Balanced Network Flows. VI. Polyhedral Descriptions

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This paper discusses the balanced circulation polytope, that is, the convex hull of balanced circulations of a given balanced flow network. The LP description of this polytope is the LP description of ordinary circulations plus some odd-set constraints. The paper starts with an exposition of several classes of odd-set inequalities. These inequalities are described in terms of balanced network flows as well as matchings and put into relation to each other. Step by step, the problem of finding a cost minimum balanced circulation can be reduced to the b-matching problem. We present an LP characterization of the b-matching polytope by blossom inequalities. With a moderate effort, these odd sets are lifted to the setting of balanced-network flows. We finish with the dualization of the derived LP formulation, an introduction of reduced-cost labels, and a corresponding optimality condition. © 2001 John Wiley & Sons, Inc.

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30. PRELIMINARIES

This paper continues our discussion of balanced network flows which can be viewed as a network-flow description of matching problems. In [4, 5, 6], we presented the algorithmic concepts available for nonweighted problems. In [7], one can find a duality theory for nonweighted matching problems which does not use polyhedral descriptions.

Part V [8] of this series discusses the relationships between ordinary flows, fractional balanced flows, and (integral!) balanced flows of a balanced flow network N. The outcome was an algorithm which applies as a starting heuristic to algorithms for nonweighted as well as weighted problems. That paper gave a first glimpse of polyhedral descriptions. In particular, we discussed the polyhedron $\mathscr{F}(N)$ of fractional balanced circulations, which is defined by the constraints

$$\begin{aligned} &(p1a) && lower(a) \le f(a) && \text{for all arcs } a \in A(N),\\ &(p1b) && f(a) \le cap(a) && \text{for all arcs } a \in A(N),\\ &(p2) && f(a) = f(a') && \text{for all arcs } a \in A(N),\\ &(p3) && e(v) = 0 && \text{for all nodes } v \in V(N). \end{aligned}$$

We do not repeat all of the basic definitions given in the previous parts, but only the notion of pseudobasic circulations, which is essential. So, let f be some fractional balanced circulation on N.

If $2f$ is integral, then f is called **half-integral**. An arc $a \in A(N)$ is called **free** if and only if $rescap_f(a)$ and also $rescap_f(\bar{a})$ are strictly positive. A **free** path is a path that entirely consists of free arcs.

We call a cycle in N **odd** if and only if it is simple and contains arcs a and $\bar{a}'$ always pairwise. An odd-cycle Q can be written as $Q = q \circ \bar{q}'$, where q is a strictly simple vv'-path and $v \in Q$ is arbitrary.

In the previous parts, we considered bipartite balanced networks. Then, the path q indeed has odd length. Later, we will show that the bipartiteness requirement is immaterial.

We call a fractional balanced flow f **pseudobasic** if and only if f is half-integral, and the fractional arcs form pairwise disjoint odd cycles. We refer to these cycles $Q_1, Q_2, \dots, Q_r$ as the **odd cycle system** associated with f.

In [8], we showed that f is a vertex of the polytope $\mathscr{F}(N)$ if and only if every free cycle in $N(f)$ is odd. Thus, any vertex of the polytope $\mathscr{F}(N)$ is pseudobasic.

As an example, consider the balanced network given in Figure 1 with unit capacities. The reader may verify that exactly three pseudobasic circulations exist: f_0, the zero flow, $f_1 := \frac{1}{2}\chi^p$, where $p := (u, u', v, v', u)$, and $f_2 := \chi^q$, where $q := (u, u', v', u)$.

Note that f_0 and f_2 are integral, but f_1 is not. It turns out that f_0, f_1 and f_2 are the vertices of $\mathscr{F}(N)$.

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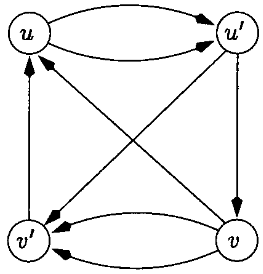

FIG. 1. A balanced-flow network.

31. ODD-SET INEQUALITIES

We start the characterization of balanced circulation polytopes by specifying some sets of feasible (and partially redundant) inequalities.

Let $V(N) = U \uplus \overline{U}$ and $U = U'$. We call the node set U **nontrivial** if and only if $N[U]$ as well as $N[\overline{U}]$ contain at least one odd cycle and if $N[U]$ is connected. An edge set $\tilde{A} \subseteq A(N)$ is called a **nontrivial edge cut** if one can partition $V(N) = U \uplus \overline{U}$ so that $U = U'$, $\tilde{A} = [U, \overline{U}]$, and U is nontrivial.

Our aim is to find appropriate cuts which separate the odd cycles of a pseudobasic flow and, by that, to separate this pseudobasic flow from the integral solutions. The first statement is a special case of Lemma 27.6 in [8]:

Corollary 31.1. *Let N be a balanced flow network, f be a balanced circulation on N, and $V(N) = U \uplus \overline{U}$, $U = U'$. Then, $f(U, \overline{U})$ is even.*

Corollary 31.2. (Cut Inequalities).
Let

$$\begin{aligned} \mathscr{U}_{cut}(N) &:= \{U : V(N) = U \uplus \overline{U},\ U = U', \\ &\qquad U \textit{ nontrivial},\ cap(U, \overline{U}) \textit{ odd}\}, \\ \mathscr{L}_{cut}(N) &:= \{U : V(N) = U \uplus \overline{U},\ U = U', \\ &\qquad U \textit{ nontrivial},\ lower(U, \overline{U}) \textit{ odd}\}. \end{aligned}$$

Every balanced circulation satisfies the conditions

$$\begin{array}{lll} (p4a) & f(U, \overline{U}) \le cap(U, \overline{U}) - 1 & \textit{for } U \in \mathscr{U}_{cut}(N) \\ (p4b) & f(U, \overline{U}) \ge lower(U, \overline{U}) + 1 & \textit{for } U \in \mathscr{L}_{cut}(N). \end{array}$$

□

If one has $N = N_{\mathscr{M}}$, where $\mathscr{M}$ is an instance of the perfect 1-matching problem, the restrictions (p4b) are the well-known cut constraints for the perfect matching polytope.

Unfortunately, N admits no odd cuts in our introductory example, that is, the circulation f_1 cannot be separated by odd-cut inequalities.

It turns out that (p4a) and (p4b) appear as extreme cases of a more general set of inequalities:

Corollary 31.3. (Skew-cut Inequalities).
For partitions $V(N) = U \uplus \overline{U}$, $U = U'$ and $[U, \overline{U}] = A_1 \uplus A_2$, let

$$scap(A_1, A_2) := lower(A_2) - cap(A_1).$$

Denote

$$\begin{aligned} \mathscr{O}_{skew}(N) &:= \{(A_1, A_2) : \\ &\qquad A_1 \uplus A_2 \textit{ nontrivial},\ scap(A_1, A_2) \textit{ odd}\}. \end{aligned}$$

Every balanced circulation satisfies the conditions (p4c):

$$f(A_2) - f(A_1) \ge scap(A_1, A_2) + 1 \qquad \textit{for } (A_1, A_2) \in \mathscr{O}_{skew}(N).$$

Proof: Observe that $f(A_2) - f(A_1) = f(U, \overline{U}) - 2f(A_1)$, which is even by Corollary 31.1. □

If a skew cut satisfies (p4c) with equality, we say it is **tight** with respect to the circulation f.

The polytope which is defined by the constraints (p1a), (p1b), (p2), (p3), and (p4c) is called the **balanced circulation polytope** and denoted by $\mathscr{P}(N)$. We will show that $\mathscr{P}(N)$ is the convex hull of all balanced circulations. In the remainder of this section, we will discuss the odd-set constraints which are relevant for particular matching problems.

Let N be bipartite and $W \subseteq Outer(N)$. If not stated otherwise, we associate with W the sets $U := W \uplus W'$ and $\overline{U} := V(N) \setminus U$. We say that W is **nontrivial** if and only if U is nontrivial.

Lemma 31.4. (Comb Inequalities).
Let $W \subseteq Outer(N)$ be nontrivial. For partitions $[W, \overline{U}] = A_1 \uplus A_2$, let

$$\zeta(W, A_1) := cap(\overline{U}, W) + cap(A_1) - lower(A_2).$$

Denote

$$\begin{aligned} \mathscr{O}_{comb}(N) &:= \{(W, A_1) : W \textit{ nontrivial},\ A_1 \subseteq [W, \overline{U}], \\ &\qquad \zeta(W, A_1) \textit{ odd}\}. \end{aligned}$$

Every balanced circulation satisfies the conditions (p4d):

$$f(W, W') + 2f(A_1) \le cap(W', W) + \zeta(W, A_1) - 1 \qquad \textit{for } (W, A_1) \in \mathscr{O}_{comb}(N).$$

Proof: Adding the flow-conservation conditions for the nodes in W yields

$$\begin{aligned} & f(W, W') + 2f(A_1) \\ &\quad = f(W', W) + f(\overline{U}, W) - f(W, \overline{U}) + 2f(A_1) \\ &\quad = f(W', W) + f(\overline{U}, W) + f(A_1) - f(A_2). \end{aligned}$$

Observe that $f(W, W')$ and $cap(W', W)$ are even. Hence, we have

$$\begin{aligned} & f(W, W') + 2f(A_1) \\ &\quad \le cap(W', W) + cap(\overline{U}, W) + cap(A_1) - lower(A_2), \end{aligned}$$

and the inequality is strict if $\zeta(W, A_1)$ is odd. □

This set of inequalities is a generalization of the facet-generating constraints for the 2-factor polytope where the arcs in A_1 are **teeth** with pairwise different end nodes and W is called the **handle**. These 2-factor comb constraints also have been embedded into a powerful set of inequalities for the TSP (see [3]). Letting $A_1 = \emptyset$ in Lemma 31.4, we obtain the following important special case:

Corollary 31.5. (Blossom Inequalities). *For $W \subseteq Outer(N)$ nontrivial, let $\theta(W) := cap(\overline{U}, W) - lower(W, \overline{U})$. Denote*

$$\mathcal{O}_{blossom}(N) := \{W \subseteq V(N):\ W \text{ nontrivial},\ \theta(W) \text{ odd}\}.$$

Every balanced circulation satisfies the conditions (p4e):

$$f(W, W') \leq cap(W', W) + \theta(W) - 1 \quad \text{for } W \in \mathcal{O}_{blossom}(N).$$

Let $\mathcal{M}$ be a subgraph network with degree sequences a, b, underlying graph G, and $N = N_{\mathcal{M}}$. In what follows, we need some notation which is familiar in matching theory (Lovasz and Plummer [11] is our reference):

By $\gamma_G(W)$, we denote all arcs with both end nodes in $W \subseteq V(G)$. By $\delta_G(W)$, we denote all arcs in $E(G)$ with exactly one end node in W. Since G is obvious here, we will omit the subscript.

Partition $V(G) = W \uplus \overline{W}$, $W = W_1 \uplus W_2$ and $\gamma(W) = E_1 \uplus E_2$. Let A_1, A_2 denote the images of the arc sets E_1, E_2 under the construction of $N_{\mathcal{M}}$.

Using this notation, the inequalities (p4a)–(p4e) become

$$\begin{aligned}
&(m4a)\quad deg_x(W) + x(\delta(W)) \leq c(\delta(W)) + b(W) - 1\\
&(m4b)\quad deg_x(W) + x(\delta(W)) \geq a(W) + 1\\
&(m4c)\quad deg_x(W_2) - deg_x(W_1) + x(E_2) - x(E_1)\\
&\qquad\qquad \geq a(W_2) - b(W_1) - c(E_1) + 1\\
&(m4d)\quad 2x(\gamma(W)) + 2x(E_1) \leq b(W) + c(E_1) - 1\\
&(m4e)\quad 2x(\gamma(W)) \leq b(W) - 1.
\end{aligned}$$

Note that W is nontrivial if $G(\mathcal{M})[W]$ is connected and contains an odd-length cycle. The subgraph $G(\mathcal{M})[V(\mathcal{M}) \setminus W]$ may be bipartite! The explicit translation of the respective odd sets for (m4a)–(m4e) is left to the reader.

Lemma 31.6. *Let $\mathcal{M}$ be a subgraph network and $N = N_{\mathcal{M}}$. All the constraints (p4a)–(p4e) with $t \in U$ are simultaneously redundant.*

Proof: In the case of (p4a)–(p4c), we can exchange U with $\overline{U}$, and A_1 with A_1' to obtain an equivalent inequality. The constraints (p4e) specialize the constraints (p4d) which are discussed in what follows:

Let $V(\mathcal{M}) = W \uplus \overline{W}$, $U := W \uplus W'$, $\overline{U} = \overline{W} \uplus \overline{W}' \uplus \{s, t\}$, and $A_1 \subseteq [W, \overline{W}']$. Note that $f(W', W) = cap(W', W) = 0$. The reader may check that

$$\zeta(W, A_1) = b(W) + c(E_1) = \zeta(\overline{W} \uplus \{t\}, A_1').$$

The comb inequality formulated for (W, A_1) is

$$f(W, W') + 2f(A_1) \leq b(W) + c(E_1) - 1. \tag{1}$$

Adding the flow-conservation equalities for the nodes in W and $\overline{W}$ yields

$$f(W, \overline{W}') = f(s, W) - f(W, W') \geq a(W) - f(W, W'), \tag{2}$$

and, respectively, $f(\overline{W}, W') = f(s, \overline{W}) - f(\overline{W}, \overline{W}')$, and, hence,

$$f(W, W') = f(\overline{W}, \overline{W}') - f(s, \overline{W}) + f(s, W). \tag{3}$$

If we apply Eqs. (3), $f(ts) = f(s, \overline{W}) + f(s, W)$, $f(A_1) = f(A_1')$, and the inequality $f(s, \overline{W}) \leq b(\overline{W})$, we obtain

$$f(\overline{W}, \overline{W}') + f(ts) + 2f(A_1') \leq 2b(\overline{W}) + b(W) + c(E_1) - 1, \tag{4}$$

which is the redundant (!) comb inequality formulated for $(\overline{W} \uplus \{t\}, A_1')$. □

It follows that (p4b) and (m4b), (p4c) and (m4c), (p4d) and (m4d), as well as (p4e) and (m4e) are equivalent.

If we let $W_2 = \emptyset$ and replace $deg_x(W) = 2x(\gamma(W)) + x(\delta(W))$, we obtain the constraints (m4d) as a subset of (m4c). On the other hand, we observe that the constraints

$$(m4f)\quad deg_x(W) + x(E_2) - x(E_1) \geq a(W) - c(E_1) + 1$$

can be obtained from (m4c) by letting $W_1 = \emptyset$.

We show by an example that a fractional factor may be separated by the constraints (m4f) where the constraints (m4d) fail. This example eventually shows that (p4d) is less restrictive than is (p4c):

Let $\mathcal{M}$ be defined on the complete graph with node set $\{1, 2, 3, 4\}$. Let $c \equiv 1$, $b(1) = b(3) = b(4) = a(2) = a(3) = a(4) = 2$, $b(2) = 3$ and $a(1) = 1$. With small effort, the reader may check that the fractional factor $x_{12} = x_{23} = x_{13} = \frac{1}{2}$, $x_{24} = x_{34} = 1$, $x_{14} = 0$ is not a linear combination of (integral) factors.

This solution is separated by the inequality $deg_x(V) \geq a(V) + 1 = 8$ which is among (m4f). A careful inspection would show that all odd-comb constraints are satisfied. The details are left to the reader.

With a slight modification of this example, one can show that (m4f) is also less restrictive than is (m4c): Put $c(e_{12}) := 2$ and $x(e_{12}) := \frac{3}{2}$. This fractional factor is separated by the blossom inequality $deg_x(V) \leq b(V) - 1 = 8$. On the other hand, (m4f) is satisfied.

Theorem 31.7. *Let $\mathscr{M}$ be a subgraph network with degree sequences $a \equiv b$, $b(V)$ even and $N = N_{\mathscr{M}}$. Then, the inequalities (m4b) and (m4e) are equivalent and the inequalities (m4f) and (m4d) are equivalent.*

Proof: Let $V(\mathscr{M}) = W \uplus \overline{W}$, $U = W \uplus W'$, $\overline{U} = \overline{W} \uplus \overline{W}' \uplus \{s,t\}$, and $A_1 \subseteq [W, \overline{W}']$. We have

$$\begin{array}{rcll} lower(U,\overline{U}) = a(W) & \equiv & b(W) = \theta(W) & \mod 2, \\ scap(A_1,A_2) = a(W) - c(E_1) & \equiv & c(E_1) + b(W) = \zeta(W,A_1) & \mod 2. \end{array}$$

Hence, $W \in \mathscr{O}_{blossom}(N)$ if and only if $W \in \mathscr{L}_{cut}(N)$, and $(W, A_1) \in \mathscr{O}_{comb}(N)$ if and only if $(A_1, A_2) \in \mathscr{O}_{skew}(N)$. By Eq. (2), we obtain

$$f(U,\overline{U}') = f(W',t) + f(W,\overline{W}') = 2a(W) - f(W,W') \quad (5)$$

and, hence,

$$\begin{array}{rll} & f(U,\overline{U}') \ge a(W) + 1 & (p4b) \\ \iff & f(W,W') \le b(W) - 1 & (p4e) \end{array}$$

and

$$\begin{array}{rll} & f(U,\overline{U}') - 2f(A_1) \ge a(W) - c(E_1) + 1 & (p4c) \\ \iff & f(W,W') + 2f(A_1) \le a(W) + c(E_1) - 1. & (p4d) \end{array}$$

□

By the main theorem, it will turn out that the inequalities (m4f) and (m4c) are likewise equivalent.

32. MATCHING POLYTOPES

As a preparation for the general setting, we derive complete characterizations of b-matching problems. The polyhedral results are well known ([2, 12]), but worth a second reading from this perspective.

First, let $\mathscr{M}$ be an instance of the perfect b-matching problem. The **perfect b-matching polytope** $\mathscr{P}(G,b)$ is defined by the constraints

$$\begin{array}{lll} (m1a) & x(e) \ge 0 & \text{for } e \in E(\mathscr{M}), \\ (m3) & x(\delta(v)) = b(v) & \text{for } v \in V(\mathscr{M}), \\ (m4b) & x(\delta(W)) \ge 1 & \text{for } W \in \mathscr{O}_{cut}(\mathscr{M}), \end{array}$$

where

$$\mathscr{O}_{cut}(\mathscr{M}) \;:=\; \{W \subseteq V(\mathscr{M}) : \; W \text{ nontrivial, } b(W) \text{ odd}\}.$$

By Lemma 31.6 and Theorem 31.7, it is evident that a vertex of $\mathscr{P}(G,b)$ corresponds to a vertex of the polytope which is defined by the constraints (p1a), (p1b), (p2), (p3), and (p4b).

Theorem 32.1. *Let $\mathscr{M}$ be an instance of the perfect b-matching problem with a nonintegral vertex x in $\mathscr{P}(N_{\mathscr{M}})$. Let $\mathscr{M}$ be chosen with $|V(\mathscr{M})| + |E(\mathscr{M})|$ minimum. Then, no cut constraint (m4b) is tight at x.*

Proof: Let $V(\mathscr{M}) = W \uplus \overline{W}$, $U = W \uplus W'$, $\overline{U} = \overline{W} \uplus \overline{W}' \uplus \{s,t\}$, and f be the flow on $N_{\mathscr{M}}$ corresponding to x, and assume that $f(U,\overline{U}) = lower(U,\overline{U}) + 1$ is even. By Eq. (5), we have that $f(W,W')$ and $f(\overline{W},\overline{W}')$ are also even.

We contract the node sets W, W' to nodes w and w' as follows: Identify the nodes $W \sim w$ and put $a(w), b(w) := a(W) - f(W,W')$ for the new node w. All arcs with both end nodes in W are deleted, whereas the other arcs incident with W may be present as parallel arcs in the resulting instance $\mathscr{M}_1$. We consider the flow f_1 on $N_1 = N_{\mathscr{M}_1}$ defined by

$$\begin{array}{rcl} f_1(sw) & = & f(s,W) - f(W,W'), \\ f_1(w't) & = & f(W',t) - f(W,W'), \\ f_1(ts) & = & f(ts) - f(W,W'), \end{array}$$

and $f_1(a) = f(a)$ for the other arcs of $N_1 = N_{\mathscr{M}_1}$. We contract $\overline{W}$ likewise to obtain the subgraph network $\mathscr{M}_2$, the balanced-flow network $N_2 = N_{\mathscr{M}_2}$, and a respective flow f_2. Since U is nontrivial, the instance size strictly decreases for $\mathscr{M}_1$ and $\mathscr{M}_2$.

Observe that the odd cuts in $\mathscr{M}_1$ and $\mathscr{M}_2$ correspond to odd cuts in the original subgraph network $\mathscr{M}$. Hence, f_1 is feasible for $\mathscr{P}(\mathscr{M}_1)$ and f_2 is feasible for $\mathscr{P}(\mathscr{M}_2)$. By the minimality of $\mathscr{M}$, we may write

$$f_1 = \sum_{i=1}^{r} \mu_i g_i, \; \sum_{i=1}^{r} \mu_i = 1, \; \mu_i > 0,$$

$$f_2 = \sum_{i=1}^{s} \nu_i h_i, \; \sum_{i=1}^{s} \nu_i = 1, \; \nu_i > 0,$$

where $g_1, g_2, \ldots, g_r$ are balanced circulations on N_1 and $h_1, h_2, \ldots, h_s$ are balanced circulations on N_2. Note that f, f_1, f_2, and all of the coefficients are rational and choose a common denominator M. We can rewrite

$$Mf_1 = \sum_{i=1}^{M} \tilde{g}_i, \; Mf_2 = \sum_{i=1}^{M} \tilde{h}_i,$$

where the $\tilde{g}_i$'s, $\tilde{h}_i$'s are the g_i's, h_i's with possible repetitions. Furthermore,

$$f(U,\overline{U}) = lower(U,\overline{U}) + 1$$

implies that

$$\tilde{g}_i(U,\overline{U}) = lower(U,\overline{U}) + 1,$$

$$\tilde{h}_i(U,\overline{U}) = lower(U,\overline{U}) + 1,$$

that is, for a given i, there is a unique arc a_i with $\tilde{g}_i(a_i) = lower(a_i) + 1$ and $\tilde{g}_i(a) = lower(a)$ for the other arcs $a \in [U,\overline{U}]$. Since $f_1 \equiv f_2$ on the arcs in $[U,\overline{U}]$, we can reorder the $\tilde{h}_i$'s so that $\tilde{h}_i(a_i) = lower(a_i) + 1$ and $\tilde{h}_i(a) = lower(a)$ for the other arcs $a \in [U,\overline{U}]$. We obtain

$$\begin{aligned} f(a) &= \left\{ \begin{array}{l} f_1(a), \text{ if } a \in N[U] \\ f_2(a), \text{ if } a \notin N[U] \end{array} \right\} \\ &= \frac{1}{M} \left\{ \begin{array}{l} \sum_{i=1}^{M} \tilde{g}_i(a), \; a \in N[U] \\ \sum_{i=1}^{M} \tilde{h}_i(a), \; a \notin N[U] \end{array} \right\} \\ &= \frac{1}{M} \sum_{i=1}^{M} \tilde{f}_i(a), \end{aligned}$$

where
$$\tilde{f}_i(a) = \left\{ \begin{array}{ll} \tilde{g}_i(a), & \text{if } a \in N[U] \\ \tilde{h}_i(a), & \text{if } a \notin N[U] \end{array} \right\}.$$

By the reordering of the $\tilde{h}_i$'s, the f_i's satisfy the flow-conservation equalities and, hence, are balanced circulations on the network N. By Corollary 0, the blossom inequalities are satisfied so that $\tilde{f}_i \in \mathscr{P}(N)$, the final contradiction. □

This "gluing" technique is due to Schrijver [13]. We now merely need evidence that a tight cut constraint exists. The argument given in Schrijver [13] can be replaced by some separation rule for pseudobasic solutions.

Corollary 32.2. *The vertices of the perfect b-matching polytope $\mathscr{P}(G, b)$ are integral.*

Proof: Let $\mathscr{M}$ be an instance for the perfect b-matching problem with graph G and degree sequence b. Suppose that x is a nonintegral vertex of the polytope $\mathscr{P}(G, b)$ and that there is no smaller instance with nonintegral vertices. Let f be the flow on $N_{\mathscr{M}}$ corresponding to x.

Then, by Theorem 32.1 and Lemma 31.6, no cut inequality and no blossom inequality is tight. As in the proof of Theorem 27.1 in [8], it turns out that f is pseudobasic. By Corollary 27.7 in [8], f has at least two odd cycles which obviously do not traverse the nodes s, t.

Let Q be an arbitrary odd cycle Q. Let $W \subseteq V(\mathscr{M})$ be the set of nodes which can be reached from Q by free arcs. Note that W does not meet an odd cycle other than Q, since, otherwise, f would not be pseudobasic.

Denote $\overline{W} = V(\mathscr{M}) \setminus W$. Since the nodes in W' can also be reached from Q by free arcs, and since $f(uv') < cap(uv') = \infty$ for all $u \in W$, $v \in \overline{W}$, we have that $f(W, \overline{W}') = 0$. But, then, $f(W, W') = b(W)$ is odd. Since $\overline{W}$ is met by an odd cycle other than Q, W is nontrivial and f violates the corresponding blossom inequality, a contradiction. □

Next, let $\mathscr{M}$ be an instance of the perfect c-capacitated b-matching problem. The **perfect c-capacitated b-matching polytope** $\mathscr{P}(G, b, c)$ is defined by the constraints

$(m1a)$	$x(e) \geq 0$	for $e \in E(\mathscr{M})$,
$(m1b)$	$x(e) \leq c(e)$	for $e \in E(\mathscr{M})$,
$(m3)$	$x(\delta(v)) = b(v)$	for $v \in V(\mathscr{M})$,
$(m4d)$	$x(\delta(W)) + c(E_1) - 2x(E_1) \geq 1$	for $(W, E_1) \in \mathscr{O}_{comb}(\mathscr{M})$,

where
$$\begin{aligned} \mathscr{O}_{comb}(\mathscr{M}) := \{ & (W, E_1) : W \text{ nontrivial}, \\ & E_1 \subseteq \delta(W),\ b(W) + c(E_1) \text{ odd} \}. \end{aligned}$$

Again, it is evident that the bijection from the fractional matchings of $\mathscr{M}$ onto the fractional balanced circulations on $N_{\mathscr{M}}$ maps $\mathscr{P}(G, b, c)$ to the polytope, which is defined by the constraints (p1a), (p1b), (p2), (p3), and (p4d).

Theorem 32.3. *The vertices of the perfect c-capacitated b-matching polytope $\mathscr{P}(G, b, c)$ are integral.*

Proof: Construct an instance $\tilde{\mathscr{M}}$ of the perfect b-matching problem as follows: Replace every edge $e = \{u, v\} \in E(\mathscr{M})$ by the three edges $e_1 = \{v_e^0, v_e^1\}$, $e_2 = \{v_e^1, v_e^2\}$, $e_3 = \{v_e^2, v_e^3\}$, where $v_e^0 := u$, $v_e^3 := v$, and v_e^1, v_e^2 are new nodes with $b(v_e^1), b(v_e^2) := c(e)$. A factor y of $\tilde{\mathscr{M}}$ turns into a factor x of $\mathscr{M}$ if we put $x(e) := y(e_1)$.

To see this, note that $y(e_1) = c(e) - y(e_2) = y(e_3)$ holds for every fractional factor y of $\tilde{\mathscr{M}}$. Hence, the transformation of the factors is affine and bijective and preserves the polyhedral geometry. We merely need to translate the odd set constraints from $\tilde{\mathscr{M}}$ to $\mathscr{M}$.

Let $\tilde{W} \subseteq V(\tilde{\mathscr{M}})$ so that $b(\tilde{W})$ is odd. Denote $W := \tilde{W} \cap V(\mathscr{M})$,

$$E_1 := \{e \in E(\mathscr{M}) :\ e_2 \in \delta(\tilde{W}), e_1, e_3 \notin \delta(\tilde{W})\},$$

and $E_2 := \delta(W) \setminus E_1$. Let $e \in E(\mathscr{M})$. Note that $e \in \delta(W)$ if and only if one of

(a) $e_2, e_1, e_3 \in \delta(\tilde{W})$,	(b) $e_2 \in \delta(\tilde{W}), e_1, e_3 \notin \delta(\tilde{W})$,
(c) $e_3 \in \delta(\tilde{W}), e_1, e_2 \notin \delta(\tilde{W})$,	(d) $e_1 \in \delta(\tilde{W}), e_2, e_3 \notin \delta(\tilde{W})$

is true, and $e \notin \delta(W)$ if and only if one of

(e) $e_2, e_1, e_3 \notin \delta(\tilde{W})$,	(f) $e_2 \notin \delta(\tilde{W}), e_1, e_3 \in \delta(\tilde{W})$,
(g) $e_3 \notin \delta(\tilde{W}), e_1, e_2 \in \delta(\tilde{W})$,	(h) $e_1 \notin \delta(\tilde{W}), e_2, e_3 \in \delta(\tilde{W})$

is true. If (a), (g), or (h) is true, then $\tilde{x}(\delta(\tilde{W})) \geq c(e) \geq 1$ holds for any fractional b-matching of $\tilde{\mathscr{M}}$. If (d) is true, we put $\tilde{W} := \tilde{W} \oplus \{v_e^1, v_e^2\}$ to obtain an equivalent constraint which satisfies (c). If (f) is true, then $\tilde{W} := \tilde{W} \oplus \{v_e^1, v_e^2\}$ decreases $\tilde{x}(\delta(\tilde{W}))$. In all these cases, $\tilde{W}$ would be redundant. Hence, we can assume (b), (c), or (e) for every arc $e \in \delta(W)$. Then, (m4b) is equivalent with

$$1 \leq \tilde{x}(\delta(\tilde{W})) = \sum_{e \in E_1} \tilde{x}(e_2) + \sum_{e \in E_2} \tilde{x}(e_3) = x(E_2) + c(E_1) - x(E_1)$$

and

$$b(\tilde{W}) = b(W) + 2c(E_2) + c(E_1) \equiv b(W) + c(E_1) \mod 2.$$ □

33. PROBLEM EQUIVALENCE

So far, we utilized the reduction of matching problems to balanced network-flow problems. We point out a reduction mechanism which works in the opposite direction and which is similar to the reduction of bidirected flows ([2, 1]):

If N is nonbipartite, we must specify a partition $V(N) = Inner(N) \uplus Outer(N)$ so that $Outer(N)' = Inner(N)$. In that case, the shown reduction is not unique!

For every node $v \in Inner(N)$, we put

$$\begin{aligned} cap^+(v) &:= \sum_{a^+ = v} cap(a), \\ cap^-(v) &:= \sum_{a^- = v} cap(a), \\ lower^+(v) &:= \sum_{a^+ = v} lower(a), \end{aligned}$$

$$lower^-(v) := \sum_{a^-=v} lower(a),$$
$$cap(v) := cap^+(v) + cap^-(v).$$

Construct an instance $\mathcal{M}(N)$ of the capacitated b-matching problem as follows: A node pair $v \in Inner(N)$, $v' \in Outer(N)$ is mapped to a pair of nodes v^+, v^- which are joined by an arc e_v.

A complementary arc pair $a = uv'$, $a' = vu'$ is mapped to a single edge e_a. This edge is incident with u^- $[v^-]$ if and only if $u \in Inner(N)$ $[v \in Inner(N)]$ and with u^+ $[v^+]$ otherwise. For certain integers $K(v) \geq cap(v)$, assign

$$\begin{aligned} b(v^+) &:= K(v) - lower^+(v), \\ b(v^-) &:= K(v) - lower^-(v), \\ c(e_v) &:= \infty, \\ c(e_a) &:= cap(a) - lower(a). \end{aligned}$$

Actually, we chose $K(v)$ so that $b(v^-)$ becomes an even number.

If we let $Inner(N) = \{u, v\}$ in the introductory example, Figure 2 depicts the graph for the resulting capacitated matching problem. The reader is asked to compute the c-labels and the b-labels.

A flow f on the flow network N and a matching x of the subgraph network $\mathcal{M}(N)$ can be transformed by

$$f(a), f(a') := x(e_a) + lower(a) \tag{6}$$

and

$$x(e_a) := f(a) - lower(a), \tag{7}$$
$$x(e_v) := K(v) - \sum_{a^+=v} f(a), \tag{8}$$

respectively.

Lemma 33.1.

(a) By Eq. (6), a (fractional) factor of $\mathcal{M}(N)$ is mapped to a (fractional) balanced circulation on N.
(b) By Eqs. (7) and (8), a (fractional) balanced circulation on N is mapped to a (fractional) factor of $\mathcal{M}(N)$.
(c) Both mappings are affine, bijective, and inverse to each other.

Proof: Let x be a fractional factor of $\mathcal{M}(N)$, $v \in Inner(N)$, and f, the fractional balanced flow obtained

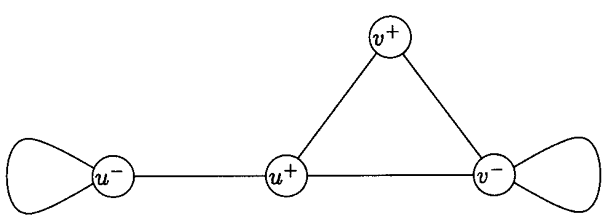

FIG. 2. A reduced matching problem.

by Eq. (6). Observe that

$$\begin{aligned} \sum_{a^+=v} f(a) &= \sum_{e_a \in \delta(v^+)} \{x(e_a) + lower(a)\} \\ &= b(v^+) - x(e_v) + lower^+(v) \\ &= K(v) - x(e_v) = b(v^-) - x(e_v) + lower^-(v) \\ &= \sum_{e_a \in \delta(v^-)} \{x(e_a) + lower(a)\} = \sum_{a^-=v} f(a). \end{aligned} \tag{9}$$

Due to the symmetry in f, this shows the flow-conservation property not only for the inner nodes, but also for the outer nodes of N. This is Assertion (a).

Equation (8) also shows that f is mapped to x by (7) and (8) again. Hence, the mapping of Assertion (a) is injective. To see Assertion (b) and the surjectivity, one checks that

$$x(e_v) + \sum_{e_a \in \delta(v^+)} e_a = b(v^+)$$

and

$$x(e_v) + \sum_{e_a \in \delta(v^-)} e_a = b(v^-),$$

which is essentially Eq. (8). □

Letting $c(e_v) := \infty$ is not precise if we want to compare the asymptotic complexities of the original problem and the transformed problem. If we put $c(e_v) := K(v)$, it turns out that the number of nodes, the number of arcs, and the total sum of capacities increase by a constant factor; the details are left to the reader.

Hence, from the view of computational complexity, the problem of finding a (minimum-cost) balanced circulation and the (weighted) perfect b-matching problem are equivalant in a very strong sense.

Our main interest in this problem reduction is the specification of a complete system of inequalities for the (minimum-cost) balanced circulation problem:

Theorem 33.2. *The polytope $\mathcal{P}(N)$ is the convex hull of balanced circulations.*

Proof: Let f be a fractional balanced circulation on N, and x, the corresponding fractional factor of $\mathcal{M}(N)$. Let $\tilde{\mathcal{M}}$ be the instance of the perfect b-matching problem obtained by the reduction principle of Theorem 32.3. [We do not replace the edges e_u, $u \in Inner(N)$ which already have $c(e_u) = \infty$.]

Let $\tilde{W} \subseteq V(\tilde{\mathcal{M}})$ so that $b(\tilde{W})$ is odd. For every node $u \in Inner(N)$, we have

$$\begin{aligned} x(e_u) &= K(u) - \sum_{a^+=u} f(a) \geq cap(u) - cap^+(u) \\ &= cap^-(u) \geq \sum_{a^-=u} f(a) \geq \sum_{a^-=u} x(e_a). \end{aligned}$$

In the case of $e_u \in \delta(\tilde{W})$, putting $\tilde{W} := \tilde{W} \oplus \{u^-\}$ can only decrease $y(\delta(\tilde{W}))$. Since $b(u^-)$ is even, this does not change the parity of $b(\tilde{W})$. If we ignore redundant constraints, we can assume that $u^+ \in \tilde{W}$ if and only if $u^- \in \tilde{W}$. Let W, E_1, E_2 be chosen as in the proof of Theorem 0. Denote

$$\begin{aligned}\hat{W} &:= \{v \in \mathit{Outer}(N) : v^+ \in \tilde{W}\},\\ U &:= \hat{W} \oplus \hat{W}',\\ A_1 &:= \{a \in A(N) : e_a \in E_1, a^- \in U\},\\ A_2 &:= \{a \in A(N) : e_a \in E_2, a^- \in U\}.\end{aligned}$$

The inequality

$$x(E_2) + c(E_1) - x(E_1) \geq 1$$

is equivalent to

$$f(A_2) - \mathit{lower}(A_2) + \mathit{cap}(A_1) - f(A_1) \geq 1.$$

We also have

$$\begin{aligned}b(W) &\equiv \sum_{v\in\hat{W}} \{b(v^+) - b(v^-)\}\\ &= \sum_{v\in\hat{W}} \{\mathit{lower}^+(v) - \mathit{lower}^-(v)\}\\ &= \sum_{a^+\in\hat{W}} \mathit{lower}(a) - \sum_{a^-\in\hat{W}} \mathit{lower}(a)\\ &\equiv \sum_{a^-\in\hat{W}'} \mathit{lower}(a) + \sum_{a^-\in\hat{W}} \mathit{lower}(a)\\ &\equiv \mathit{lower}(U, \overline{U}) \mod 2\end{aligned}$$

and

$$\begin{aligned}c(E_1) = \mathit{cap}(A_1) - \mathit{lower}(A_1)&\\ &\equiv \mathit{cap}(A_1) + \mathit{lower}(A_1) \mod 2,\end{aligned}$$

which eventually shows the identity

$$\begin{aligned}&\mathit{lower}(A_2) - \mathit{cap}(A_1)\\ &\quad = \mathit{lower}(U, \overline{U}) - \mathit{cap}(A_1) - \mathit{lower}(A_1)\\ &\quad \equiv b(W) + c(E_1) \mod 2.\end{aligned}$$

□

All the described mappings between fractional factors and fractional balanced circulations are bijective and affine. Hence, vertices are mapped to vertices and facets are mapped to facets.

Note that all reduction mechanisms are polynomial. The only superlinear step is the reduction of the c-capacitated b-matching problem to the ordinary b-matching problem. Hence, a polynomial algorithm for balanced circulations essentially is a polynomial b-matching algorithm.

We emphasize that the problem reduction to the capacitated b-matching problem works even if the capacity bounds are (partially) negative.

34. DUALITY

Up to this point, our discussion of balanced flows was strictly primal or, as in Part (IV), concerned combinatorial dual problems. To establish primal-dual algorithms for balanced network-flow problems (which is the standard approach in matching theory), the explicit specification of an LP-dual is crucial. We start with the primal problem (LP) in the most natural (but somewhat redundant) description:

minimize

$$\sum_{a\in A(N)} c(a)f(a)$$

subject to

$$\begin{array}{lll}(p1a) & f(a) \geq \mathit{lower}(a) & \forall\ a \in A(N)\\ (p1b) & f(a) \leq \mathit{cap}(a) & \forall\ a \in A(N)\\ (p2) & f(a) = f(a') & \forall\ a \in A(N)\\ (p3) & e(v) = 0 & \forall\ v \in V(N)\\ (p4) & f(A_2) - f(A_1) \geq \mathit{scap}(A_1, A_2) + 1 & \forall\ (A_1, A_2) \in \mathcal{O}(N).\end{array}$$

The dual of this linear program formally is the following problem (DLP):

maximize

$$\sum_{a\in A(N)} \{\mathit{lower}(a)\alpha(a) - \mathit{cap}(a)\beta(a)\} + \sum_{(A_1,A_2)\in\mathcal{O}(N)} \{\mathit{scap}(A_1, A_2) + 1\}\phi(A_1, A_2)$$

subject to

$$\begin{array}{ll}(d1) & \alpha(a) - \beta(a) + \psi(a) - \psi(a') + \pi(a^+) - \pi(a^-)\\ & \quad - \sum_{(A_1,A_2)\in\mathcal{O}(N)} \chi^{A_1,A_2}(a)\, \phi(A_1, A_2) = c(a) \quad \forall\ a \in A(N)\\ & \alpha \geq 0,\ \beta \geq 0,\ \phi \geq 0.\end{array}$$

Here, χ^{A_1,A_2} is the **incidence vector** of the skew cut (A_1, A_2). It is defined as $\chi^{A_1,A_2} := +1$ for $a \in A_1$, $\chi^{A_1,A_2} := -1$ for $a \in A_2$, and $\chi^{A_1,A_2} := 0$ for the noncut arcs.

In terms of these two linear programming problems, one has the following complementary slackness optimality conditions:

$$\begin{array}{lrl}(cs1a) & \alpha(a)\{f(a) - \mathit{lower}(a)\} &= 0\\ (cs1b) & \beta(a)\{\mathit{cap}(a) - f(a)\} &= 0\\ (cs2) & \phi(A_1, A_2)\{f(A_2) - f(A_1) - \mathit{scap}(A_1, A_2) - 1\} &= 0.\end{array}$$

This description of the dual is still a little bit clumsy. But observe that an arbitrary dual solution $\Delta = (\alpha, \beta, \psi, \pi, \phi)$ can be symmetrized as follows:

$$\begin{aligned}\dot{\alpha}(a) &:= \frac{1}{2}\{\alpha(a) + \alpha(a')\},\\ \dot{\beta}(a) &:= \frac{1}{2}\{\beta(a) + \beta(a')\},\\ \dot{\psi}(a) &:= 0,\\ \dot{\pi}(v) &:= \frac{1}{2}\{\pi(v) - \pi(v')\},\\ \dot{\phi}(A_1, A_2) &:= \frac{1}{2}\{\phi(A_1, A_2) + \phi(A_1', A_2')\}.\end{aligned}$$

It is easy to see that Δ and $\dot{\Delta} = (\dot{\alpha}, \dot{\beta}, \dot{\psi}, \dot{\pi}, \dot{\phi})$ have equal value and that the symmetrized solution satisfies the nonnegativity requirements for $\dot{\alpha}$, $\dot{\beta}$, $\dot{\phi}$. To see that the constraint (d1) holds for $\dot{\Delta}$ and some $a \in A(N)$, one merely has to consider the sum of (d1) for the arcs a and a' regarding Δ.

Hence, one can add to (DLP) the constraint $\psi \equiv 0$ and introduce symmetry constraints to the dual program. These modifications do not change the optimal objective value.

Theorem 34.1. *In the polyhedral description of $\mathscr{P}(N)$, all the symmetry constraints (p2) are simultaneously redundant.*

Proof: Omit the constraints (p2) and choose a vertex $\dot{f}$ of the resulting polytope. We can choose a cost function c so that $\dot{f}$ is the unique optimum for the respective problem (LP*).

Choose a optimum solution Δ for the dual (DLP) of the original problem. Symmetrize Δ to obtain an optimum $\dot{\Delta}$ for the dual of the modified problem (LP*). Let f be an integral optimum of (LP) including the symmetry constraints.

It turns out that f, $\dot{f}$, Δ, and $\dot{\Delta}$ have equal objective values. This implies that f is an optimum for (LP*) and, hence, $f = \dot{f}$. □

It is convenient to have some notion of reduced cost labels. In accordance with [10], we call

$$c_\pi^\phi(a) \;:=\; c(a) + \pi(a^-) - \pi(a^+) + \sum\nolimits_{(A_1,A_2)\in\mathscr{O}(N)} \chi^{A_1,A_2}(a)\,\phi(A_1,A_2)$$

the **modified cost** of the arc a. Note that the modified cost labels are the reduced cost labels known from linear programming. The **reduced cost labels** known from ordinary network-flow problems (which coincide with the reduced cost labels for the respective LP formulations) are obtained from the modified cost labels by putting $\phi :\equiv 0$.

If π and ϕ are symmetric, the modified-cost labels are balanced. Even more, in optimum solutions, α and β are also symmetric. The result is the following program (DLP2):

maximize

$$\sum_{a\in A(N)} \{lower(a)\alpha(a) - cap(a)\beta(a)\} + \sum_{(A_1,A_2)\in\mathscr{O}(N)} \{scap(A_1,A_2)+1\}\phi(A_1,A_2)$$

subject to

$$\begin{array}{lll}
(d1) & \alpha(a) - \beta(a) = c_\pi^\phi(a) & \forall\ a \in A(N)\\
(d2) & \pi(v) = -\pi(v') & \forall\ v \in V(N)\\
(d3) & \phi(A_1,A_2) = \phi(A_1',A_2') & \forall\ (A_1,A_2) \in \mathscr{O}(N)\\
 & \alpha \ge 0,\ \beta \ge 0,\ \phi \ge 0. &
\end{array}$$

Note that this problem is not really an LP-dual of (LP), but rather a **combinatorial dual problem**. By the new symmetry constraints, the modified-cost labels are again (fractional) balanced. Throughout the later discussion of algorithms, we only consider dual solutions which are **fractional balanced** or even half-integral balanced.

We can extend the definition of reduced and modified-cost labels to backward arcs by putting $c_\pi^\phi(\overline{a}) := -c_\pi^\phi(a)$ and then obtain the following optimality statement:

Theorem 34.2. *Let f be a balanced circulation on a balanced flow network N, and c, an arc cost function. Then, the following statements are equivalent:*

(a) f is optimal.
(b) $N(f)$ does not admit a valid cycle of negative length.
(c) There are vectors π and $\phi \ge 0$ so that

$$\begin{array}{lll}
(cs1) & c_\pi^\phi(a) \ge 0, & \textit{if } rescap_f(a) > 0,\\
(cs2) & \phi(A_1,A_2) = 0, & \textit{if } (A_1,A_2) \in \mathscr{O}(N) \textit{ is not tight.}
\end{array}$$

Proof: The equivalence of (a) and (b) is Theorem 7.1 in [4]. The equivalence of (a) and (c) is a mere reformulation of the slackness conditions (cs1a), (cs1b), and (cs2). □

For algorithmic purposes, one would like more explicit dual solutions in (c) where only a small number of ϕ's are strictly positive. In the traditional setting, one would introduce **shrinking families** at this point.

An exhaustive discussion of shrinking families will come up with a primal-dual algorithm for the minimum-cost balanced *st*-flow problem. This is, in fact, the next milestone in our investigation of balanced network flows [9].

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