar{b}_2(\xi_1)\right)\\
r_2(\xi_1) &= \cos\left(\bar{b}_1(\xi_1)-\bar{b}_2(\xi_1)+\bar{b}_3(\xi_1)\right)\\
r_3(\xi_1) &= \cos\left(\bar{b}_1(\xi_1)-\bar{b}_2(\xi_1)-\bar{b}_4(\xi_1)\right)\\
r_4(\xi_1) &= \cos\left(\bar{b}_1(\xi_1)+\bar{b}_3(\xi_1)\right)\\
r_5(\xi_1) &= \cos\left(\bar{b}_1(\xi_1)-\bar{b}_2(\xi_1)+\right.\\
&\qquad\left.\bar{b}_3(\xi_1)-\bar{b}_4(\xi_1)\right)\\
r_6(\xi_1) &= \sin\left(\frac{1}{2}\bar{b}_3(\xi_1)+\xi_1\right)\\
r_7(\xi_1) &= \sin\left(-\frac{1}{2}\bar{b}_3(\xi_1)+\xi_1\right)
\end{aligned}
$$

$$r_8(\xi_1) = \sin\Big(\bar{b}_1(\xi_1) - \bar{b}_2(\xi_1) + \frac{1}{2}\bar{b}_3(\xi_1) + \xi_1\Big).$$