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A DECOMPOSITION THEOREM FOR MAXIMUM WEIGHT BIPARTITE MATCHINGS[*]

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Abstract. Let G be a bipartite graph with positive integer weights on the edges and without isolated nodes. Let n, N, and W be the node count, the largest edge weight, and the total weight of G. Let $k(x, y)$ be $\log x/\log(x^2/y)$. We present a new decomposition theorem for maximum weight bipartite matchings and use it to design an $O(\sqrt{n}W/k(n, W/N))$-time algorithm for computing a maximum weight matching of G. This algorithm bridges a long-standing gap between the best known time complexity of computing a maximum weight matching and that of computing a maximum cardinality matching. Given G and a maximum weight matching of G, we can further compute the weight of a maximum weight matching of $G - \{u\}$ for all nodes u in $O(W)$ time.

Key words. all-cavity matchings, maximum weight matchings, minimum weight covers, graph algorithms, unfolded graphs

AMS subject classifications. 05C05, 05C70, 05C85, 68Q25

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1. Introduction.

Let $G = (X, Y, E)$ be a bipartite graph with positive integer weights on the edges. A *matching* of G is a subset of node-disjoint edges of G. Let $\mathrm{mwm}(G)$ (respectively, $\mathrm{mm}(G)$) denote the maximum weight (respectively, cardinality) of any matching of G. A *maximum weight* matching is one whose weight is $\mathrm{mwm}(G)$. Let N be the largest weight of any edge. Let W be the total weight of G. Let n and m be the numbers of nodes and edges of G; to avoid triviality, we maintain $m = \Omega(n)$ throughout the paper.

The problem of finding a maximum weight matching of a given G has a rich history. The first known polynomial-time algorithm is the $O(n^3)$-time Hungarian method [15]. Fredman and Tarjan [5] used Fibonacci heaps to improve the time to $O(n(m + n \log n))$. Gabow [6] introduced scaling to solve the problem in $O(n^{3/4} m \log N)$ time by taking advantage of the integrality of edge weights. Gabow and Tarjan [7] improved the scaling method to further reduce the time to $O(\sqrt{n}m \log(nN))$. For the case where the edges all have weight 1, i.e., $N = 1$ (and $W = m$), Hopcroft and Karp [11] gave an $O(\sqrt{n}W)$-time algorithm, and Feder and Motwani [4] improved the time complexity to $O(\sqrt{n}W/k(n, m))$, where $k(x, y) = \log x/\log(x^2/y)$. It has remained open whether the gap between the running times of the Gabow–Tarjan algorithm and the latter two algorithms can be closed for the case where $W = o(m \log(nN))$.

We resolve this open problem in the affirmative by giving an $O(\sqrt{n}W/k(n, W/N))$-time algorithm for general W. Note that $W/N = m$ when all the edges have the same weight. The algorithm does not use scaling but instead employs a novel decomposition theorem for weighted bipartite matchings (Theorem 2.2). We also use the theorem to

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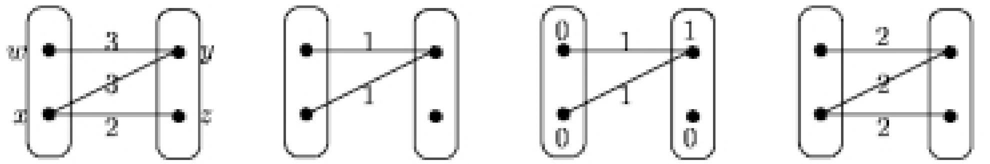


(a) G (b) G_h (c) C_h (d) $G_h^{\triangle}$

FIG. 1. *Consider* $h = 1$. G *is decomposed into* G_h *and* $G_h^{\triangle}$; C_h *is a minimum weight cover of* G_h.

solve the *all-cavity maximum weight matching* problem which, given G and a maximum weight matching of G, asks for $\text{mwm}(G - \{u\})$ for all nodes u in G. This problem has applications to tree comparisons [2, 14]. The case where $N = 1$ has been studied by Chung [2]. Recently, Kao, Lam, Sung, and Ting [12] gave an $O(\sqrt{n}m \log N)$-time algorithm for general N. This paper presents a new algorithm that runs in $O(W)$ time.

Section 2 presents the decomposition theorem and uses it to compute the weight of a maximum weight matching. Section 3 gives an algorithm to construct a maximum weight matching. Section 4 solves the all-cavity matching problem.

2. The decomposition theorem.

In section 2.1, we state the decomposition theorem and use the theorem to design an algorithm to compute the weight $\text{mwm}(G)$ in $O(\sqrt{n}W/k(n, W/N))$ time. In section 2.2, we prove the decomposition theorem. In section 3, we further construct a maximum weight matching itself within the same time bound.

2.1. An algorithm for computing mwm(G).

Let $V(G)$ be the node set of G, i.e., $X \cup Y$. Let $w(u, v)$ denote the weight of an edge $uv \in G$; if u is not adjacent to v, let $w(u, v) = 0$. A *cover* of G is a function $C : X \cup Y \to \{0, 1, 2, \ldots\}$ such that $C(x) + C(y) \geq w(x, y)$ for all $x \in X$ and $y \in Y$. Let $w(C) = \sum_{z \in X \cup Y} C(z)$ be the weight of C. C is a *minimum weight* cover if $w(C)$ is the smallest possible. Let $\text{mwc}(G)$ denote the weight of a minimum weight cover of G. A minimum weight cover is a dual of a maximum weight matching as stated in the next fact.

FACT 2.1 (see [1]). *Let* C *be a cover and* M *be a matching of* G. *The following statements are equivalent.*

1. C *is a minimum weight cover and* M *is a maximum weight matching of* G.
2. $\sum_{uv \in M} w(u, v) = \sum_{u \in X \cup Y} C(u)$.
3. *Every node in* $\{u \mid C(u) > 0\}$ *is matched by some edge in* M, *and* $C(u) + C(v) = w(u, v)$ *for all* $uv \in M$.

For an integer $h \in [1, N]$, we divide G into two lighter bipartite graphs G_h and $G_h^{\triangle}$ as follows:

- G_h is formed by the edges uv of G with $w(u, v) \in [N - h + 1, N]$. Each edge uv in G_h has weight $w(u, v) - (N - h)$. For example, G_1 is formed by the heaviest edges of G, and the weight of each edge is exactly one.
- Let C_h be a minimum weight cover of G_h. $G_h^{\triangle}$ is formed by the edges uv of G with $w(u, v) - C_h(u) - C_h(v) > 0$. The weight of uv is $w(u, v) - C_h(u) - C_h(v)$.

An example is depicted in Figure 1. Note that the total weight of G_h and $G_h^{\triangle}$ is at most W.

The next theorem is the decomposition theorem.

THEOREM 2.2. $\mathrm{mwm}(G) = \mathrm{mwm}(G_h) + \mathrm{mwm}(G_h^{\triangle})$; *in particular,* $\mathrm{mwm}(G) = \mathrm{mm}(G_1) + \mathrm{mwm}(G_1^{\triangle})$.

Proof. See section 2.2. □

Theorem 2.2 suggests the following recursive algorithm to compute $\mathrm{mwm}(G)$.

PROCEDURE Compute-MWM(G).

1. Construct G_1 from G.
2. Compute $\mathrm{mm}(G_1)$ and find a minimum weight cover C_1 of G_1.
3. Construct $G_1^{\triangle}$ from G and C_1.
4. If $G_1^{\triangle}$ is empty, then return $\mathrm{mm}(G_1)$; otherwise, return $\mathrm{mm}(G_1)$+Compute-MWM($G_1^{\triangle}$).

THEOREM 2.3. Compute-MWM(G) *finds* $\mathrm{mwm}(G)$ *in* $O(\sqrt{n}W/k(n, W/N))$ *time.*

Proof. The correctness of Compute-MWM follows from Theorem 2.2. Below, we analyze the running time. We initialize a maximum heap [3] in $O(m)$ time to store the edges of G according to their weights. Let $T(n, W, N)$ be the running time of Compute-MWM excluding this initialization. Let L be the set of the heaviest edges in G. Then Step 1 takes $O(|L| \log m)$ time. In Step 2, we can compute $\mathrm{mm}(G_1)$ in $O(\sqrt{n}|L|/k(n, |L|))$ time [4]. From this matching, C_1 can be found in $O(|L|)$ time [1]. Let L_1 be the set of the edges of G adjacent to some node u with $C_1(u) > 0$; i.e., L_1 consists of the edges of G whose weights are reduced in $G_1^{\triangle}$. Let $\ell_1 = |L_1|$. Step 3 updates every edge of L_1 in the heap in $O(\ell_1 \log m)$ time. As $L \subseteq L_1$, Steps 1 to 3 altogether use $O(\sqrt{n}\ell_1/k(n, \ell_1))$ time. Since the total weight of $G_1^{\triangle}$ is at most $W - \ell_1$, Step 4 uses at most $T(n, W - \ell_1, N')$ time, where $N' < N$ is the maximum edge weight of $G_1^{\triangle}$. In summary, for some positive integer $\ell_1 \le W$,

$$T(n, W, N) = O(\sqrt{n}\ell_1/k(n, \ell_1)) + T(n, W - \ell_1, N'),$$

where $T(n, 0, N') = 0$. By recursion, for some positive integers $\ell_1, \ell_2, \ldots, \ell_p$ with $p \le N$ and $\sum_{1 \le i \le p} \ell_i = W$,

$$\begin{aligned} T(n, W, N) &= O\left(\sqrt{n}\left(\frac{\ell_1}{k(n, \ell_1)} + \frac{\ell_2}{k(n, \ell_2)} + \cdots + \frac{\ell_p}{k(n, \ell_p)}\right)\right) \\ &= O\left(\frac{\sqrt{n}}{\log n}\left(\left(\sum_{1 \le i \le p} \ell_i\right) \log n^2 - \sum_{1 \le i \le p} \ell_i \log \ell_i\right)\right). \end{aligned}$$

Since $x \log x$ is convex, by Jensen's inequality [10],

$$\sum_{1 \le i \le p} \ell_i \log \ell_i \ge \left(\sum_{1 \le i \le p} \ell_i\right) \log \frac{\sum_{1 \le i \le p} \ell_i}{p} \ge W \log \frac{W}{N}.$$

Therefore,

$$\begin{aligned} T(n, W, N) &= O\left(\frac{\sqrt{n}}{\log n}\left(W \log n^2 - W \log \frac{W}{N}\right)\right) \\ &= O\left(\frac{\sqrt{n}W}{\log n / \log(n^2/\frac{W}{N})}\right) = O\big(\sqrt{n}W/k(n, W/N)\big). \end{aligned}$$ □

2.2. Proof of Theorem 2.2. This section proves the statement that $\mathrm{mwm}(G) = \mathrm{mwm}(G_h) + \mathrm{mwm}(G_h^{\triangle})$, where $G_h^{\triangle}$ is defined according to an arbitrary minimum weight cover C_h of G_h. By Fact 2.1, it suffices to prove $\mathrm{mwc}(G) = w(C_h) + \mathrm{mwc}(G_h^{\triangle})$.

To show the direction $\mathrm{mwc}(G) \leq w(C_h) + \mathrm{mwc}(G_h^{\triangle})$, note that any cover D of $G_h^{\triangle}$ augmented with C_h gives a cover C of G, where $C(u) = C_h(u) + D(u)$ for each node u of G. Then $C(u) + C(v) \geq w(u, v)$ for all edges uv of G. Thus, $\mathrm{mwc}(G) \leq w(C_h) + \mathrm{mwc}(G_h^{\triangle})$.

To show the direction $w(C_h) + \mathrm{mwc}(G_h^{\triangle}) \leq \mathrm{mwc}(C)$, let C be a minimum weight cover of G. A node u of G is called *bad* if $C(u) < C_h(u)$. Lemma 2.4 below shows that G must have a minimum weight cover C allowing no bad node. Then we can construct a cover D of $G_h^{\triangle}$ as follows. For each node u of G, define $D(u) = C(u) - C_h(u)$, which must be at least 0. D is a cover of $G_h^{\triangle}$ because for any edge uv of $G_h^{\triangle}$, $D(u) + D(v) = C(u) + C(v) - C_h(u) - C_h(v) \geq w(u, v) - C_h(u) - C_h(v)$. Note that $w(D) = w(C) - w(C_h)$. Thus, $\mathrm{mwc}(G_h^{\triangle}) \leq w(C) - w(C_h)$, or equivalently, $\mathrm{mwc}(G_h^{\triangle}) + w(C_h) \leq \mathrm{mwc}(G)$.

The next lemma concludes the proof of Theorem 2.2.

LEMMA 2.4. *There exists a minimum weight cover of G such that no node of G is bad.*

Proof. Suppose, for the sake of contradiction, that every minimum weight cover allows some bad node. Then we can obtain a contradiction by constructing another minimum weight cover with no bad node.

Let C be a minimum weight cover of G with u as a bad node, i.e., $C(u) < C_h(u)$. Recall that C_h is a minimum weight cover of G_h. Consider a maximum weight matching M of G_h. By Fact 2.1, since $C_h(u) > C(u) \geq 0$, u is matched by an edge in M, say, to a node v, and $C_h(u) + C_h(v) = w(u, v) - (N - h)$. We call v the *mate* of u. Note that v cannot be a bad node; otherwise, $C(u) + C(v) < w(u, v) - (N - h) \leq w(u, v)$ and a contradiction occurs.

Since C is a cover of G, $C(u) + C(v) \geq w(u, v)$. Thus, $C(v) \geq w(u, v) - C(u) \geq N - h + C_h(u) + C_h(v) - C(u)$. Define another cover C' of G as follows. For each bad node defined by C, let v be the mate of u, define $C'(u) = C_h(u)$ and $C'(v) = C(v) - (C_h(u) - C(v))$. Note that u is not a bad node with respect to C', and neither is v since $C'(v) \geq N - h + C_h(v) \geq C_h(v)$. For all other nodes x, $C'(x)$ is the same as $C(x)$. Therefore, if C' is a cover of G, C' allows no bad node. Also, $w(C') = w(C)$.

It remains to prove that C' is a cover of G. By the definition of C', $C'(v) < C(v)$ if and only if v is the mate of a bad node with respect to C. Suppose C' is not a cover of G. Then there exists an edge vt such that $C'(v) + C'(t) \leq w(v, t)$ and v is the mate of a bad node. Recall that the latter implies that $C'(v) \geq N - h + C_h(v)$. In other words,

$$C'(t) < w(v, t) - C'(v) \leq w(v, t) - (N - h) - C_h(v).$$

We can derive a contradiction as follows.

Case 1: $w(v, t) \leq N - h$. Then $C'(t) < -C_h(v) \leq 0$, which contradicts that $C'(t) \geq C_h(t) \geq 0$.

Case 2: $w(v, t) > N - h$. Then G_h contains the edge vt and $C_h(v) + C_h(t) \geq w(v, t) - (N - h)$. Thus, $C'(t) < w(v, t) - (N - h) - C_h(v) \leq C_h(t)$, which contradicts the fact that C' allows no bad node.

In conclusion, C' is a cover of G. Together with the fact that $w(C) = w(C')$, we obtain the desired contradiction that C' is a minimum weight cover of G with no bad node. Lemma 2.4 follows. $\square$

3. Construct a maximum weight matching. The algorithm in section 2.1 only computes the value of $\mathrm{mwm}(G)$. To report the edges involved, we show below how to first construct a minimum weight cover of G in $O(\sqrt{n}W/k(n, W/N))$ time and then use this cover to construct a maximum weight matching in $O(\sqrt{n}m/k(n, m))$ time. Thus, the time required to construct a maximum weight matching is $O(\sqrt{n}W/k(n, W/N))$.

Lemma 3.1. *Assume that* h, G_h, C_h, *and* $G_h^{\triangle}$ *are defined as in section* 2. *Let* $C_h^{\triangle}$ *be any minimum weight cover of* $G_h^{\triangle}$. *If* D *is a function on* $V(G)$ *such that for every* $u \in V(G)$, $D(u) = C_h(u) + C_h^{\triangle}(u)$, *then* D *is a minimum weight cover of* G.

Proof. Consider any edge uv of G. If uv is not in $G_h^{\triangle}$, then $w(u, v) \le C_h(u) + C_h(v) \le D(u) + D(v)$. Assume that uv is in $G_h^{\triangle}$. Note that its weight in $G_h^{\triangle}$ is $w(u, v) - C_h(u) - C_h(v)$. Since $C_h^{\triangle}$ is a cover, $C_h^{\triangle}(u) + C_h^{\triangle}(v) \ge w(u, v) - C_h(u) - C_h(v)$. Thus, $D(u) + D(v) = C_h(u) + C_h^{\triangle}(u) + C_h(v) + C_h^{\triangle}(v) \ge w(u, v)$. It follows that D is a cover of G. To show that D is a minimum weight one, we observe that

$$\begin{aligned} \textstyle\sum_{u \in V(G)} D(u) &= \textstyle\sum_{u \in V(G)} C_h(u) + C_h^{\triangle}(u) \\ &= \textstyle\sum_{u \in V(G)} C_h(u) + \sum_{u \in V(G)} C_h^{\triangle}(u) \\ &= \mathrm{mwm}(G_h) + \mathrm{mwm}(G_h^{\triangle}) && \text{by Fact 2.1} \\ &= \mathrm{mwm}(G) && \text{by Theorem 2.2.} \end{aligned}$$

By Fact 2.1, D is minimum. □

By Lemma 3.1, a minimum weight cover of G can be computed using a recursive procedure similar to Compute-MWM as follows.

Procedure Compute-Min-Cover(G).

1. Construct G_1 from G.
2. Find a minimum weight cover C_1 of G_1.
3. Construct $G_1^{\triangle}$ from G and C_1.
4. If $G_1^{\triangle}$ is empty, then return C_1; otherwise, let $C_1^{\triangle} =$ Compute-Min-Cover($G_1^{\triangle}$) and return D, where for all nodes u in G, $D(u) = C_1(u) + C_1^{\triangle}(u)$.

Theorem 3.2. Compute-Min-Cover(G) *correctly computes a minimum weight cover of* G *in* $O(\sqrt{n}W/k(n, W/N))$ *time.*

Proof. The correctness of Compute-Min-Cover(G) follows from Lemma 3.1. For the time complexity, the analysis is similar to that of Theorem 2.3. □

Now, we show how to recover a maximum weight matching of G from a minimum weight cover D of G.

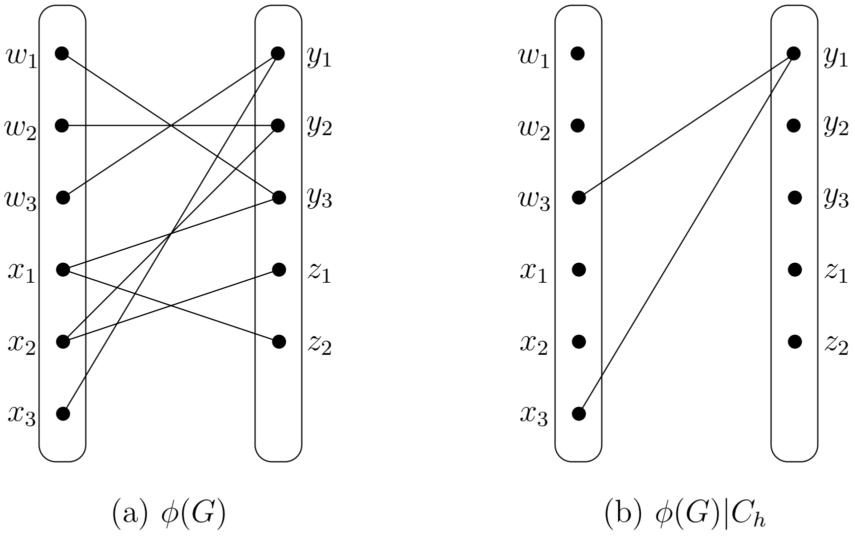
Procedure Recover-Max-Matching(G, D).

1. Let H be the subgraph of G that contains all edges uv with $w(u, v) = D(u) + D(v)$.
2. Make two copies of H. Call them H^a and H^b. For each node u of H, let u^a and u^b denote the corresponding nodes in H^a and H^b, respectively.
3. Union H^a and H^b to form H^{ab}, and add to H^{ab} the set of edges $\{u^a u^b \mid u \in V(H),\ D(u) = 0\}$.
4. Find a maximum cardinality matching K of H^{ab} and return the matching $K^a = \{uv \mid u^a v^a \in K\}$.

Theorem 3.3. Recover-Max-Matching(G, D) *correctly computes a maximum weight matching of* G *in* $O(\sqrt{n}m/k(n, m))$ *time.*

Proof. The running time of Recover-Max-Matching(G, D) is dominated by the construction of K. Since H^{ab} has at most $2n$ nodes and at most $3m$ edges, K can be constructed in $O(\sqrt{n}m/k(n, m))$ time using the Feder–Motwani algorithm [4].

It remains to show that K^a is a maximum weight matching of G. First, we argue that H^{ab} has a perfect matching. Let M be a maximum weight matching of G. By



(a) $\phi(G)$ (b) $\phi(G)|C_h$

FIG. 2. (a) *The unfolded graph* $\phi(G)$ *of the bipartite graph given in Figure* 1(a). (b) *With respect to the cover* C_h *defined in Figure* 1(c)*, the node* y_1 *in* $\phi(G)$ *is the only node satisfying the condition that* $1 \le C_h(y)$*. Thus,* $\phi(G)|C_h$ *comprises only the edges incident to* y_1*.*

Fact 2.1, $D(u) + D(v) = w(u, v)$ for every edge $uv \in M$. Therefore, M is also a matching of H. Let U be the set of nodes in H unmatched by M. By Fact 2.1, $D(u) = 0$ for all $u \in U$. Let Q be $\{u^a u^b \mid u \in U\}$. Let $M^a = \{u^a v^a \mid uv \in M\}$ and $M^b = \{u^b v^b \mid uv \in M\}$. Note that $Q \cup M^a \cup M^b$ forms a matching in H^{ab} and every node in H^{ab} is matched by either Q, M^a, or M^b. Thus, H^{ab} has a perfect matching.

Since K is a maximum cardinality matching of H^{ab}, K must be a perfect matching. For every node u with $D(u) > 0$, u^a must be matched by K. Since there is no edge between u^a and any x^b in H^{ab}, there exists some v^a with $u^a v^a \in K$. Thus, every node u with $D(u) > 0$ must be matched by some edge in K^a. Therefore, $\sum_{uv \in K^a} w(u, v) = \sum_{u \in X \cup Y, D(u)>0} D(u) = \sum_{u \in X \cup Y} D(u) = \text{mwm}(G)$, and K^a is a maximum weight matching of G. $\square$

4. All-cavity maximum weight matchings.

In section 4.1, we introduce the notion of an *unfolded graph*. In section 4.2, we use this notion to design an algorithm which, given a weighted bipartite graph G and a maximum weight matching of G, computes $\text{mwm}(G - \{u\})$ for all nodes u in G using $O(W)$ time.

4.1. Unfolded graphs.

The *unfolded graph* $\phi(G)$ of G is defined as follows.

- For each node u of G, $\phi(G)$ has α copies of u, denoted as $u^1, u^2, \ldots, u^\alpha$, where α is the weight of the heaviest edge incident to u.
- For each edge uv of G, $\phi(G)$ has the edges $u^1 v^\beta, u^2 v^{\beta-1}, \ldots, u^\beta v^1$, where $\beta = w(u, v)$.

See Figure 2(a) for an example. Let M be a matching of G. Consider M as a weighted bipartite graph; then, by definition, $\phi(M) = \bigcup_{uv \in M} \{u^1 v^\beta, \ldots, u^\beta v^1 \mid \beta = w(u, v)\}$ is a matching of $\phi(G)$. The number of edges in $\phi(M)$ is equal to the total weight of the edges in M, i.e., $|\phi(M)| = \sum_{uv \in M} w(u, v)$. The next lemma relates G and $\phi(G)$.

LEMMA 4.1. *Assume that M is a maximum weight matching of G.*

1. $\text{mwm}(G) = \text{mm}(\phi(G))$.
2. *The set $\phi(M)$ is a maximum cardinality matching of $\phi(G)$.*

Proof. Statement 4.1 follows from Statement 4.1. Statement 4.1 is proved as follows. Since M is a maximum weight matching of G, $\text{mwm}(G) = \sum_{uv \in M} w(u, v) = |\phi(M)| \leq \text{mm}(\phi(G))$. By Fact 2.1, $\text{mwm}(G) \geq \text{mm}(\phi(G))$ if and only if $\text{mwc}(G) \geq \text{mwc}(\phi(G))$. We prove the latter as follows. Given a minimum weight cover C of G, we can obtain a cover C' of $\phi(G)$ as follows. For any node u of G, $C'(u^i) = 1$ if $C(u) > 0$ and $i \leq C(u)$; otherwise, $C'(u^i) = 0$. Note that $w(C') = w(C) = \text{mwc}(G)$. Therefore, $\text{mwc}(G) \geq \text{mwc}(\phi(G))$ and $\text{mwm}(G) \geq \text{mm}(\phi(G))$. ☐

4.2. An algorithm for all-cavity maximum weight matchings. Let M be a given maximum weight matching of G.

By Lemma 4.1(2), $\phi(M)$ is a maximum cardinality matching of $\phi(G)$. In light of this maximality, we say that a path in $\phi(G)$ is *alternating* for $\phi(M)$ if (1) its edges alternate between being in $\phi(M)$ and being not in $\phi(M)$ and (2) in the case the first (respectively, last) node is matched by $\phi(M)$, the path contains the matched edge of u as the first (respectively, last) edge. The length of an alternating path is its number of edges. An alternating path may have zero length; in this case, the path contains exactly one unmatched node. An alternating path P can modify $\phi(M)$ to another matching, i.e., $(\phi(M) \cup P) - (\phi(M) \cap P)$. If P is of even length, the resulting matching has the same size as $\phi(M)$. If P is of odd length, P modifies M to a strictly smaller or bigger matching; yet the latter is impossible because $\phi(M)$ is maximum. Intuitively, we would like to maximize the size of the resultant matching and even-length alternating paths are preferred.

Our new algorithm for computing $\text{mwm}(G - \{u\})$ is based on the observation that $\text{mwm}(G - \{u\})$ can be determined by detecting the smallest i such that u^i has an even-length alternating path for $\phi(M)$. Details are as follows.

Definition. For each u^i in $\phi(G)$, let $\rho(u^i) = 0$ if there is an even-length alternating path for $\phi(M)$ starting from u^i; otherwise, let $\rho(u^i) = 1$.

The following lemma states a monotone property of $\rho(u^i)$ over different i's.

LEMMA 4.2. *Consider any node u in G. Let $u^1, u^2, \ldots, u^\beta$ be its corresponding nodes in $\phi(G)$. If $\rho(u^i) = 0$, then $\rho(u^j) = 0$ for all $j \in [i, \beta]$. Furthermore, there exist $\beta - i + 1$ node-disjoint even-length alternating paths $P_i, P_{i+1}, \ldots, P_\beta$ for $\phi(M)$, where each P_j starts from u^j.*

Proof. As $\rho(u^i) = 0$, let $P_i = u_0^{a_0}, v_0^{b_0}, u_1^{a_1}, v_1^{b_1}, \ldots, u_{p-1}^{a_{p-1}}, v_{p-1}^{b_{p-1}}, u_p^{a_p}$ be a shortest even-length alternating path for $\phi(M)$, where $u_0^{a_0} = u^i$.

Based on P_i, we can construct an even-length alternating path P_{i+1} for $\phi(M)$ starting from u^{i+1} as follows. If u^{i+1} is not matched by $\phi(M)$, P_{i+1} is simply a path of zero length. From now on, we assume that u^{i+1} is matched by $\phi(M)$. As P is of even length, $u_p^{a_p}$ is not matched by $\phi(M)$. Then, by the definition of $\phi(M)$, $u_p^{a_p+1}$ is also not matched by $\phi(M)$. Let h be the smallest integer in $[1, p]$ such that $u_h^{a_h+1}$ is not matched by $\phi(M)$. Notice that, for all $\ell < h$, $u_\ell^{a_\ell+1}$ is matched to $v_\ell^{b_\ell-1}$; furthermore, $\phi(G)$ contains an edge between $v_\ell^{b_\ell-1}$ and $u_{\ell+1}^{a_{\ell+1}+1}$. Thus, $P_{i+1} = u^{i+1}, v_0^{b_0-1}, u_1^{a_1+1}, v_1^{b_1-1}, \ldots, u_h^{a_h+1}$ is an even-length alternating path for $\phi(M)$. Similarly, for $j = i + 2, \ldots, \beta$, we can use P_i to define an even-length alternating path P_j for $\phi(M)$ starting from u^j. By construction, $P_i, P_{i+1}, \ldots, P_\beta$ are node-disjoint. ☐

The next lemma is the basis of our cavity matching algorithm. It shows that given $\text{mwm}(G)$ (i.e., the weight of M), we can compute $\text{mwm}(G - \{u\})$ from the values $\rho(u^i)$, and all the $\rho(u^i)$'s can be found in $O(W)$ time.

LEMMA 4.3.

1. $\sum_{1 \leq i \leq \beta} \rho(u^i) = \text{mwm}(G) - \text{mwm}(G - \{u\})$.
2. *For all* $u^i \in \phi(G)$, $\rho(u^i)$ *can be computed in* $O(W)$ *time in total.*

Proof. The two statements are proved as follows.

Statement 1. Let k be the largest integer such that $\rho(u^k) = 1$. By Lemma 4.2, $\rho(u^i) = 1$ for all $1 \leq i \leq k$, and 0 otherwise. Note that if $\rho(u^i) = 1$, u^i must be matched by $\phi(M)$. Thus, $\sum_{1 \leq i \leq \beta} \rho(u^i) = k$. Below, we prove the following two equalities:

(1) $\text{mm}(\phi(G) - \{u^1, \ldots, u^k\}) = \text{mm}(\phi(G)) - k$.

(2) $\text{mm}(\phi(G) - \{u^1, \ldots, u^\beta\}) = \text{mm}(\phi(G) - \{u^1, \ldots, u^k\})$.

Then, by Lemma 4.1, $\text{mwm}(G) = \text{mm}(\phi(G))$ and $\text{mwm}(G - \{u\}) = \text{mm}(\phi(G) - \{u^1, \ldots, u^\beta\})$. Thus, $\text{mwm}(G) - \text{mwm}(G - \{u\}) = k$ and Statement 1 follows.

To show equality (1), let H be the set of edges of $\phi(M)$ incident to u^i with $1 \leq i \leq k$. Let $M' = \phi(M) - H$. Then, $|M'| = |\phi(M)| - k$. We claim that M' is a maximum cardinality matching of $\phi(G) - \{u^1, \ldots, u^k\}$. Hence, $\text{mwm}(\phi(G) - \{u^1, \ldots, u^k\}) = |\phi(M)| - k$, and equality (1) follows. We prove the claim by contradiction. Suppose M' is not a maximum cardinality matching of $\phi(G) - \{u^1, \ldots, u^k\}$. Then, there exists an alternating path P that can modify M' to a larger matching of $\phi(G) - \{u^1, \ldots, u^k\}$ [8, 9]; in particular, the length of P must be odd and both of its endpoints are not matched by M'. P must start from some node v^j with $u^i v^j \in \phi(M)$ and $i < k$; otherwise, P is alternating for $\phi(M)$ in G and $\phi(M)$ cannot be a maximum cardinality matching of $\phi(G)$. Let Q be a path formed by joining $u^i v^j$ with P. Q is an even-length alternating path for $\phi(M)$ starting from u^i in $\phi(G)$. This contradicts the fact that there is no even-length alternating path for $\phi(M)$ starting from u^i for $i < k$.

To show equality (2), we first note that $\text{mm}(\phi(G) - \{u^1, \ldots, u^\beta\}) \leq \text{mm}(\phi(G) - \{u^1, \ldots, u^k\})$. It remains to prove the other direction. By Lemma 4.2, we can find $\beta - k$ node-disjoint even-length alternating paths $P_{k+1}, \ldots, P_\beta$ for $\phi(M)$, which start from $u^{k+1}, \ldots, u^\beta$. P_j starts at u^j. Let $M'' = (\phi(M) \cup (P_{j+1} \cup \cdots \cup P_\beta)) - (\phi(M) \cap (P_{j+1} \cup \cdots \cup P_\beta))$. Note that $|M''| = |\phi(M)|$ and there are no edges in M'' incident to any of $u^{k+1}, \ldots, u^\beta$. M'' is a matching of $\phi(G) - \{u^{k+1}, \ldots, u^\beta\}$ and $M'' - H$ of $\phi(G) - \{u^1, \ldots, u^\beta\}$. $|M'' - H| \geq | M''| - k = |\phi(M)| - k$. Since $\text{mm}(\phi(G) - \{u^1, \ldots, u^k\}) = |\phi(M)| - k$ by equality (1), it follows that $\text{mm}(\phi(G) - \{u^1, \ldots, u^\beta\}) \geq |M'' - H| \geq \text{mm}(\phi(G) - \{u^1, \ldots, u^k\})$. Therefore, equality (2) holds.

Statement 2. We want to determine whether $\rho(u^i) = 0$ for all nodes $u^i \in \phi(G)$ in $O(W)$ time. By definition, $\rho(u^i) = 0$ if and only if there is an even-length alternating path for $\phi(M)$ starting from u^i. Let us partition the nodes of $\phi(G)$ into two parts: $\phi(X) = \{u^i \in \phi(G) \mid u \in X\}$ and $\phi(Y) = \{u^i \in \phi(G) \mid u \in Y\}$. Below, we give the details of computing $\rho(u^i)$ for all $u^i \in \phi(X)$. The case where $u^i \in \phi(Y)$ is symmetric.

Let D be a directed graph over the node set $\phi(X)$. D contains an edge $u^i v^j$ if there exists a node $w^k \in \phi(Y)$ such that $u^i w^k \in \phi(G) - \phi(M)$ and $w^k v^j \in \phi(M)$. Consider any node v^j of D that is unmatched by $\phi(M)$. A directed path in D from v^j to a node u^i corresponds to a path in $\phi(G)$, which is indeed an even-length alternating path for $\phi(M)$ starting from u^i. Therefore, for any $u^i \in \phi(X)$, $\rho(u^i) = 0$ if and only if u^i is reachable from some node in D that is unmatched by $\phi(M)$. We can identify all such u^i by using a depth-first search on D starting with all the nodes unmatched by M. The time required is $O(|D|)$. As $|D| \leq | \phi(G)| = W$, the lemma follows. □

The following procedure computes $\text{mwm}(G - \{u\})$ for all nodes u of G. Let M be a maximum weight matching of G.

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PROCEDURE Compute-All-Cavity($(G, M)$).
  1. Construct $\phi(G)$ and $\phi(M)$.
  2. For every $j \in [0, n/2]$, determine $A_j$ from $\phi(M)$.
  3. For every node $u^i$ of $\phi(G)$, if $u^i \in \bigcup_j A_j$ then $\rho(u^i) = 0$; otherwise $\rho(u^i) = 1$.
  4. For every node $u$ of $G$, compute $\mathrm{mwm}(G - \{u\}) = \mathrm{mwm}(G) - \sum_{1 \le i \le \beta} \rho(u^i)$,
     where $u^1, u^2, \ldots, u^\beta$ are the nodes corresponding to $u$ in $\phi(G)$.
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THEOREM 4.4. Compute-All-Cavity(G, M) *correctly computes* $\mathrm{mwm}(G - \{u\})$ *for all* u *of* G *in* $O(W)$ *time.*

Proof. The proof follows from Lemma 4.3 □

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