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ON THE TRANSLOCATION OF MASSES

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Foreword

The following paper is reproduced from a Russian journal of the character of our own Proceedings of the National Academy of Sciences, *Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS*, 1942, Volume XXXVII, No. 7–8. The author is one of the most distinguished of Russian mathematicians. He has made very important contributions in pure mathematics in the theory of functional analysis, and has made equally important contributions to applied mathematics in numerical analysis and the theory and practice of computation. Although his exposition in this paper is quite terse and couched in mathematical language which may be difficult for some readers of *Management Science* to follow, it is thought that this presentation will: (1) make available to American readers generally an important work in the field of linear programming, (2) provide an indication of the type of analytic work which has been done and is being done in connection with rational planning in Russia, (3) through the specific examples mentioned indicate the types of interpretation which the Russians have made of the abstract mathematics (for example, the potential and field interpretations adduced in this country recently by W. Prager were anticipated in this paper).

It is to be noted, however, that the problem of determining an effective method of actually acquiring the solution to a specific problem is *not* solved in this paper. In the category of development of such methods we seem to be, currently, ahead of the Russians.—A. CHARNES, *Northwestern Technological Institute and The Transportation Center.*

R will denote a compact metric space, though some of the following definitions and results are valid in more general spaces.

Let $\Phi(e)$ be a mass distribution, i.e. a set function possessing the following properties: 1) $\Phi(e)$ is defined on Borel sets in R, 2) $\Phi(e)$ is non-negative, $\Phi(e) \geqslant 0$, 3) $\Phi(e)$ is absolutely additive, i.e. if $e = e_1 + e_2 + \cdots$, $e_i e_k = 0 (i \neq k)$, then $\Phi(e) = \Phi(e_1) + \Phi(e_2) + \cdots$. Let further $\Phi'(e')$ be another mass distribution, such that $\Phi(R) = \Phi'(R)$. Under the translocation of masses we shall understand the function $\Psi(e, e')$ defined on the pairs of B-sets $e, e' \in R$ and such that: 1) $\Psi(e, e')$ is non-negative and absolutely additive in each of its arguments, 2) $\Psi(e, R) \equiv \Phi(e); \Psi(R, e') \equiv \Phi'(e')$.

Let a continuous non-negative function $r(x, y)$ be given that represents the work expended in transferring a unit mass from x to y.

By the work required for transferring the given mass distributions will be understood

$$W(\Psi, \Phi, \Phi') = \int_R \int_R r(x, x')\Psi(de, de') = \lim_{\lambda \to 0} \sum_{i,k} r(x_i, x'_k)\Psi(e_i, e'_k)$$

where e_i are mutually disjoint and $\sum_1^n e_i = R$; e'_k are also mutually disjoint and $\sum_1^m e'_k = R$, $x_i \in e_i$, $x'_k \in e'_k$, λ being the maximum of diameter e_i ($i = 1, 2, \cdots, n$) and diameter e'_k ($k = 1, 2, \cdots, m$). This integral does evidently exist.

$$W(\Phi, \Phi') = \inf_{\Psi} W(\Psi, \Phi, \Phi')$$

is said to be the minimal translocation work. Since the totality of functions $\{\Psi\}$ is compact, there exists a function Ψ_0 that realizes this minimum, i.e. such that

$$W(\Phi, \Phi') = W(\Psi_0, \Phi, \Phi').$$

This function, it is true, is not unique. Such a translocation is said to be minimal translocation.

We shall further say that the translocation Ψ from x to y is not zero, and write $x \to y$ if $\Psi(U_x, U_y) > 0$ for any neighbourhoods U_x and U_y of the points x and y. The translocation Ψ will be called potential if there exists a function $U(x)$ such that: 1) $| U(x) - U(y) | \leqslant r(x, y)$; 2) $U(y) - U(x) = r(x, y)$ when $x \to y$.

Theorem. The translocation Ψ is minimal if and only if it is potential.

Sufficiency of the Condition. Let Ψ_0 be a potential translocation and U the corresponding potential. According to property 2) of U

$$\begin{aligned} W(\Psi_0, \Phi, \Phi') &= \int_R \int_R r(x, y)\Psi_0(de, de') = \int_R \int_R [U(y) - U(x)]\Psi_0(de, de') \\ &= \int_R \int_R U(y)\Psi_0(de, de') - \int_R \int_R U(x)\Psi_0(de, de') \\ &= \int_R U(y)\Phi'(de') - \int_R U(x)\Phi(de), \end{aligned}$$

while for any other function Ψ

$$\begin{aligned} W(\Psi, \Phi, \Phi') &= \int_R \int_R r(x, y)\Psi(de, de') \geqslant \int_R \int_R [U(y) - U(x)]\Psi(de, de') \\ &= \int_R U(y)\Phi'(de') - \int_R U(x)\Phi(de), \end{aligned}$$

i.e. $W(\Psi, \Phi, \Phi') \geqslant W(\Psi_0, \Phi, \Phi')$ and Ψ_0 is thus minimal.

Necessity of the Condition. Let Ψ_0 be a minimal translocation. Consider a set of points $\xi_0, \xi_1, \cdots$ dense in R. Denote by D_n the minimal point set containing ξ_n and such that $x \in D_n$ and $x \to y$, or $y \to x$, imply $y \in D_n$. It is obvious that if $y \in D_n$, there exists then a system of points x_i, y_i such that $\xi_0 = x_0 \to y_1$, $x_1 \to y_1$, $x_1 \to y_2$, $\cdots$, $x_{n-1} \to y_n$, $x_n \to y_n (y_n = y)$ (or a similar chain with the inverse direction of arrows at the beginning or at the end). In our case we set

$$U(y) = \sum_1^n r(x_{i-1}, y_{i-1}) - \sum_1^n r(x_i, y_i).$$

It is readily verified that U does not depend on the choice of the joining chain and that properties 1) and 2) of the potential are valid for U, if $x, y \in D_0$. Indeed, it is easy to show that if one of these circumstances failed to take place, Ψ_0 might be replaced by another translocation involving less work, which contradicts the assumed minimality of Ψ_0.

Suppose the function U to be already defined on the domains $D_0, D_1, \cdots, D_{n-1}$.

If the point r_n belongs to $D_0 + D_1 + \cdots + D_{n-1}$, then U is defined in r_n and on the whole D_n as well. Otherwise we define $V(x)$ on the domain D_n in the same way as U on D_0, provided that r_0 is now replaced by r_n. Further we choose the number μ satisfying the inequalities

$$\inf_{\substack{x \in D_0+\cdots+D_{n-1} \\ y \in D_n}} \{U(x) - V(y) - r(x, y)\} \leqslant \mu$$

$$\leqslant \inf_{\substack{x \in D_0+\cdots+D_{n-1} \\ y \in D_n}} \{U(x) - V(y) + r(x, y)\}.$$

The existence of such μ is again implied by the minimality of Ψ_0. Then we put for $x \in D_n$: $U(x) = V(x) + \mu$. The function U is thus defined on the set $D_0 + D_1 + \cdots$. As the latter is everywhere dense in R, U is defined on the whole space R according to the property 2), so that properties 1) and 2) are preserved. We see that Ψ_0 is a potential translocation.

The theorem just demonstrated makes it easy for one to prove that a given mass translocation is or is not minimal. He has only to try and construct the potential in the way outlined above. If this construction turns out to be impossible, i.e. the given translocation is not minimal, he at least will find himself in the possession of the method how to lower the translocation work and eventually come to the minimal translocation.

It is interesting to investigate the space of mass distributions with the metric defined as $W(\Phi, \Phi')$ [in the case, when $r(x, y)$ coincides with the distance $\rho(x, y)$]. This method of metrization of the space considered seems to be the most natural in some respects.

Finally, we give two practical problems, for the solution of which our theorem may be applied.

Problem 1. Location of consumption stations with respect to production stations. Stations $A_1, A_2, \cdots, A_m$, attached to a network of railways deliver goods to an extent of $a_1, a_2, \cdots, a_m$ carriages per day respectively. These goods are consumed at stations $B_1, B_2, \cdots, B_n$ of the same network at a rate of $b_1, b_2, \cdots, b_n$ carriages per day respectively ($\Sigma a_i = \Sigma b_k$). Given the costs $r_{i,k}$ involved in moving one carriage from station A_i to station B_k, assign the consumption stations such places with respect to the production stations as would reduce the total transport expenses to a minimum.

The solution of this and more complicated problems of the same type the reader will find exposed in detail in a paper by L. Kantorovitch and M. Govurin which is soon to be published.

Problem 2. Levelling a land area. Given the relief of the locality, namely, the equations $z = f(x, y)$ and $z = f_1(x, y)$ for the earth surface, before and after the levelling $\left[\text{under the condition } \int\int f(x, y)\,dx\,dy = \int\int f_1(x, y)\,dx\,dy\right]$ and the cost of transport of 1 m^3 of earth from point (x, y) to point (x_1, y_1); find a plan of transporting the earth masses with the least total expense.