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FACES OF MATCHING POLYHEDRA

by

William R. Pulleyblank

A Thesis
Submitted in Partial Fulfilment
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for the Degree of
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The University of Waterloo requires the signature of all persons using this thesis. Please sign below, and give address and date.

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Abstract

Let $G = (V, E, \psi)$ be a finite loopless graph, let $b = (b_i : i \in V)$ be a vector of positive integers. A feasible matching is a vector $x = (x_j : j \in E)$ of nonnegative integers such that for each node i of G, the sum of the x_j over the edges j of G incident with i is no greater than b_i. The matching polyhedron $P(G, b)$ is the convex hull of the set of feasible matchings.

In Chapter 3 we describe a version of Edmonds' blossom algorithm which solves the problem of maximizing $c \cdot x$ over $P(G, b)$ where $c = (c_j : j \in E)$ is an arbitrary real vector. This algorithm proves a theorem of Edmonds which gives a set of linear inequalities sufficient to define $P(G, b)$.

In Chapter 4 we prescribe the unique subset of these inequalities which are necessary to define $P(G, b)$, that is, we characterize the facets of $P(G, b)$. We also characterize the vertices of $P(G, b)$, thus describing the structure possessed by the members of the minimal set X of feasible matchings of G such that for any real vector $c = (c_j : j \in E)$, $c \cdot x$ is maximized over $P(G, b)$ by a member of X.

In Chapter 5 we present a generalization of the blossom algorithm which solves the problem: maximize $c \cdot x$ over a face F of $P(G, b)$ for any real vector $c = (c_j : j \in E)$. In other words, we find a feasible matching x of G which satisfies the constraints obtained by replacing an arbitrary subset of the inequalities which define $P(G, b)$ by

equations and which maximizes $c \cdot x$ subject to this restriction. We also describe an application of this algorithm to matching problems having a hierarchy of objective functions, so called "multi-optimization" problems.

In Chapter 6 we show how the blossom algorithm can be combined with relatively simple initialization algorithms to give an algorithm which solves the following postoptimality problem. Given that we know a matching $x^0 \in P(G, b)$ which maximizes $c \cdot x$ over $P(G, b)$, we wish to utilize x^0 to find a feasible matching $x' \in P(G, b')$ which maximizes $c \cdot x$ over $P(G, b')$, where $b' = (b'_i : i \in V)$ is a vector of positive integers and $c = (c_j : j \in E)$ is an arbitrary real vector.

In Chapter 7 we describe a computer implementation of the blossom algorithm described herein.

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Chapter 1

Introduction and Foundations

1.1. Introduction

Let $G = (V, E, \psi)$ be a finite loopless graph, where V is the set of nodes of G, E is the set of edges of G and ψ is the incidence function of G which maps E into the set of all two element subsets of V. For each $i \in V$, let b_i be a positive integer. A feasible matching is a vector $x = (x_j: j \in E)$ of nonnegative integers such that for each node i of G, the sum of the x_j over the edges j of G incident with i is no greater than b_i. The matching polyhedron $P(G, b)$ is the bounded polyhedron containing all feasible matchings of G and all of whose vertices are feasible matchings of G. (In other words, $P(G, b)$ is the convex hull of the set of feasible matchings.) In this thesis we examine several different aspects of the faces of $P(G, b)$.

The later sections of Chapter 1 consist of a summary of the basic results from various fields of mathematics which are assumed to be known, we also introduce all our basic notation and terminology.

In Chapter 2 we develop the general polyhedral theory used in later chapters. This topic is developed from the point of view of studying systems of linear inequalities. The facets of a polyhedron are the faces of the polyhedron which have dimension one less than the dimension of the polyhedron itself. In characterizing the facets of matching polyhedra in Chapter 4 we make extensive use of (2.2.15), which states that a proper face F of a polyhedron P of dimension d is a facet of P if and only if F contains d + 1 affinely independent elements. In Theorems (2.3.25), (2.3.30), (2.3.31), (2.3.32) and (2.3.34) we discuss the connection between the facets of a polyhedron and a minimal set of inequalities necessary to define the polyhedron. We show in (2.3.32) that if P is a polyhedron of full dimension, then the facets of P determine, up to multiplication by a positive constant, the minimal subset of inequalities needed to define P. Since matching polyhedra are of full dimension, this is the case in which we are interested.

We discuss the vertices of polyhedra in the last section of Chapter 2 and prove three fundamental results. First (Theorem 2.4.1)), the vertices of a polyhedron P are precisely those elements $v \in P$ for which there is some linear objective function c such that v is the unique member of P maximizing $c \cdot x$ over P. Second (Theorem (2.4.5)), if P is a bounded polyhedron then for any linear objective function c, there is a vertex v of P which maximizes $c \cdot x$ over P. Third (Theorem (2.4.10)), any nonempty bounded polyhedron is equal to the convex hull of its vertices.

Chapter 2 is largely expository, however the point of view taken in this chapter is somewhat different from standard references on polyhedra (Grünbaum [G1], Rockafellar [R1] and Stoer, Witzgall [S1]) and tends to emphasize the relationship between polyhedra and linear programming.

In Chapter 3 we describe a version of the so called blossom algorithm (Edmonds [E1], [E2], [E3], [E4]). This algorithm finds a matching $x^0 \in P(G, b)$ which maximizes $c \cdot x$ over $P(G, b)$. In fact the algorithm described solves a somewhat more general problem, it maximizes $c \cdot x$ over a face F of $P(G, b)$ obtained by requiring the sum of the x_j on the edges j incident with node i to be exactly equal to b_i for all nodes i belonging to some subset W of V.

For any node $i \in V$ we let $\delta(i)$ denote the set of edges of G incident with i. For any $S \subseteq V$ we let $\gamma(S)$ denote the set of edges of G having both ends in S. For any $J \subseteq E$ we let $x(J)$ denote $\sum_{j \in J} x_j$ and for any $W \subseteq V$ we let $b(W)$ denote $\sum_{i \in W} b_i$. The feasible matchings of G are the integer solutions of the linear system

$$(1.1.1) \quad x_j \geq 0 \text{ for all } j \in E,$$

$$(1.1.2) \quad x(\delta(i)) \leq b_i \text{ for all } i \in V.$$

Clearly if we let P be the polyhedron defined by (1.1.1) and (1.1.2) then $P \supseteq P(G, b)$. In fact, if G is bipartite or if b_i is even for all $i \in V$ then $P = P(G, b)$. However in general there are vertices of P which are not vertices of $P(G, b)$ and thus have fractional components. Consequently there are generally some linear objective functions which when maximized over P, attain their maximum for a member x of P having fractional components. It can be seen that if x is a noninteger vertex of P then every component of x

is either integer or half integer valued and the edges j for which x_j are half integer valued form the edge sets of node disjoint odd polygons.

The blossom algorithm proves a theorem of Edmonds, that

$P(G, b) = \{(x_j \in \mathbb{R} : j \in E): x$ satisfies (1.1.1) and (1.1.2) and

$$(1.1.3) \quad x(\gamma(S)) \le q_S \quad \text{for all } S \in Q\}$$

where $Q \equiv \{S \subseteq V: b(S)$ is odd, $|S| \ge 3\}$ and $q_S \equiv 1/2(b(S)-1)$ for all $S \in Q$. It is not difficult to see that every feasible matching of G satisfies the constraints (1.1.3); it is more difficult to see that this set of constraints is sufficient to define $P(G, b)$, that is, that all vertices of the polyhedron defined by (1.1.1)-(1.1.3) are integer valued.

The blossom algorithm makes use of the weak duality theorem of linear programming and the principle of complementary slackness to prove the optimality of the matching which it finds. For any linear objective function c it produces an <u>integer</u> solution x^0 to the linear program: maximize $c \cdot x$ subject to x satisfying (1.1.1)-(1.1.3). It also produces a solution y^0 to the dual linear program and shows that x^0 and y^0 satisfy the complementary slackness conditions for optimality. Thus, where d is the objective function of the dual linear program, $c \cdot x^0 = d \cdot y^0$. By the weak duality theorem of linear programming, any solution x of (1.1.1)-(1.1.3) must satisfy $c \cdot x \le d \cdot y^0$, therefore x^0 is an optimal solution to the linear program: maximize

$c \cdot x$ subject to (1.1.1)-(1.1.3). Since every feasible matching x of G satisfies (1.1.1)-(1.1.3) it follows that x^0 is the optimal matching we require.

From this it easily follows that $P(G, b)$ is the solution set of (1.1.1)-(1.1.3), for if v is any vertex of the polyhedron defined by (1.1.1)-(1.1.3) then there is some linear objective function maximized over that polyhedron only by v. But we have seen that every linear objective function is maximized by an integer solution of (1.1.1)-(1.1.3), hence all the vertices of this polyhedron are feasible matchings.

The set of inequalities (1.1.3) is generally far larger than is necessary to define $P(G, b)$; as was mentioned if G is bipartite then none of them are necessary. In view of the structure of the vertices of P, the solution set of (1.1.1) and (1.1.2), it has been surmised that all of the constraints (1.1.3) which are really necessary are those for which S is the node set of an odd polygon. Unfortunately, these are generally not enough; if we just add these inequalities to our linear system (1.1.1)-(1.1.2) then we usually introduce new fractional vertices having a more complex structure than those possessed by P. In Chapter 4 of this thesis, by considering the structure of G and the value of b, we prescribe the minimal subset of the inequalities (1.1.3) which must be added to (1.1.1) and (1.1.2) to obtain $P(G, b)$.

Since $P(G, b)$ is of full dimension there is a direct correspondence between the facets of $P(G, b)$ and the inequalities necessary to define $P(G, b)$, namely $\{x \in P(G, b): ax = \alpha\}$ is a facet of $P(G, b)$ if and only if the inequality $ax \leq \alpha$ (or a positive multiple of $ax \leq \alpha$) is necessary to define $P(G, b)$. Thus in Chapter 4 when we characterize the facets of $P(G, b)$ we are in fact prescribing which of the inequalities (1.1.1)-(1.1.3) are necessary to define $P(G, b)$. We prove

<u>Theorem (4.1.2)</u>. <u>For every $j \in E$, $\{(x_j : j \in E) \in P(G, b): x_j = 0\}$ is a facet of $P(G, b)$.</u>

In other words all the constraints (1.1.1) are essential for defining $P(G, b)$.

However, some of the constraints (1.1.2) are not necessary. For any $i \in V$ we let $N(i)$ be the set of nodes of G adjacent to i. If v, w are nodes of G such that $N(v) = \{w\}$, $N(w) = \{v\}$ and $b_w = b_v$ then we call the connected component of G spanned by $\{v, w\}$ a <u>balanced edge</u>.

<u>Theorem (4.2.1)</u>. <u>For any $i \in V$, $\{(x_j : j \in E) \in P(G, b): x(\delta(i)) = b_i\}$ is a facet of $P(G, b)$ if and only if</u>

<u>i is a node of a balanced edge</u>

<u>or</u>

<u>$b(N(i)) > b_i$ and if $b(N(i)) = b_i + 1$ then $\gamma(N(i)) = \phi$.</u>

A salient feature of the blossom algorithm is the "shrinking" process applied to certain subgraphs of G, effectively reducing the size of the problem under consideration. It is implicit in the blossom algorithm that the set Q in (1.1.3) can be replaced by the set $Q^0 \equiv \{S \subseteq V\colon G[S]$ is shrinkable$\}$ where $G[S]$ is the subgraph of G induced by S, that is $G[S] \equiv (S, \gamma(S), \psi|\gamma(S))$. We prove that all we need add is a connectivity condition to the condition of shrinkability and we have the essential inequalities of the sort (1.1.3).

Theorem (4.3.46). For any $S \subseteq V$ such that $G[S]$ is shrinkable, $\{x \in P(G, b)\colon x(\gamma(S)) = q_S\}$ is a facet of $P(G, b)$ if and only if $G[S]$ contains no cutnode v for which $b_v = 1$.

The necessity of our conditions of both Theorem (4.2.1) and Theorem (4.3.46) is proved by constructing $|E|$ affinely independent feasible matchings of G which belong to the facet of $P(G, b)$. We define a near perfect matching of G deficient at $v \in V$ to be a matching x of G which satisfies

$$x(\delta(i)) = b_i \quad \text{for all} \quad i \in V - \{v\},$$
$$x(\delta(v)) = b_v - 1.$$

A feasible matching x of G will satisfy $x(\gamma(S)) = q_S$ if and only if $\bar{x}$, the restriction of x to $\gamma(S)$, is a near perfect matching of $G[S]$. Thus when constructing feasible matchings of G which satisfy $x(\gamma(S)) = q_S$, our first step is to be able to construct a large number of near perfect matchings of $G[S]$.

We say that G is b-critical if for every node v of G there is a near perfect matching of G which is deficient at v. These $|V|$ near perfect matchings can be seen to be linearly independent, but we usually require a much larger set of linearly independent near perfect matchings. However we show that if a graph G is b-critical and contains no cutnode v for which $b_v = 1$ then G has as many linearly independent near perfect matchings as it has edges. This we prove by showing (Theorem (4.4.2)) that a graph G is b-critical if and only if G is shrinkable. We also prove that these conditions are equivalent to G being connected, $b(V)$ being odd and the empty set being the only subset of V which violates Tutte's condition (3.10.34) for the existence of a perfect matching.

Thus we obtain two more facet characterization theorems (4.4.15), (4.4.17). In particular we have the following.

Theorem. <u>For any $S \subseteq V$ such that $b(S)$ is odd and $|S| \geq 3$, $F \equiv \{x \in P(G, b): x(\gamma(S)) = q_S\}$ is a facet of $P(G, b)$ if and only if</u>

<u>$G[S]$ is b-critical and contains no cutnode v such that $b_v = 1$</u>

<u>or</u>

<u>F is a facet of the sort described in Theorem (4.2.1)</u>.

As a result of this theorem we can see very easily that if G is bipartite then none of the inequalities (1.1.3) need be added to define $P(G, b)$, for let S be any subset of V such that $b(S)$ is odd and $|S| \geq 3$. Then there must be a part T of $G[S]$ for which $b(T) < 1/2\ b(S)$. Obviously we cannot construct a near perfect matching of

$G[S]$ deficient at a node v belonging to T and consequently $G[S]$ cannot be b-critical.

There is a close relationship between polyhedron theory and min-max theorems; whenever we know a set of linear inequalities sufficient to define a polyhedron, linear programming duality immediately provides us with a min-max theorem and we have already discussed how we use a min-max theorem proved by the blossom algorithm to establish the matching polyhedron. We discuss the min-max theorem proved by the blossom algorithm in Section 3.10 and show how it implies theorems of Berge [B2] and Tutte [T1], [T2], [T3].

When we know the facets of a polyhedron, we are able to obtain a "best possible" min-max theorem. In Theorems (4.4.20) we describe such a theorem. We also show how the min-max theorems proved by the blossom algorithm can be combined with our characterization of b-critical graphs to obtain strengthenings of Tutte's theorems, in particular, we derive the following theorem concerning the existence of perfect 1-matchings (matchings x which satisfy $x(\delta(i)) = 1$ for all $i \in V$).

Theorem (4.4.22) $G = (V, E, \psi)$ has a perfect 1-matching if and only if for every $X \subseteq V$ such that $G[V - X]$ consists of 1-critical components, the number of components of $G[V - X]$ is no greater than $|X|$.

In Theorem (4.5.3) we characterize the vertices of $P(G, b)$ and show that every matching produced by the blossom algorithm is a vertex of $P(G, b)$. Since the vertex set of $P(G, b)$ is the smallest subset X of $P(G, b)$ such that for any linear function c, $c \cdot x$ is maximized over $P(G, b)$ by a member of X, this shows that the blossom algorithm makes use of as small a subset of $P(G, b)$ as possible when solving matching problems. As we saw in Chapter 2, every member of a bounded polyhedron can be expressed as a convex combination of its vertices, in (4.5.21) we describe an algorithm which will express any feasible matching of G which is not a vertex of $P(G, b)$ as a convex combination of two other members of $P(G, b)$. We also describe how this algorithm can be used to express any $x \in P(G, b)$ as a convex combination of a subset of the vertices of $P(G, b)$.

In Chapter 5 we consider the problem of maximizing $c \cdot x$ over any face F of $P(G, b)$ where $c = (c_j : j \in E)$ is an arbitrary real vector. That is, we are given sets $J \subseteq E$, $W \subseteq V$ and $N \subseteq Q$ and we wish to maximize $c \cdot x$ over all $x = (x_j : j \in E) \in P(G, b)$ which satisfy

(1.1.4) $x_j = 0$ for all $j \in J$,

(1.1.5) $x(\delta(i)) = b_i$ for all $i \in W$,

(1.1.6) $x(\gamma(S)) = q_S$ for all $S \in N$.

For any $J \subseteq E$, $W \subseteq V$ and $N \subseteq Q$ we let $F(J, W, N) \equiv \{(x_j : j \in E) \in P(G, b) : x$ satisfies (1.1.4)-(1.1.6)$\}$. The algorithm proposed to solve this problem consists of two parts. The first part described in Section 5.2 is a preconditioning process which finds sets $J' \subseteq E$, $W' \subseteq V$

and $N' \subseteq Q$ such that $F(J, W, N) = F(J', W', N')$ and N' has the property that for any $S, T \in N'$ such that $S \cap T \neq \phi$, either $S \subseteq T$ or $T \subseteq S$. (We call such a family of sets a nested family of sets.) The second part of the algorithm described in Section 5.4, can then be used to solve the equivalent problem. The algorithm is a generalization of the blossom algorithm of Chapter 3 and an upper bound on the amount of work performed by this algorithm in solving a problem maximize $c \cdot x$ over $F(J', W', N') \subseteq P(G, b)$ is of the same order as the amount of work performed by the blossom algorithm in solving $c \cdot x$ over $P(G, b)$.

In Section 5.5 we describe how this problem of maximizing $c \cdot x$ over a face F of $P(G, b)$ can be reduced to the problem of maximizing a new objective function c' over $P(G, b)$. This so called "Big-M" method is attractive theoretically, but in practice the number of significant digits in the components of c' tends to increase rather rapidly and so this method does have limitations as a practical method.

In Section 5.6 we discuss multi-optimization matching problems, matching problems in which we have a sequence $c_1, c_2, \ldots, c_k$ of objective functions and wish to solve the following problem. Let $X_0 \equiv P(G, b)$ and for each $i \in \{1, 2, \ldots, k\}$ let

$$X_i \equiv \{x \in X_{i-1}: c^i x \text{ is maximized over } X_{i-1}\}.$$

We wish to find a matching $x^* \in X_k$. We show how the face optimization algorithm of this chapter can be used to solve this sort of problem and various generalizations of this problem.

In Chaper 6 we discuss a post optimality problem. We assume that we know a matching $x^0 \varepsilon P(G, b)$ which maximizes $c \cdot x$ over $P(G, b)$ and we wish to find a matching $x^* \varepsilon P(G, b')$ which maximizes $c \cdot x$ over $P(G, b')$ where $b' = (b_i' : i \varepsilon V)$ is a vector of positive integers. Since the parameters G and c of our original problem are unchanged in the new problem, we would hope that we could make use of x^0 so as to be able to solve the new problem more quickly than by simply reapplying the blossom algorithm.

In this chapter we describe a relatively simple initialization procedure which can be combined with the blossom algorithm when we know x^0 and an optimal dual solution y^0 to the original problem, so that an upper bound on the amount of work performed in finding x^* depends upon the value of $|b - b'|$ in essentially the same way as the upper bound on the amount of work performed by the blossom algorithm depended on the value of b.

Finally, in Chapter 7, we discuss a computer implementation of the blossom algorithm and describe some experimental results.

1.2 Set Theory and General Notation

We use the symbol "$\equiv$" to indicate a definition and reserve the symbol "$=$" for denoting the equality of two objects.

If X and Y are sets we denote the union and intersection of X and Y by $X \cup Y$ and $X \cap Y$ respectively. We let $X - Y$ denote the set theoretic difference, that is

$$X - Y \equiv \{x \in X: x \notin Y\}.$$

We denote the empty set by ϕ. Expressions involving $\cup$, $\cap$, $-$ should be evaluated from left to right, thus

$$X \cup Y \cap Z - V$$

should be taken to be

$$((X \cup Y) \cap Z) - V.$$

If R is a set of sets, we will let

$$\cup(R) \equiv \bigcup_{X \in R} X$$

and

$$\cap(R) \equiv \bigcap_{X \in R} X.$$

We let $|X|$ denote the cardinality of X.

We let $\mathbb{R}$ denote the set of real numbers. For any $X \subseteq \mathbb{R}$ we let

$$\max X \equiv \max_{x \in X} x$$

and

$$\min X \equiv \min_{x \in X} x.$$

Where $X = (x_i : i \in I)$ is an indexed set of members of $\mathbb{R}$, we let

$$\Sigma X \equiv \sum_{i \in I} x_i .$$

For any $x \in \mathbb{R}$, $[x]$ denotes the largest integer no greater than x. $[x]$ is sometimes called the floor of x or the integer part of x.

We use $X \subseteq Y$ to denote "X is a subset of Y" and we use $X \subset Y$ to denote "X is a proper subset of Y"(thus $X \neq Y$).

If ψ is a function mapping a set X into a set Y, then for any $S \subseteq X$ we let $\psi|S$ denote the restriction of ψ to S. That is $\bar{\psi} = \psi|S$ is the function mapping S into Y defined by

$$\bar{\psi}(s) \equiv \psi(s) \text{ for all } s \in S.$$

We always use the words maximal and minimal in the sense of set inclusion. Thus if R is a family of sets we say that X is a maximal member of R if there is no $Y \in R$ such that $Y \supseteq X$. Similarly X is a minimal member of R if there is no $Y \in R$ such that $Y \subseteq X$.

We denote the cartesian product of two sets X and Y by $X \times Y$. Thus

$$X \times Y \equiv \{(x, y) : x \in X, y \in Y\}.$$

1.3 Graph Theory.

Standard references on graph theory are Berge [B3], Busacker and Saaty [B5] and Harary [H2]. For our purpose a graph G is an ordered triple (V, E, ψ) where V and E

are finite sets and ψ is a function mapping E into the set of two element subsets of V. The members of V are called <u>nodes</u>, the members of E are called <u>edges</u>, and ψ is called the <u>incidence function</u>. We say that $j \in E$ <u>meets</u> $v \in V$ or j and v are <u>incident</u> if $v \in \psi(j)$. We say that $v, w \in V$ are <u>adjacent</u> if there is $j \in E$ such that $\psi(j) = \{v, w\}$. If $\{v, w\} = \psi(j)$ then v and w are called the <u>ends</u> of j. If H is any graph we let $V(H)$, $E(H)$ and ψ_H denote the node set, edge set and incidence function of H respectively.

(1.3.1) A <u>track τ in $G = (V, E, \psi)$ from v_0 to v_n</u> is a sequence

$$v_0, j_1, v_1, j_2, v_2, \ldots, j_n, v_n \text{ for some } n \geq 0$$

such that

$$v_i \in V \text{ for } i \in \{0, 1, \ldots, n\},$$

$$j_i \in E \text{ for } i \in \{1, 2, \ldots, n\},$$

$$\psi(j_i) = \{v_{i-1}, v_i\} \text{ for } i \in \{1, 2, \ldots, n\}.$$

We call n the <u>length</u> of τ, we say that τ is <u>odd</u> or <u>even</u> according as the length of τ is even or odd. We let $E(\tau)$ denote $\{j_i: i \in \{1, 2, \ldots, n\}\}$ and $V(\tau)$ denote $\{v_i: i \in \{0, 1, \ldots, n\}\}$. For any $j_i \in E(\tau)$ we call j_i an <u>even</u> edge of τ if i is even and an <u>odd</u> edge of τ if i is odd. Edges occurring more than once in τ may be both even and odd.

A track τ induces an ordering on the nodes in $V(\tau)$ and edges in $E(\tau)$. Thus for any $P \subseteq V(\tau)$ we say that v is the <u>first node</u> in $V(\tau) \cap P$ if $s = \min\{i \in \{0,1,2,\ldots,n\}: v_i \in P\}$ and $v = v_s$. We define <u>last node</u> and <u>first</u> and last edge analogously.

(1.3.2) A path is a track π of length n for which $|V(\pi)| = n + 1$. In other words, no node occurs more than once.

A path π is said to be maximal with a given property if no other path having that property has π as a subsequence. (Obviously there is no such thing as a maximal track.)

A graph $G = (V, E, \psi)$ is said to be connected if for every $\{v, w\} \in V$ there is a path (track) π in G joining v to w.

A graph H is said to be a subgraph of $G = (V, E, \psi)$ if $V(H) \subseteq V$, $E(H) \subseteq E$ and $\psi_H = \psi|E(H)$. In this case we say that G contains H. A maximal connected subgraph of G is called a component of G.

The distance between nodes v and w belonging to the same component of G is defined to be the length of the shortest path joining v and w.

Let $G = (V, E, \psi)$ be any graph. For any $S \subseteq V$ we let $\delta_G(S)$ denote the coboundary of S, that is

$$(1.3.3) \quad \delta_G(S) \equiv \{j \in E: |S \cap \psi(j)| = 1\}.$$

When S consists of a single element v, then we abbreviate $\delta_G(\{v\})$ by $\delta_G(v)$. For any $v \in V$ we call $|\delta_G(v)|$ the valence of v. For any $S \subseteq V$ we let $\gamma_G(S)$ denote the set of edges of G having both ends in S, thus

$$(1.3.4) \quad \gamma_G(S) \equiv \{j \in E: \psi(j) \subseteq S\}.$$

We abbreviate δ_G and γ_G by δ and γ respectively.

(1.3.5) Let $S \subseteq V$. We let $G[S]$ denote the graph $(S, \gamma(S), \psi|\gamma(S))$. We call $G[S]$ the subgraph of G <u>induced</u> by S.

(1.3.6) A <u>polygon</u> is a connected graph P such that $|\delta_P(v)| = 2$ for all $v \in V(P)$. If $|E(P)|$ is even then we say that P is an <u>even polygon</u>, otherwise we call P an <u>odd polygon</u>.

(1.3.6a) Let P be a polygon and let $w \in V(P)$. Let τ be a track in P from w to w such that $V(\tau) = V(P)$, $E(\tau) = E(P)$ and the length of τ is as small as possible with this property. We call τ a <u>track from w to w induced by P</u>. Intuitively, τ is the track obtained by travelling once around the polygon P, starting at w.

(1.3.7) A graph $G = (V, E, \psi)$ is <u>bipartite</u> if V can be partitioned into $V_1 \cup V_2$ and $E = \delta(V_1) = \delta(V_2)$. Any $S \subseteq V$ such that $\delta(S) = E$ and $\gamma(S) = \phi$ is called a <u>part</u> of G.

(1.3.8) <u>Theorem</u>. (König [Kl] p. 170) <u>G is bipartite if and only if G contains no odd polygon</u>.

(1.3.9) A <u>cutnode</u> v of $G = (V, E, \psi)$ is a node $v \in V$ such that $G[V - \{v\}]$ has more components than G. G is <u>nonseparable</u> if G is connected and has no cutnode. A <u>block</u> is a maximal nonseparable subgraph of G. It is easily seen that

(1.3.10) every polygon of G is a subgraph of a block of G,

that is, no polygon can have edges from different blocks.

An <u>isthmus</u> of G is an edge $j \in E$ such that $(V, E - \{j\}, \psi|E - \{j\})$ has more components than G.

(1.3.11) A <u>forest</u> is a graph which contains no polygons, a <u>tree</u> is a connected forest. A tree T is said to be <u>trivial</u> if $|V(T)| \leq 1$. The following results are well known.

(1.3.12) <u>Theorem</u>. <u>Every nontrivial tree has at least two nodes of valence 1</u>.

(1.3.13) <u>Theorem</u>. <u>If T is a tree then $|E(T)| = |V(T)| - 1$</u>.

1.4 <u>Linear Algebra</u>.

Let J be a finite set. We let $\mathbb{R}^J \equiv \{(x_j : j \in J): x_j \in \mathbb{R}$ for all $j \in J\}$. We let 0 denote the vector which is zero in every component.

(1.4.1) A set $X \subseteq \mathbb{R}^J$ is said to be <u>linearly independent</u> if whenver $\sum_{x \in X} \alpha_x x = 0$ for some $(\alpha_x \in \mathbb{R} : x \in X)$ we have $\alpha_x = 0$ for all $x \in X$. Otherwise X is <u>linearly dependent</u>.

(1.4.2) Let $X \subseteq \mathbb{R}^J$. A <u>basis</u> of X is a maximal linearly independent subset of X. The following result is well known.

(1.4.3) Theorem. (Birkhoff & MacLane [B4], Ch. 7, §4). All bases of $X \subseteq \mathbb{R}^J$ have the same cardinality called the rank of X, and the rank of X is no greater than $|J|$.

(1.4.4) If $x, y \in \mathbb{R}^J$ we let $x \cdot y$ or xy denote $\Sigma\{x_j \cdot y_j : j \in J\}$.

(1.4.5) The null space of $X \subseteq \mathbb{R}^J$ is defined to be $\{y \in \mathbb{R}^J : y \cdot x = 0$ for all $x \in X\}$. We define the nullity of X to be the rank of the null space of X. The following is a basic result.

(1.4.6) Theorem. (Birkhoff & MacLane [B4], Ch. VIII, Theorem 11). For any $X \subseteq \mathbb{R}^J$, the rank of X plus the nullity of X equals $|J|$.

(1.4.7) If $x, y \in \mathbb{R}^J$, we say $x \leq y$ if $x_j \leq y_j$ for all $j \in J$. We say $x < y$ if $x_j < y_j$ for all $j \in J$.

(1.4.8) Let I, J be finite sets. If $A \subseteq \mathbb{R}^{I \times J}$ is the matrix $(a_{ij} \in \mathbb{R} : i \in I, j \in J)$ then for any $S \subseteq I$ we let A_S denote $(a_{ij} : i \in S, j \in J)$. Similarly if $b = (b_i : i \in I) \in \mathbb{R}^I$, we denote $(b_i : i \in S)$ by b_S. If S is a single element v we abbreviate $A_{\{v\}}$ by A_v. If $x = (x_j : j \in J) \in \mathbb{R}^J$ we define the product Ax to be the vector $y = (y_i : i \in I) \in \mathbb{R}^I$ where $y_i = A_i \cdot x$ for all $i \in I$.

We define the transpose of A, denoted by A^T to be the matrix $(a'_{ji}: j \in J,\ i \in I) \in \mathbb{R}^{J\times I}$ where $a'_{ji} = a_{ij}$ for all $i \in I,\ j \in J$.

(1.4.9) By the <u>rank of A</u> and <u>nullity of A</u> (written rank(A), nullity(A)) we mean the rank and nullity respectively of $\{A_i: i \in I\}$ as defined in (1.4.3) and (1.4.5).

We call $\{A_i: i \in I\}$ the <u>rows</u> of A; and $\{(a_{ij}: i \in I): j \in J\}$ the <u>columns</u> of A.

1.5 <u>Linear Programming</u>

Let I, J be finite sets, let $H \subseteq I$ and let $K \subseteq J$. Let $A \in \mathbb{R}^{I\times J}$, $b \in \mathbb{R}^I$ and $c \in \mathbb{R}^J$. A (primal) <u>linear programming problem</u> is

(1.5.1) maximize $c \cdot x$

for $x \in \mathbb{R}^J$ satisfying

(1.5.2) $x_K \geq 0$,

(1.5.3) x_{J-K} unrestricted in sign,

(1.5.4) $A_H x \leq b_H$,

(1.5.5) $A_{I-H} x = b_{I-H}$.

The <u>dual linear program</u> (Dantzig [D1] p. 126) is the linear program

(1.5.6) minimize $b \cdot y$

for $y \in \mathbb{R}^I$ satisfying

(1.5.7) $y_H \geq 0$,

(1.5.8) y_{I-H} unrestricted in sign,

(1.5.9) $A_K^T y \geq c_K$,

(1.5.10) $A_{J-K}^T y = c_{J-K}$.

Texts on linear programming generally show how a problem of the form (1.5.1)-(1.5.5) or (1.5.6)-(1.5.10) can be reduced to a problem in which $K = J$ and $H = \phi$ or $H = I$. (e.g. Dantzig [D1] p. 85-89). The following theorems are then usually proved for problems in these canonical forms. These results can be easily extended to apply to linear programs in the forms (1.5.1)-(1.5.5) or (1.5.6)-(1.5.10).

A vector $x \in \mathbb{R}^J$ satisfying (1.5.2)-(1.5.5) is called a <u>feasible solution</u> to the primal problem. A vector $y \in \mathbb{R}^I$ which satisfies (1.5.7)-(1.5.10) is called a <u>feasible dual solution</u>.

A feasible primal solution x^0 which maximizes $c \cdot x$ for all feasible primal solutions is called an <u>optimal primal solution</u>; an <u>optimal dual solution</u> is defined analagously.

The following is a fundamental theorem of linear programming (See Dantzig [D1] p. 120 Theorem 1).

(1.5.11) <u>Theorem</u>. <u>For any linear programming problem exactly one of the following situations occurs</u>.

i) <u>There exists no feasible solution</u>.

ii) <u>For any $\alpha \in \mathbb{R}$ there is a feasible solution x such that $c \cdot x > \alpha$</u>.

iii) <u>There is an optimal feasible solution</u>.

The following theorems give the relationship between the values of $c \cdot x$ and $b \cdot y$ for primal and dual feasible solutions.

(1.5.12) *Weak L.P. Duality Theorem* (Dantzig [D1] p. 130)

If x is a feasible primal solution and y is a feasible dual solution then $c \cdot x \leq b \cdot y$.

(1.5.13) *Corollary*. *If for any $\alpha \in \mathbb{R}$ there is a feasible dual solution y such that $b \cdot y \leq \alpha$ then there is no feasible primal solution.*

(1.5.14) *Strong L.P. Duality Theorem* (Dantzig [D1] p. 129 Theorem 1, p. 134, Theorems 2, 3).

If there is a feasible primal solution and an upper bound $c \cdot x$ over for all feasible primal solutions x then there is an optimal primal solution x^0 and an optimal dual solution y^0 and $c \cdot x^0 = b \cdot y^0$.

(1.5.15) *Corollary (Farkas' Lemma)* (Dantzig [D1] p. 137, Theorem 6.)

Let $A \in \mathbb{R}^{I \times J}$, $b \in \mathbb{R}^I$. There exists $x \in \mathbb{R}^J$ such that $x \geq 0$ and $Ax = b$ if and only if there is no $y \in \mathbb{R}^I$ such that $A^T y \leq 0$ and $b \cdot y > 0$.

The following theorem is used extensively in later chapters. It is the tool used to prove optimality of the solutions produced by the matching algorithms.

(1.5.16) Complementary Slackness Theorem (Dantzig [D1] p. 135,136).

A feasible solution x^0 to (1.5.2)-(1.5.5) and a feasible solution y^0 to (1.5.7)-(1.5.10) are optimal if and only if

(1.5.17) *$x_j^0 > 0$ implies $A_j^T y^0 = c_j$ for all $j \in K$,*

(1.5.18) *$y_i^0 > 0$ implies $A_i x^0 = b_i$ for all $i \in H$.*

Proof. For any feasible solution x to (1.5.2)-(1.5.5) and any feasible solution y to (1.5.7)-(1.5.10) we define

(1.5.19) $$f(x,y) \equiv x \cdot (A^T y - c) + y \cdot (b - Ax)$$

$$= x_K(A_K^T y - c_K) + y_H(b_H - A_H x)$$

(1.5.20) $$= \sum_{j \in K} x_j(A_j^T y - c_j) + \sum_{i \in H} y_i(b_i - A_i x)$$

by (1.5.5) and (1.5.10). By (1.5.2), (1.5.4), (1.5.7) and (1.5.9) every term in (1.5.20) is the product of nonnegative factors so

(1.5.21) $f(x, y) \geq 0$.

Moreover,

(1.5.22) $f(x, y) = 0$ if and only if one factor in each term of (1.5.20) is zero.

Simplifying (1.5.19) gives

(1.5.23) $f(x, y) = b \cdot y - c \cdot x$.

(Note that (1.5.21) and (1.5.23) together prove (1.5.12)).

If x^0 and y^0 satisfy (1.5.17) and (1.5.18) then by (1.5.22) $f(x^0, y^0) = 0$. Therefore, by (1.5.21) and (1.5.23) x^0 and y^0 are optimal solutions.

If x^0 and y^0 are optimal solutions then by (1.5.13) (Strong L.P. Duality) $b \cdot y^0 = c \cdot x^0$ so by (1.5.23), $f(x^0, y^0) = 0$. Therefore by (1.5.22), x^0 and y^0 must satisfy (1.5.17) and (1.5.18).□

Notice that the sufficiency of (1.5.17) and (1.5.18) were easily proved, however we required the strong duality theorem of linear programming to prove their necessity. In the applications we make use of complementary slackness in proving optimality of the matchings produced by the blossom algorithm and the face optimization algorithm, all we require is the sufficiency of (1.5.17) and (1.5.18) for the algorithm in fact produces solutions x^0 and y^0 satisfying (1.5.17) and (1.5.18).

1.6 Integer Programming and Good Algorithms.

When studying algorithms it is often desirable to be able to establish an upper bound on the amount of work performed by the algorithm as a function of the size of the problem. An elementary step of an algorithm is any step performed by the algorithm which does not depend on the size of the problem, for example adding two numbers , comparing two numbers, seeing whether an edge of a graph meets a node of a graph. Thus an algorithm will, in solving a problem, perform a certain number of elementary steps. If there is some constant K such that the number of these elementary steps which can be performed in solving a problem P whose size is measured by the parameters $r_1, r_2, \ldots, r_n$ is no greater than $K \cdot f(r_1, r_2, \ldots, r_n)$ where f is some function of $r_1, r_2, \ldots, r_n$ then we say that an upper bound on the amount of work performed by the algorithm is of the order $f(r_1, r_2, \ldots, r_n)$.

In this thesis, when discussing bounds on algorithms, we make a "fixed-word" assumption, namely that the time required to perform arithmetic operations (addition, subtraction, division by two) on two numbers is independent of the number of digits in the numbers. This is the way in which most large computers operate, the number of significant digits to be considered becomes a constraint as to whether or not a problem is solvable rather than a factor in the time taken to solve the problem.

Following the terminology of Edmonds [E1] we call an algorithm "good" if there is an upper bound on the amount of work performed by the algorithm that is of the order $p(r_1, r_2, \ldots, r_n)$ where $p(r_1, r_2, \ldots, r_n)$ is a polynomial function of $r_1, r_2, \ldots, r_n$.

Consider the problem (1.5.1)-(1.5.5) with the added restriction

$$(1.6.1) \quad x_j \text{ is integer valued for all } j \in J.$$

Such a problem is called an <u>integer programming problem</u>. Although it does not have a polynomial bound, the famous Simplex Algorithm of Dantzig, does provide a practical method of solving reasonably large linear programming problems.

CHAPTER 2

Basic Polyhedral Theory

In this chapter we define polyhedra and develop some of their basic properties which are used in later chapters. In particular we prove two theorems characterizing the facets of a polyhedron which are used extensively in Chapter 4.

This treatment of the subject, suggested by J. Edmonds, is most similar to that of Stoer, Witzgall [S1]. Other standard references are Grünbaum [G1] and Rockafellar [R1]. The advantage of our approach for present purposes is that it tends to emphasize the relationship between polyhedral theory and linear programming and it is in fact this relationship which prompts our interest in special classes of polyhedra.

2.1 Polyhedra and their Faces

Let I and J be finite sets, let $A = (a_{ij}: i \in I, j \in J) \in \mathbb{R}^{I \times J}$ and let $b = (b_i: i \in I) \in \mathbb{R}^I$. We call the set of linear inequalities $Ax \le b$ a <u>linear system</u> and define a <u>polyhedron</u> to be the solution set of any linear system. We define the polyhedron

$$P(A, b) \equiv \{x \in \mathbb{R}^J: Ax \le b\}.$$

We take A, b, I and J to be defined as above throughout the rest of this chapter.

If there is $i \in I$ such that $A_i = 0$ then either $b_i < 0$ in which case $P(A, b) = \phi$ or else $b_i \ge 0$ and

$P(A, b) = P(A_{I-\{i\}}, b_{I-\{i\}})$. Therefore we will henceforth assume that $A_i \neq 0$ for all $i \in I$ (that is, the matrix A has no zero rows).

If K is a finite set, $A' \in \mathbb{R}^{K \times J}$ and $b' \in \mathbb{R}^K$ then

$$P \equiv \{x \in \mathbb{R}^J: Ax \leq b, A'x = b'\}$$

is the same set as

$$Q \equiv \{x \in \mathbb{R}^J: Ax \leq b, A'x \leq b', (-A')x \leq -b'\}.$$

Since Q is a polyhedron, we have

(2.1.1) any $P \subseteq \mathbb{R}^J$ which is the solution set of a finite system of linear inequalities and linear equations is a polyhedron.

For any $I' \subseteq I$ we define

(2.1.2) $f(I') \equiv \{x \in P(A, b): A_{I'}x = b_{I'}\}$.

By (2.1.1) $f(I')$ is a polyhedron and is called a <u>face</u> of $P(A, b)$. The fact that the faces of $P(A, b)$ depend on the polyhedron, not the linear system $Ax \leq b$ is shown in (2.1.5). The empty set is also taken to be a face of every polyhedron.

It is clear that

(2.1.3) every face of a face of a polyhedron P is itself a face of P,

also,

(2.1.4) the intersection of any collection of faces of a polyhedron P is itself a face of P; if $I^k \subseteq I$ for $k \in K$ we have $\bigcap_{k \in K} f(I^k) = f(\bigcup_{k \in K} I^k)$.

There is associated with every linear system $Ax \leq b$ a unique maximal set $I^0 \subseteq I$ for which $P(A, b) = f(I^0)$ (since for any $t \in I$, either there exists $x^t \in P(A, b)$ such that $A_t x^t < b_t$ in which case $t \notin I^0$ or no such x^t exists and $t \in I^0$). We call I^0 the <u>equality set</u> of $Ax \leq b$. We say that I^1 is the <u>equality set</u> of a face F of $P(A, b)$ if I^1 is the maximal subset of I such that $F = f(I^1)$.

It is easily seen that there are many different sets of linear inequalities which define the same polyhedron. However the faces of the polyhedron depend only upon the polyhedron itself and not upon the choice of inequalities. This we now prove by showing that a nonempty subset F of a polyhedron P is a face of P if and only if there is some linear function c which is maximized over P by precisely the members of F.

(2.1.5) <u>Theorem</u>. $F \subseteq P(A, b)$ <u>is a nonempty face of $P(A, b)$ if and only if</u>

(2.1.6) <u>there is $c \in \mathbb{R}^J$ and $\alpha \in \mathbb{R}$ such that $cx = \alpha$ for all $x \in F$ and $cx < \alpha$ for all $x \in P(A, b) - F$.</u>

<u>Proof</u>. First we prove the necessity of (2.1.6), let F be a nonempty face of $P(A, b)$, let I^0 be the equality

set of F. Then for each $x \in P(A, b) - F$ there is some $t(x) \in I^0$ such that

$$(2.1.7) \qquad A_{t(x)}x < b_{t(x)} .$$

If $I^0 = \phi$ we take $c_j \equiv 0$ for all $j \in J$, otherwise take $c_j \equiv \Sigma(a_{ij}: i \in I^0)$ for all $j \in J$.

For any $x \in F$,

$$\Sigma(c_j x_j: j \in J) = \Sigma(a_{ij}x_j: i \in I^0, j \in J) = \Sigma(b_i: i \in I^0)$$

since I^0 is the equality set of F. For any $x \in P(A,b)-F$ we have

$$\Sigma(c_j x_j: j \in J) = \Sigma(a_{ij}x_j: i \in I^0-\{t(x)\}, j \in J) + \Sigma(a_{t(x)j}x_j: j\in J)$$

$$< \Sigma(b_i: i \in I^0) \text{ by } (2.1.7).$$

Thus if we take $\alpha \equiv \Sigma(b_i: i \in I^0)$, α and c so defined satisfy (2.1.6).

We now prove the sufficiency. Let F be a nonempty subset of P, let c and α be as in (2.1.6). Then the linear program

$$\text{maximize } c \cdot x$$

for

$$Ax \leq b$$

has an upper bound. So by the strong linear programming duality theorem (1.5.14) there is an optimal solution $y^0(y^0: i \in I)$ to the dual linear program

$$\text{minimize } b \cdot y$$

$$y \geq 0$$

$$A^T_y = c .$$

By complementary slackness (1.5.16) a solution x to $Ax \le b$ maximizes cx if and only if $A_i x = b_i$ for all $i \in I$ such that $y_i^0 \ne 0$. Thus $F = f(\{i \in I: y_i \ne 0\})$ and the proof is complete.□

We obtain the following result by combining (2.1.6) and (1.5.10).

(2.1.8) <u>Theorem</u>. <u>Let $c \in \mathbb{R}^J$. If there is $\alpha \in \mathbb{R}$ such that $c \cdot x \le \alpha$ for all x belonging to a nonempty polyhedron $P(A, b)$ then there is a face F of $P(A, b)$ such that x^0 maximizes $c \cdot x$ for $x \in P(A, b)$ if and only if $x^0 \in F$.</u>

<u>Proof</u>. Since $P(A, b) \ne \phi$ and since $c \cdot x \le \alpha$ for all $x \in P(A, b)$ it follows from (1.5.11) that there is $x^0 \in P(A, b)$ such that $c \cdot x^0 = \max\{c \cdot x: x \in P(A, b)\}$. Let $F \equiv \{x \in P(A, b): c \cdot x = c \cdot x^0\}$. By (2.1.5) F is a face of $P(A, b)$.□

Let I^0 be the equality set of $Ax \le b$. We call $x \in P(A, b)$ an <u>interior point</u> of $P(A, b)$ if $A_{I-I^0} x < b_{I-I^0}$.

(2.1.9) <u>Proposition</u>. <u>Every nonempty polyhedron has an interior point</u>.

<u>Proof</u>. Suppose I^0 is the equality set of $Ax \le b$ and $P(A, b) \ne \phi$. If $I^0 = I$ then any $x \in P(A, b)$ is trivially an interior point. Otherwise for each $t \in I - I^0$ there must be $x^t \in P(A, b)$ such that

$$\begin{array}{ll} & A_{I^0} x^t = b_{I^0} \\ (2.1.10) & A_t x^t < b_t \\ & A_{I^t} x^t \le b_{I^t} \end{array}$$

where $I^t = I - I^0 - \{t\}$ for otherwise t would be in the equality set of $P(A, b)$. Let

$$\bar{x} \equiv \Sigma(x^t : t \in I - I^0)/|I - I^0|.$$

It follows immediately from (2.1.10) that

$$A_{I^0}\bar{x} = b_{I^0},$$

$$A_{I-I^0}\bar{x} < b_{I-I^0}$$

so $\bar{x}$ is an interior point of $P(A, b)$ as required.□

2.2 Dimension and a First Facet Characterization

Let $Ax \le b$ have equality set I^0. If $P(A, b) = \phi$ then we define the dimension of $P(A, b)$ to be -1. Otherwise we define the dimension of $P(A, b)$ to be

$$|J| - \text{rank}\,(A_{I^0}).$$

We show in (2.2.14) that dimension depends only on the polyhedron not on the linear system which defines the polyhedron. We denote the dimension of a polyhedron P by $\dim(P)$. It follows from (1.4.9) and (1.4.3) that if $P \ne \phi$, $\dim(P) \ge 0$.

Clearly every polyhedron P is a face of itself called an <u>improper face</u>. All other faces including the empty face, are called <u>proper faces</u>.

If $\dim(P(A, b)) = |J|$, that is if $P(A, b) \neq \phi$ and $\text{rank}(A_{I^0}) = 0$ where I^0 is the equality set of $Ax \leq b$, then we say that $P(A, b)$ is of <u>full dimension</u>.

First we show that the dimension of every proper face of a polyhedron P is less than $\dim(P)$.

(2.2.1) <u>Proposition.</u> <u>Let F be a proper face of $P(A, b)$. Then $\dim(F) \leq \dim(P(A, b)) - 1$.</u>

<u>Proof</u>. Since $P(A, b)$ has a proper face, $P(A, b)$ is nonempty. If $F = \phi$ then the result is trivial. Assume $F \neq \phi$, let I^0 be the equality set of $Ax \leq b$, let I' be the equality set of F. Then $I^0 \subseteq I'$ and $\text{rank}(A_{I^0}) \leq \text{rank}(A_{I'})$. Suppose

$$\text{(2.2.2)} \qquad \text{rank}(A_{I^0}) = \text{rank}(A_{I'}) .$$

Then a row basis of A_{I^0} is a row basis of $A_{I'}$ hence for any $t \in I' - I^0$, A_t is a linear combination of rows of A_{I^0}. If b_i is not equal to the same linear combination of the components of b_{I^0} then $F = \phi$, contradictory to our assumption. Otherwise, for any $x \in \mathbb{R}^J$ satisfying $A_{I^0}x = b_{I^0}$ we also have $A_t x = b_t$ so $t \in I^0$, contradictory to the choice of t. Hence (2.2.2) must be false,

$$\text{rank}(A_{I^0}) + 1 \leq \text{rank}(A_{I'})$$

and the result now follows from the definition of dimension.□

Let $\{x^k : k \in K\} \subseteq \mathbb{R}^J$. We say that $x^k : k \in K$ are <u>affinely independent</u> if for any $(\alpha_k \in \mathbb{R} : k \in K)$ such that

$$\Sigma(\alpha_k x^k : k \in K) = 0$$

and

$$\Sigma(\alpha_k : k \in K) = 0$$

we have $\alpha_k = 0$ for all $k \in K$. If $x^k : k \in K$ are not affinely independent then we say that they are <u>affinely dependent</u>.

Let $\{x^k : k \in K\} \subseteq \mathbb{R}^J$. We say that $x \in \mathbb{R}^J$ is an <u>affine combination</u> of $\{x^k : k \in K\}$ if there exist $\alpha_k \in \mathbb{R}$ for $k \in K$ such that

$$x = \Sigma(\alpha_k x^k : k \in K)$$

and

$$\Sigma(\alpha_k : k \in K) = 1$$

The following is an immediate consequence of these definitions

(2.2.3) <u>Proposition</u>. <u>The vectors $x^k \in \mathbb{R}^J : k \in K$ are affinely independent if and only if no x^h for $h \in K$ is an affine combination of $\{x^k : k \in K - \{h\}\}$.</u>

The following proposition relates affine independence to linear independence.

(2.2.4) <u>Proposition</u>. <u>The vectors $x^k \in \mathbb{R}^J : k \in K$ are affinely independent if and only if for any $h \in K$, the vectors $x^k - x^h : k \in K-\{h\}$ are linearly independent.</u>

Proof. Suppose $x^k: k \in K$ are affinely independent, let $h \in K$ and let $K' \equiv K - \{h\}$. Let $(\alpha_k \in \mathbb{R} : k \in K')$ be such that

$$\Sigma(\alpha_k(x^k - x^h): k \in K') = 0$$

Then

$$-\Sigma(\alpha_k: k \in K')x^h + \Sigma(\alpha_k x^k: k \in K') = 0$$

and

$$-\Sigma(\alpha_k: k \in K') + \Sigma(\alpha_k: k \in K') = 0$$

so since $x^k: k \in K$ are affinely independent we must have $\alpha_k = 0$ for all $k \in K$ and the vectors $x^k - x^h: k \in K - \{h\}$ are linearly independent.

Conversely, suppose that for $h \in K$ the vectors $x^k - x^h: k \in K' \equiv K - \{h\}$ are linearly independent. Let $(\alpha_k \in \mathbb{R} : k \in K)$ be such that

$$\Sigma(\alpha_k x^k: k \in K) = 0 \tag{2.2.5}$$

$$\Sigma(\alpha_k: \quad k \in K) = 0. \tag{2.2.6}$$

Then by (2.2.6) $\alpha_h = -\Sigma(\alpha_k: k \in K')$ so (2.2.5) implies $-\Sigma(\alpha_k x^h: k \in K') + \Sigma(\alpha_k x^k: k \in K') = 0$ or $\Sigma(\alpha_k(x^k - x^h): k \in K') = 0$. Since $(x^k - x^h): k \in K'$ are linearly independent we have $\alpha_k = 0$ for all $k \in K'$. Hence, by (2.2.6), $\alpha_h = 0$ and so $x^k: k \in K$ are affinely independent and the proof is complete.□

Note that affine independence is implied by linear independence and affine dependence implies linear dependence.

For $V \subseteq \mathbb{R}^J$ we define the affine rank of V to be the cardinality of a largest affinely independent subset of V. In view of (2.2.4) and (1.4.3),

(2.2.7) the affine rank of $V \subseteq \mathbb{R}^J$ is no greater than $|J| + 1$.

We now prove a theorem which relates the affine rank of a polyhedron to its dimension and thus shows that the dimension of a polyhedron is determined irrespective of the linear system.

(2.2.8) Lemma. If $\dim(P(A, b)) = k$ then $P(A, b)$ contains $k + 1$ affinely independent elements.

Proof. If $k = -1$ then $P(A, b) = \phi$ and the result is trivial. Otherwise $k \geq 0$ and $P(A, b) \neq \phi$. Let I^0 be the equality set of $Ax \leq b$. By (2.19) $P(A, b)$ has an interior point x which satisfies

(2.2.9) $$A_{I^0}x = b_{I^0} ,$$

(2.2.10) $$A_{I-I^0}x < b_{I-I^0} .$$

If $k = 0$ then $\{x\}$ is the set of affinely independent elements we require. Suppose $k \geq 1$. Since $\dim(P(A, b)) = k$, $\operatorname{rank}(A_{I^0}) = |J| - k$. Therefore by (1.4.6) $\operatorname{nullity}(A_{I^0}) = k$. Hence there are k linearly independent vectors $y^1, y^2, \ldots, y^k \in \mathbb{R}^J$ such that

$$(2.2.11) \qquad A_{I^0} y^i = 0 \quad \text{for} \quad i \in \{1, 2, \ldots, k\}.$$

Let $t \in \{1, 2, \ldots, k\}$. In view of (2.2.10) there is $\epsilon_t > 0$ such that

$$A_{I-I^0}(x + \epsilon_t y^t) \leq b_{I-I^0}$$

since $A_{I-I^0}(x + \epsilon_t y^t) = A_{I-I^0}x + \epsilon_t(A_{I-I^0}y^t)$.

Then
$$A_{I^0}(x + \epsilon_t y^t) = A_{I^0}x + \epsilon_t A_{I^0} y^t$$
$$= b_{I^0}$$

by (2.2.9) and (2.2.11). Thus the vectors x, $x + \epsilon_1 y^1$, $x + \epsilon_2 y^2, \ldots, x + \epsilon_k y^k$ all belong to $P(A, b)$. Moreover, since $y^1, y^2, \ldots, y^k$ are linearly independent and since $\epsilon_t > 0$ for all $t \in \{1, 2, \ldots, k\}$, $\epsilon_1 y^1, \epsilon_2 y^2, \ldots, \epsilon_k y^k$ are linearly independent. Hence by (2.2.4) $x, x + \epsilon_1 y^1, \ldots, x + \epsilon_k y^k$ are affinely independent and the proof is complete.□

(2.2.12) <u>Lemma</u>. <u>If $P(A, b)$ contains $k + 1$ affinely independent members then $\dim(P(A, b)) \geq k$.</u>

<u>Proof</u>. If $k \leq 0$ the result is trivial, assume $k \geq 1$. Let $x^0, x^1, \ldots, x^k$ be affinely independent members of $P(A, b)$. Then if I^0 is the equality set of $Ax \leq b$ we have

$$(2.2.13) \qquad A_{I^0} x^i = b_{I^0} \quad \text{for} \quad i \in \{0, 1, \ldots, k\}.$$

By (2.2.4) the vectors $x^1 - x^0, x^2 - x^0, \ldots, x^k - x^0$

are linearly independent. Moreover by (2.2.13)

$$A_{I^0}(x^i - x^0) = A_{I^0}x^i - A_{I^0}x^0$$

$$= b_{I^0} - b_{I^0} = 0$$

for $i \in \{1, 2, \ldots, k\}$. Hence $\mathrm{nullity}(A_{I^0}) \geq k$ and so $\mathrm{rank}(A_{I^0}) \leq |J| - k$. Thus $\dim(F) = |J| - \mathrm{rank}(A_{I^0}) \geq k$. □

We can now combine these two lemmas to obtain the following theorem.

(2.2.14) Theorem. The dimension of $P(A, b)$ is one less than the affine rank of $P(A, b)$.

We showed (2.1.5) that the faces of a polyhedron P are independent of the choice of inequalities used to represent P. A consequence of (2.2.14) is that the dimension of a polyhedron is also independent of the choice of inequalities since the affine rank does not depend on the set of inequalities used to define the polyhedron.

If F is a face of $P(A, b)$ and $\dim(F) = \dim(P(A,b))-1$ then F is called a facet of $P(A, b)$.

In Chapter 4 we make extensive use of the following corollary of (2.2.14).

(2.2.15) Corollary. If F is a proper face of a polyhedron P of dimension d then F is a facet of P if and only if F contains d affinely independent elements.

Proof. The result is a combination of (2.2.1) and (2.2.14).□

2.3 Second Facet Characterization

We prove in this section that the facets of a polyhedron P are precisely the maximal proper faces of P. We also show that the facets of P correspond in a certain sense to a minimal collection of inequalities required to define P. We then discuss the specialization of this theorem to the case in which P is of full dimension as this is the situation which we study in chapter 4.

(2.3.1) Theorem. Let $P \equiv P(A, b)$ be nonempty and let I^0 be the equality set of $Ax \leq b$. Let $I' \subseteq I - I^0$. Let $P' \equiv P(A_{I-I'}, b_{I-I'})$. Then $P \neq P'$ if and only if $I' \cup I^0$ contains the equality set of a nonempty proper face of P.

Proof. Clearly $P \subseteq P'$, suppose there is some $y \in P' - P$. Then for some nonempty $K \subseteq I'$ we have

$$A_{I-K}y \leq b_{I-K} \tag{2.3.2}$$

and

$$A_K y > b_K . \tag{2.3.3}$$

By (2.1.9) P has an interior point w, that is, w satisfies

$$A_{I-I^0}w < b_{I-I^0} , \tag{2.3.4}$$

$$A_{I^0}w = b_{I^0} . \tag{2.3.5}$$

Therefore we can choose $\lambda \in \mathbb{R}$ satisfying $0 < \lambda < 1$ such that if we let $z \equiv \lambda w + (1 - \lambda)y$ then for some nonempty $T \subseteq K$

$$(2.3.6) \qquad A_T z = b_T \ ,$$

$$(2.3.7) \qquad A_{I-I^0-T} z < b_{I-I^0-T} \ ,$$

$$(2.3.8) \qquad A_{I^0} z \le b_{I^0} \ .$$

(Take $\lambda \equiv \max\{(A_i y - b_i)/(A_i(y - w)) : i \in K\}$ and let T be the set of $i \in K$ which attain this maximum).

By (2.3.6) - (2.3.8) $z \in P$ so

$$(2.3.9) \qquad A_{I^0} z = b_{I^0} \ .$$

By (2.3.6) and (2.3.9) $z \in f(I^0 \cup T)$ and by (2.3.7), $I^0 \cup T$ is the equality set of this face. This proves the necessity of our condition , since $I^0 \cup T \subseteq I^0 \cup K \subseteq I^0 \cup I'$.

Conversely, suppose that $I^0 \cup I'$ contains the equality set of a nonempty proper face F of P. Let K be the equality set of F. Note that $I^0 \subset K \subseteq I^0 \cup I'$. By (2.1.9) F has an interior point y, that is, y satisfies

$$(2.3.10) \qquad A_K y = b_K$$

$$(2.3.11) \qquad A_{I-K} y < b_{I-K} \ .$$

Similarly P has an interior point w, that is an element w satisfying

$$(2.3.12) \qquad A_{I^0} w = b_{I^0} \ ,$$

$$(2.3.13) \qquad A_{I-I^0} w < b_{I-I^0} \,.$$

For any $\epsilon > 0$ let $z(\epsilon) \equiv (1 + \epsilon)y - \epsilon w$. Then

$$(2.3.14) \qquad A_{I^0} z(\epsilon) = b_{I^0} \quad \text{for any } \epsilon \in \mathbb{R} \text{ by}$$

(2.3.10) and (2.3.12).

$$(2.3.15) \qquad A_{K-I^0} z(\epsilon) = b_{K-I^0} + \epsilon(A_{K-I^0} y - A_{K-I^0} w)$$

$$> b_{K-I^0} \quad \text{for any } \epsilon > 0$$

by (2.3.11) and (2.3.13).

$$A_{I-K} z(\epsilon) = A_{I-K} y + \epsilon \cdot A_{K-I^0}(y - w)$$

so in view of (2.3.11) if we choose $\bar{\epsilon} > 0$ sufficiently small we will have

$$(2.3.16) \qquad A_{I-K} z(\bar{\epsilon}) \le b_{I-K} \,.$$

Since $K \supset I^0$, by (2.3.15) $z(\bar{\epsilon}) \notin P$. Since $I - K \subseteq I - I'$, by (2.3.14) and (2.3.16) $z(\bar{\epsilon}) \in P' = P(A_{I-I'}, b_{I-I'})$. That is $P \neq P$ and the proof is complete.□

We are now in a position to prove the following theorem equating the facets of a polyhedron to its maximal proper faces.

(2.3.17) Theorem. <u>$F \neq \phi$ is a facet of $P(A, b)$ if and only if F is a maximal proper face of $P(A, b)$.</u>

<u>Proof</u>. Suppose $F \neq \phi$ is a maximal proper face of

$P(A, b)$. Then by (2.2.1)

$$(2.3.18) \quad \dim(F) \le \dim(P(A, b)) - 1.$$

Let I^0 be the equality set of $Ax \le b$, let I' be the equality set of F. Let $i \in I' - I^0$ and let $K \equiv I'-I^0-\{i\}$.

If $K \cup I^0$ $(= I' - \{i\})$ contained the equality set of a proper face F' of $P(A, b)$ then $F \subset F'$ contradicting the maximality of F. Thus by (2.3.1),

$$P(A, b) = P(A_{I-K}, b_{I-K})$$

and I^0 is the equality set of $A_{I-K}x \le b_{I-K}$. The equality set of F in $P(A_{I-K}, b_{I-K})$ is $I^0 \cup \{i\}$ so since $\text{rank}(A_{I^0 \cup \{i\}}) \le \text{rank}(A_{I^0}) + 1$ we have

$$(2.3.19) \quad \dim(F) \ge \dim(P(A, b)) - 1.$$

Combining (2.3.18) and (2.3.19) we see that F is a facet of $P(A, b)$.

Conversely, suppose that $F \ne \phi$ is a facet of $P(A, b)$. Then

$$(2.3.20) \quad \dim(F) = \dim(P(A, b)) - 1.$$

Suppose that there is a face F' of $P(A, b)$ such that $F \subset F' \subset P(A, b)$. By (2.2.1)

$$(2.3.21) \quad \dim(F') \le \dim(P(A, b)) - 1.$$

By (2.1.3) F is a face of F' and since we assume $F \subset F'$, F is a proper face of F'. Thus by (2.2.1),

$$(2.3.22) \qquad \dim(F) \le \dim(F') - 1.$$

Combining (2.3.21) and (2.3.22) we have

$$\dim(F) \le \dim(P(A, b)) - 2$$

a contradiction to (2.3.20) which proves the theorem.□

It should be noted that the hypothesis $F \ne \phi$ is indeed necessary in (2.3.17) as is shown by the following example.

Let $P \equiv \{(x_1, x_2) \in \mathbb{R}^{\{1,2\}}: x_1 + x_2 = 1\}$. Then $\dim(P) = 1$ and ϕ is the only proper face of P. But $\dim(\phi) = -1$ so ϕ is not a facet of P. This also illustrates that there do exist polyhedra having no facets.

(2.3.23) Corollary. Let P be a polyhedron, let $d \equiv \dim(P)$. Let $F \ne \phi$ be a face of P of dimension $k < d$. Then there are faces $F_{k+1}, F_{k+2}, \ldots, F_{d-1}$ of P such that

$$F \subset F_{k+1} \subset F_{k+2} \subset \ldots \subset F_{d-1} \subset P$$

$$\dim(F_j) = j \quad \text{for} \quad j \in \{k+1, k+2, \ldots, d-1\}$$

Proof. We prove by induction on $d - k$. If $d - k = 1$ then there is nothing to prove. Suppose the result is true when $d - k < t \ge 2$ and assume $d = k + t$. Let F_{d-1} be a maximal proper face of P containing F, that is

$$F \subset F_{d-1} \subset P.$$

Then $F_{d-1} \ne \phi$ so by (2.3.17) $\dim(F_d) = d - 1$. Since $(d - 1) - k < t$ there are by our induction hypothesis faces $F_{k+1}, F_{k+2}, \ldots, F_{d-2}$ of F_{d-1} such that

$$F \subset F_{k+1} \subset \ldots \subset F_{d-2} \subset F_{d-1}$$

and $\dim(F_j) = j$ for $j \in \{k+1, k+2,\ldots,d-2\}$. By (2.1.3) F_j is a face of P for $j \in \{k+1, k+2,\ldots,d-2\}$ so the result follows.□

Given the polyhedron $P(A, b)$ we may wish to find a set $I^* \subset I$ such that $P(A_{I^*}, b_{I^*}) = P(A, b)$ and I^* is minimal with this property. The next theorem characterizes such sets. First we observe the following fact.

(2.3.24) <u>Proposition. Let $F_1, F_2,\ldots,F_k$ be the facets of $P(A, b)$, let I^0 be the equality set of $Ax \le b$ and let I^i be the equality set of F_i for $i \in \{1, 2,\ldots,k\}$. Then $I^i \cap I^j = I^0$ for all distinct $i, j \in \{1, 2,\ldots,k\}$.</u>

<u>Proof</u>. Let i, j be distinct members of $\{1, 2,\ldots,k\}$ and let $K = I^i \cap I^j$. Then $I^0 \subseteq K$. Since $F_i \ne F_j$ and since both are maximal proper faces (by (2.3.17)) there are $x_i \in F_i - F_j$ and $x_j \in F_j - F_i$. Then $x_i, x_j \in f(K)$ so $F_i \ne f(K) \ne F_j$. But $f(K) \supseteq F^i \cup F^j$ so since F^i and F^j are maximal, $f(K) = P(A, b)$ so $K = I^0$, completing the proof.□

(2.3.25) <u>Theorem</u>. <u>Let F_i: $i \in K$ be the facets of a nonempty polyhedron $P(A, b)$, let I^0 be the equality set of $Ax \le b$ and let I^i be the equality set of F^i for $i \in K$. Let $I^* \subseteq I$. Then $P(A, b) = P(A_{I^*}, b_{I^*})$ if and only if</u>

$$(2.3.26)\qquad \underline{\operatorname{rank}(A_{I^0 \cap I^*}) = \operatorname{rank}(A_{I^0})}$$

<u>and</u>

(2.3.27) $\underline{(I^* \cap I^i) - I^0 \neq \phi \text{ for all } i \in K}$.

<u>Proof</u>. Suppose I^* satisfies (2.3.26) and (2.3.27). Then the rows of $A_{I^0 \cap I^*}$ are a basis of the rows of A_{I^0}. Hence for any $t \in I^0 - I^*$, A_t must be a linear combination of rows of $A_{I^0 \cap I^*}$ and b_t must be the same linear combination of the rows of $b_{I^0 \cap I^*}$ or we would have $P(A, b) = \phi$. Thus if $x \in \mathbb{R}^J$ satisfies $A_{I^0 \cap I^*}x = b_{I^0 \cap I^*}$ then it also satisfies $A_{I^0}x = b_{I^0}$. Hence

$$(2.3.28) \quad P(A_{I^*}, b_{I^*}) = P(A_{I^0 \cup I^*}, b_{I^0 \cup I^*}).$$

By (2.3.27), $(I - I^*) \cup I^0$ cannot contain the equality set of a facet of $P(A, b)$ so by (2.3.17) and (2.3.1)

$$(2.3.29) \quad P(A_{I^0 \cup I^*}, b_{I^0 \cup I^*}) = P(A, b).$$

Combining (2.3.28) and (2.3.29) proves the sufficiency of (2.3.26) and (2.3.27).

If I^* does not satisfy (2.3.26) then $\dim(P(A_{I^*}, b_{I^*})) \geq \dim(P(A, b)) + 1$ so by (2.2.14), $P(A_{I^*}, b_{I^*}) \neq P(A, b)$. If I^* does not satisfy (2.3.27) then $(I - I^*) \cup I^0$ contains the equality set of a proper face of $P(A, b)$ so by (2.3.1), $P(A, b) \neq P(A_{I^0 \cup I^*}, b_{I^0 \cup I^*})$. Since $P(A, b) \subseteq P(A_{I^0 \cup I^*}, b_{I^0 \cup I^*}) \subseteq P(A_{I^*}, b_{I^*})$ the result now follows. □

If $P(A, b)$ is a polyhedron of full dimension and I^0 is the equality set of $Ax \leq b$ then $\text{rank}(A_{I^0}) = 0$ so since we assume A has no zero rows, $I^0 = \phi$. If I' is the equality set of a facet of $P(A, b)$ then $\operatorname{rank}(A_{I'}) = 1$ so if we define for each $i \in I$

$$p(i) \equiv \{t \in I: A_t = \alpha A_i,\ b_t = \alpha b_t \text{ for some } \alpha \in \ ,\ \alpha > 0\}$$

then we can easily see that all equality sets of facets are sets of this kind. Moreover for any $i \in I$, for any $t \in p(i)$ we have $f(\{t\}) = f(\{p(i)\})$. Thus (2.3.25) specializes to the following.

(2.3.30) *Theorem*. *Let $P(A, b)$ be a polyhedron of full dimension. Then for any $K \subseteq I$, $P(A, b) = P(A_K, b_K)$ if and only if $K \cap p(i) \neq \phi$ for each $i \in I$ such that $f(\{i\})$ is a facet of $P(A, b)$.*

(2.3.31) *Corollary*. *Let $P(A, b)$ be of full dimension. Then $K \subseteq I$ is a minimal set such that $P(A, b) = P(A_K, b_K)$ if and only if for each $i \in K$, $f(\{i\})$ is a distinct facet of $P(A, b)$.*

We also have the following result.

(2.3.32) *Theorem*. *Let $P(A, b)$ be of full dimension, let $K \subseteq I$ be such that $\{f(i): i \in K\}$ is the set of facets of $P(A, b)$. Suppose $P(A', b') \supseteq P(A, b)$ where $A' \in \mathbb{R}^{I' \times J}$, $b' \in \mathbb{R}^{I'}$ and I' is a finite set. Then $P(A, b) = P(A', b')$ if and only if*

(2.3.33) *for each $i \in K$ there are $t \in I'$ and some real $\alpha > 0$ such that $A'_t = \alpha \cdot A_i$ and $b'_t = \alpha b_i$.*

<u>Proof</u>. Assume $I' \cap I = \phi$, define $\bar{A} \in \mathbb{R}^{(I' \cup I) \times J}$ and $\bar{b} \in \mathbb{R}^{I' \cup I}$ by $\bar{A}_I \equiv A$, $\bar{A}_{I'} \equiv A'$, $\bar{b}_I \equiv b$, $\bar{b}_{I'} \equiv b'$. Suppose $P(A, b) = P(A', b')$. Then $P(\bar{A}, \bar{b}) = P(A', b')$ and $\{f(i): i \in K\}$ is the set of facets of $P(\bar{A}, \bar{b})$. Hence by (2.3.30) (taking $\bar{A}, \bar{b}$ for A, b and I' for K) we see that (2.3.33) must hold.

Conversely, suppose (2.3.33) holds. By (2.3.30), $P(A, b) = P(A_K, b_K)$. Since $P(A', b') \supseteq P(A, b) = P(A_K, b_K)$, (2.3.33) clearly implies $P(A', b') = P(A_K, b_K) = P(A, b)$ and the proof is complete.□

(2.3.32) shows that the facets of a full dimensional polyhedron $P(A, b)$ determine up to a positive multiple the minimal set of inequalities of which the polyhedron is the solution set. That is, any set of inequalities defining $P(A, b)$ must contain a positive multiple of $A_i x \leq b_i$ for each i such that $f(\{i\})$ is a facet of $P(A, b)$. (2.3.31) shows that the converse also holds, if $Ax \leq b$ is a minimal set of inequalities defining a full dimensional polyhedron P, then $f(\{i\})$ is a facet of P for each $i \in I$.

This is one of the reasons for our interest in the facets of matching polyhedra. These polyhedra (see section 3.4) can be defined for a graph G by a set of inequalities which generally is far from being minimal. By characterizing the facets of matching polyhedra we are characterizing the minimal sets of inequalities necessary and sufficient to determine these polyhedra.

It may happen (as is the case with matching polyhedra) that $a_{ij} = 0$ or 1 for all $i \in I$ and $j \in J$. Then we have

$$p(i) = \{t \in I: A_i = A_t \text{ and } b_i = b_t\}$$

and we can simplify (2.3.30) as follows.

(2.3.34) Theorem. Let $P(A, b)$ be of full dimension, suppose $a_{ij} \in \{0, 1\}$ for all $i \in I$, $j \in J$. Then for any $K \subseteq I$, $P(A, b) = P(A_K, b_K)$ if and only if for each $i \in I$ such that $f(\{i\})$ is a facet of $P(A, b)$ there is $t \in K$ such that $A_i = A_t$ and $b_i = b_t$.

2.4 Vertices of Polyhedra.

In this section we prove results about vertices of polyhedra which indicate their importance to linear programming. We also show that bounded polyhedra are convex combinations of their vertices.

We say that $\hat{x} \in P$ is a vertex of the polyhedron P if $\{\hat{x}\}$ is a face of P and $\dim(\{\hat{x}\}) = 0$.

(2.4.1) Theorem. $\hat{x}$ is a vertex of $P(A, b)$ if and only if there is some $c \in \mathbb{R}^J$ such that $\hat{x}$ is the unique member of P maximizing cx for $x \in P$.

Proof. Any two distinct members of $\mathbb{R}^J$ are easily seen to be affinely independent so $F \subseteq P(A, b)$ is a face of $P(A, b)$ of dimension 0 if and only if F is a face of $P(A, b)$ and $|F| = 1$. By (2.1.5) F is a nonempty face of $P(A, b)$ if and only if there is $c \in \mathbb{R}^J$ such that cx is maximized over $P(A, b)$ by precisely the members of F. The result follows from these two facts.□

We say that a polyhedron $P \subseteq \mathbb{R}^J$ is <u>bounded</u> if there exist $\ell, u \in \mathbb{R}^J$ such that $\ell \leq x \leq u$ for all $x \in P$. A bounded polyhedron is commonly called a <u>polytope</u> (see Grunbaum [G1]).

(2.4.2) <u>Theorem</u>. <u>Let $P(A, b)$ be a nonempty bounded polyhedron. Then $P(A, b)$ has a vertex.</u>

<u>Proof</u>. Let I' be the equality set of a nonempty face F of $P(A, b)$ of minimum dimension. If $\dim(F) = 0$ then F consists of a vertex and we are finished. Otherwise if $\dim(F) > 0$ then there are by (2.1.8) an interior point x of F and by (2.2.8) an element $y \in F - \{x\}$. For any $\epsilon \in \mathbb{R}$ let $z(\epsilon) \equiv x + \epsilon \cdot (y - x)$. Then $A_{I'} z(\epsilon) = b_{I'}$ for all $\epsilon \in \mathbb{R}$. If $A_{I-I'}(y - x) \leq 0$ then $z(\epsilon) \in P(A,b)$ for all $\epsilon \in \mathbb{R}$ such that $\epsilon \geq 0$ which contradicts $P(A, b)$ being bounded. Therefore there is $i \in I - I'$ such that $A_i(y - x) > 0$. Let $\lambda^* = \min\{\frac{b_i - A_i x}{A_i(y-x)} : i \in I - I'$ and $A_i(y - x) > 0\}$. Then $z(\lambda^*) \in F$ and there is $i \in I - I'$ such that $A_i z(\lambda^*) = b_i$. Since $x \in F - F(I' \cup \{i\})$, $f(I' \cup \{i\})$ is a proper face of F; since $z(\lambda^*) \in f(I' \cup \{i\})$, $f(I' \cup \{i\}) \neq \phi$. By (2.2.1) $\dim(f(I' \cup \{i\})) \leq \dim(F) - 1$ and by (2.1.3) $f(I' \cup \{i\})$ is a face of $P(A, b)$ contradicting our choice of F. Hence $\dim(F) = 0$ and F consists of a vertex of $P(A, b)$. □

Since any face of a bounded polyhedron is itself a bounded polyhedron, we have the following corollary.

(2.4.3) <u>Corollary</u>. <u>Every nonempty face of a bounded polyhedron contains a vertex.</u>

Observe that by (2.1.5) if $c \in \mathbb{R}^J$ is such that cx has an upper bound for $x \in P(A, b)$, then this upper bound is achieved by precisely the members of some nonempty face of $P(A, b)$.

By combining this, (2.4.3), and the fact that for any $c \in \mathbb{R}^J$, $c \cdot x$ has an upper bound over a bounded polyhedron we obtain the following.

(2.4.5) Theorem. Let P be a nonempty bounded polyhedron. Then for any $c \in \mathbb{R}^J$, there is a vertex v of P which maximizes $c \cdot x$ over P.

Let K be a finite set, let $\{x^k: k \in K\} \subseteq \mathbb{R}^J$. We say that x is a convex combination of $\{x^k: k \in K\}$ if there is $(\lambda_k: k \in K) \in \mathbb{R}^K$ such that

(2.4.6) $\lambda_k \geq 0$ for all $k \in K$,

(2.4.7) $\Sigma(\lambda_k: k \in K) = 1$,

(2.4.8) $x = \Sigma(\lambda_k x^k: k \in K)$.

A set $X \subseteq \mathbb{R}^J$ is convex if every convex combination of every finite subset of X belongs to X.

(2.4.9) Proposition. Polyhedra are convex.

Proof. Let $P(A, b)$ be a polyhedron. If $P(A, b) = \phi$ then the result is trivial. If $P(A, b) \neq \phi$ let $X = \{x^k: k \in K\}$ be a finite subset of $P(A, b)$ and let x be a convex combination of X. Then there is $(\lambda_k: k \in K) \in \mathbb{R}^K$ satisfying (2.4.6)-(2.4.8). Hence

$$Ax = \Sigma(\lambda_k A\, x^k : k \in K)$$
$$\leq \Sigma(\lambda_k : k \in K)b \quad \text{by (2.4.6)}$$
$$= b \quad \text{by (2.4.7)}$$

so $x \in P(A, b)$ and (2.4.9) follows.□

If $V \subseteq \mathbb{R}^J$ then the <u>convex hull</u> of V is defined to be the set of all $x \in \mathbb{R}^J$ which are convex combinations of finite subsets of V.

(2.4.10) <u>Theorem. If $P(A, b)$ is a nonempty bounded polyhedron then $P(A, b)$ is equal to the convex hull of its set of vertices.</u>

<u>Proof</u>. Let $V = \{v_k : k \in K\}$ be the set of vertices of $P(A, b)$. Let $H(V)$ denote the convex hull of V. It follows from (2.4.9) that $H(V) \subseteq P(A, b)$.

Let $\bar{x} \in P(A, b)$. Then $\bar{x} \in H(V)$ if and only if there exists $\lambda = (\lambda_k : k \in K) \in \mathbb{R}^K$ satisfying (2.4.6), (2.4.7) and

$$\bar{x} = \Sigma(\lambda_k v_k : k \in K).$$

Suppose no such λ exists. Then by Farkas' Lemma (1.5.15) there are $y \in \mathbb{R}^J$ and $y^0 \in \mathbb{R}$ such that

(2.4.11) $\quad y \cdot v_k + y_0 \leq 0$ for $k \in K$

(2.4.12) $\quad y \cdot \bar{x} + y_0 > 0.$

Since $P(A, b)$ is bounded, by (2.4.5)

there is $\alpha \in \mathbb{R}$ such that $\alpha = \max\{y \cdot x : x \in P(A, b)\}$ and there is $h \in K$ such that $y \cdot v_h = \alpha$. By (2.4.11) $\alpha \le -y_0$ so since $\bar{x} \in P(A, b)$, $y \cdot \bar{x} \le \alpha \le -y_0$ contradictory to (2.4.12). This completes the proof.□

The number of vertices of a polyhedron is generally much larger than the dimension of the polyhedron. The following theorem due to Carathéodory [C1] shows that if x belongs to the convex hull of $S \subseteq \mathbb{R}^J$ then if r is the affine rank of S, x can be expressed as a convex combination of at most r members of S.

(2.4.13) <u>Carathéordory's Theorem</u>. <u>Let r be the affine rank of $S \subseteq \mathbb{R}^J$, let x be a member of the convex hull of S. Then there is $Y \subseteq S$ such that $|Y| \le r$ and x is a convex combination of the members of Y</u>.

<u>Proof</u>. See Stoer Witzgall [S1] p. 35.

We combine (2.4.12) with (2.4.10) and (2.2.14) to obtain

(2.4.14) <u>Theorem</u>. <u>Let P be a bounded polyhedron of dimension $d \ge 0$. Then any $x \in P$ can be expressed as a convex combination of a set of at most $d + 1$ vertices of P</u>.

Chapter 3

The Matching Problem and the Blossom Algorithm

In this chapter we describe the matching problem considered here and give a new version of the so-called blossom algorithm for solving this problem. This algorithm, which is used extensively in later chapters, is actually a combination of several other versions of the blossom algorithm. The relationship of this version to other available versions is discussed later, when sufficient terminology has been developed.

3.1 The Matching Problem.

Let V and E be finite sets, let $V^{\leq} \cup V^{=}$ be a partition of V. Let $c = (c_j : j \in E)$ be an arbitrary real vector, let $b = (b_i : i \in V)$ be a vector of positive integers. Let $A = (a_{ij} : i \in V, j \in E)$ be a matrix of zeros and ones which satisfies

$$(3.1.1) \quad \Sigma(a_{ij} : i \in V) = 2 \quad \text{for all } j \in E.$$

Then the matching problem under consideration is the following problem.

Find, if one exists, a vector $x = (x_j : j \in E) \in \mathbb{R}^E$ such that x_j is a nonnegative integer for all $j \in E$,

$$\Sigma(a_{ij}x_j : j \in E) \leq b_i \quad \text{for all } i \in V^{\leq},$$

$$\Sigma(a_{ij}x_j : j \in E) = b_i \quad \text{for all } i \in V^{=}.$$

and which maximizes $c \cdot x$ subject to these conditions.

If no such vector exists then we wish to exhibit a structure which will prove that no such vector exists.

The matching problem is, therefore, a special case of the integer programming problem (see section 1.6), the principal restriction being (3.1.1). However whereas all known algorithms for solving general integer programming problems have bounds which are exponential in the size of the input, the blossom algorithm is a method for solving matching problems whose bound is a polynomial function of the size of the input. The description of the algorithm is facilitated by interpreting the problem graphically in the following manner.

Let G be the graph (V, E, ψ) where ψ is defined by

$$\psi(j) = \{i \in V: a_{ij} = 1\} \text{ for all } j \in E.$$

In view of (3.1.1), $|\psi(j)| = 2$ for all $j \in E$. Thus G is a graph without loops having edge set E and node set V. Then the matching problem is

$$\text{(3.1.2)} \qquad \text{maximize } c \cdot x$$

where

$$\left.\begin{array}{ll} \text{(3.1.3)} & x_j \geq 0 \\ \text{(3.1.4)} & x_j \text{ integer valued} \end{array}\right\} \text{for all } j \in E$$

$$\text{(3.1.5)} \qquad x(\delta(i)) \leq b_i \text{ for all } i \in V^{\leq}$$

$$\text{(3.1.6)} \qquad x(\delta(i)) = b_i \text{ for all } i \in V^{=}$$

(See (1.3.3), (1.3.4) for the definitions of γ, δ). That is, we wish to assign a nonnegative integer x_j to each edge

j of G so that the constraints (3.1.5) and (3.1.6) are satisfied and so that $c \cdot x$ is maximized.

Throughout the remainder of this chapter $G = (V, E, \psi)$ is a graph, $b = (b_i : i \in V)$ is a vector of positive integers called degree constraints, $c = (c_j : j \in E)$ is an arbitrary real vector and $V^{\leq} \cup V^{=}$ is a partition of V.

The purpose of this chapter is to describe an algorithm, called the blossom algorithm, for solving the problem (3.1.2)-(3.1.6).

It is a version of Edmonds' blossom algorithm. In [E1] and [E3] are versions of the algorithm which solve the problem of maximizing $x(E)$ subject to x satisfying (3.1.3)-(3.1.5) taking $b_i \equiv 1$ for all $i \in V$ and $V^{=} \equiv \phi$. Another version [E2] solves the more general problem (3.1.2)-(3.1.5) where $b_i \equiv 1$ for all $i \in V$ and $V^{=} \equiv \phi$.

The description of the blossom algorithm in this chapter is based upon a version of the algorithm [E4] which solves the problem (3.1.2)-(3.1.6) taking $V^{=} \equiv V$ and allowing the b_i to be arbitrary positive integers.

This algorithm has been generalized (Johnson [J1], Edmonds, Johnson [E5] and [E6]) in other directions from those considered in this thesis. In addition a computer implementation of a generalized algorithm is available (Edmonds, Johnson, Lockhart [E7]).

We call any $x \in \mathbb{R}^E$ satisfying (3.1.3) and (3.1.4) a matching. If x also satisfies (3.1.5) and (3.1.6) then x is called a feasible b-matching or simply a feasible matching.

If x is a matching such that $x(\delta(i)) = b_i$ for all $i \in S \subseteq V$ then we say that x is a *perfect matching of S*; if $S = V$ then we may simply call x a *perfect matching of G*. For any matching x and any node i we define the *deficiency of x at i* to be $b_i - x(\delta(i))$. If x has a positive deficiency at i then we say that x is *deficient at i*. If x is deficient at i then sometimes we call i a *deficient node* relative to x. Thus x is a perfect matching of $S \subseteq V$ if S contains no deficient nodes relative to x. In Chapter 4 we will study extensively matchings having a deficiency of 1 at some node of G and having a deficiency of 0 every other node, the so-called near perfect matchings.

If $b_i = 1$ for all $i \in V$ then if x is a feasible matching, $M = \{j \in E: x_j = 1\}$ is a set of edges of G meeting each node of G at most once and each node of $V^=$ exactly once. This special case has received a great deal of attention and often is the starting point for studies of matching theory (e.g. Berge [B2], Edmonds [E1], [E2], [E3], Tutte [T2]). We call this problem the *1-matching problem* and call such a vector x a feasible *1-matching*. Several of our theorems of chapter 4 are particularly interesting for the case of 1-matchings.

(3.1.7) *Proposition*. *Let x be a matching of G which satisfies*

(3.1.8) $\underline{x(\delta(i)) \le b_i \text{ for all } i \in V}$.

Then for any $S \subseteq V$ such that $b(S)$ is odd,

$$(3.1.9) \qquad x(\gamma(S)) \le \frac{b(S)-1}{2} .$$

Proof. By (3.1.8) $\Sigma(x(\delta(i)): i \in S) \le b(S)$ and since $\Sigma(x(\delta(i)): i \in S) = 2x(\gamma(S)) + x(\delta(S))$ it follows that

$$2x(\gamma(S)) \le b(S) - x(\delta(S)) \le b(S) .$$

Since $x(\gamma(S))$ is integer valued and $b(S)$ is odd it follows that

$$2x(\gamma(S)) \le b(S) - 1$$

and (3.1.9) is immediate.□

The sets $S \subseteq V$ for which $b(S)$ is odd play an important role in matching theory where G is not bipartite. For any such set S we define

$$(3.1.10) \qquad q_S \equiv (b(S) - 1)/2 .$$

The following are two basic results concerning graphs of particularly simple structure. Notice that in both (3.1.11) and (3.1.16) we neither postulate d nor require x to be integer valued or nonnegative.

(3.1.11) Proposition. For any tree T, for any $d = (d_i: i \in V(T)) \in \mathbb{R}^{V(T)}$, for any $v \in V(T)$ there is a unique $x \in \mathbb{R}^{E(T)}$ such that

$$(3.1.12) \qquad x(\delta_T(i)) = d_i \quad \text{for all } i \in V(T) - \{v\}.$$

Proof. We prove by induction on $|V(T)|$. If $|V(T)| = 1$ or 2 the result is trivial. Assume the result true for trees having fewer than k nodes, for $k \geq 3$ and assume $|V(T)| = k$. By (1.3.12) T has a node t of valence 1 different from v, let $\{j\} = \delta_T(t)$. Clearly

(3.1.13) $x(\delta_T(t)) = d_t$ if and only if $x_j = d_t$.

Let T' be the tree obtained from T by deleting j and t, let w be the end of j in T'. Define d' by

$$d'_i \begin{cases} d_i & \text{for } i \in V(T') - \{w\} \\ d_w - d_t & \text{if } i = w \end{cases}.$$

Since $|V(T')| < k$, by our induction hypothesis

(3.1.14) there is a unique $x' \in \mathbb{R}^{E(T')}$ such that $x'(\delta_{T'}(i)) = d'_i$ for all $i \in V(T') - \{v\}$.

Define $x = (x_h : h \in E(T))$ by

$$(3.1.15)\quad x_h \equiv \begin{cases} x'_h & \text{for } h \in E(T') = E(T) - \{j\}, \\ d_t & \text{for } h = j. \end{cases}$$

By (3.1.13)-(3.1.15), x is the unique member of $\mathbb{R}^{E(T)}$ satisfying (3.1.12).□

(3.1.16) Proposition. Let B be a connected graph containing no even polygon and one odd polygon P. Then for any $d = (d_i : i \in V(B)) \in \mathbb{R}^{V(B)}$ there is a unique $x \in \mathbb{R}^{E(B)}$ such that

(3.1.17) $x(\delta(i)) = d_i$ for all $i \in V(B)$.

Proof. Let $j \in E(P)$, let B' be the graph obtained from

B by removing j. Then B' is a tree and so is bipartite, let u, v be the ends of j, let V_1 be the part see (1.3.7) of B containing $\{u, v\}$, let V_2 be the other part. Let $d' = (d'_i : i \in V(B))$ be defined by

$$d'_i \equiv \begin{cases} d_i & \text{for } i \in V(B) - \{u, w\} \\ d_i - 1/2(d(V_1) - d(V_2)) & \text{for } i \in \{u, w\}. \end{cases}$$

Then

$$(3.1.18) \quad d'(V_1) = d'(V_2).$$

By (3.1.11) there is a unique $x' \in \mathbb{R}^{E(B')}$ such that $x'(\delta_{B'}(i)) = d'_i$ for all $i \in V(B) - \{u\}$. By (3.1.18) we have $x'(\delta_{B'}(u)) = d'_u$ so if we define $x \in \mathbb{R}^{E(B)}$ by

$$x_h \equiv \begin{cases} x'_h & \text{for } h \in E(B') = E(B) - \{j\}, \\ 1/2(d(V_1) - d(V_2)) & \text{for } h = j \end{cases}$$

then x satisfies (3.1.17) as required.

Conversely, suppose $\bar{x} \in \mathbb{R}^{E(B)}$ satisfies (3.1.17). B' is bipartite so we have $\bar{x}(\delta_{B'}(V_1)) = \bar{x}(\delta_{B'}(V_2))$. Therefore we must have $\bar{x}_j = 1/2(d(V_1) - d(V_2))$. Therefore $\bar{x}|E(B')$ satisfies $\bar{x}(\delta_{B'}(i)) = d'_i$ for all $i \in V(B) - \{u\}$ so $\bar{x}|E(B') = x'$ by (3.1.11). Therefore $\bar{x} = x$ proving the uniqueness of x.□

The following six sections (3.2-3.7) are used to develop the general framework required to describe the blossom algorithm. The algorithm itself is presented in Section 3.8 and in Section 3.9 we compute a bound on the amount of work requried by the algorithm to solve a problem.

3.2 Nested Families of Sets.

Let R be a set of distinct nonempty subsets of V. We say that R is a <u>nested family</u> if for any distinct S, $T \in R$ such that $S \cap T \neq \phi$ we have $S \subset T$ or $T \subset S$. An important feature of nested families (of which we make use in establishing upper bounds on the amount of work required by various algorithms) is that they are small compared to the total number of subsets of V.

(3.2.1) <u>Theorem</u>. <u>If R is a nested family of subsets of a nonempty set V then $|R| \leq 2|V| - 1$</u>.

<u>Proof</u>. We prove by induction on $|V|$. If $|V| = 1$ then the result is obvious. Suppose the result is true when $|V| < k$ for some $k \geq 2$ and suppose $|V| = k$. Let R be a nested family of subsets of V for which $|R|$ is as large as possible. Since $S \subseteq V$ for all $S \in R$ we must have $V \in R$ or $R \cup \{V\}$ would be a larger nested family. We must also have

$$\text{(3.2.2)} \quad \{x\} \in R \text{ for every } x \in V,$$

for if there is $x \in V$ such that $\{x\} \notin R$, then $R \cup \{x\}$ is easily seen to be a larger nested family.

Let $V_1, V_2, \ldots, V_t$ be the maximal members of $R - \{V\}$. Since $|V| \geq 2$, since the members of R are distinct and by (3.2.2), (3.2.3) $t \geq 2$.

For each $i \in \{1, 2, \ldots, t\}$ let $R(V_i) = \{S \in R: S \subseteq V_i\}$. Then $R = \bigcup_{i=1}^{t} R(V_i) \cup \{V\}$ and $V = \bigcup_{i=1}^{t} V_i$. By our induction hypothesis $|R(V_i)| \leq 2|V_i| - 1$ for $i \in \{1, 2, \ldots, t\}$. Since $\bigcup_{i=1}^{t} R(V_i) \cup \{V\}$ partitions R,

$$|R| = \sum_{i=1}^{t} |R(V_i)| + 1$$

$$\leq 2 \sum_{i=1}^{t} |V_i| - t + 1$$

$$\leq 2|V| - 1$$

by (3.2.3) and since $\bigcup_{i=1}^{t} V_i$ partitions V. The theorem now follows by induction.□

If we prohibit singletons from our nested family then we have the following bound.

(3.2.4) Theorem. Let R be a nested family of subsets of V containing no singletons. Then $|R| \leq |V| - 1$.

Proof. Let R' be the family $R \cup \bigcup_{v \in V} \{v\}$. R' is easily seen to be a nested family, by (3.2.1) $|R'| \leq 2|V| - 1$. Since $|R'| = |R| + |V|$ it follows that $|R| \leq |V| - 1$.□

If R is a nested family of subsets of V then for each $S \in R$ we let

(3.2.5) $R_S = \{T \in R: T$ is a maximal proper subset of S belonging to $R\}$

and

(3.2.6) $V_S = \{v \in S: v \notin \cup(R_S)\}$.

We let

(3.2.7) $n(S) = |R_S| + |V_S|$.

Thus $n(S)$ is the number of maximal "things" which are combined to form S.

(3.2.8) Theorem. Let R be a nested family of subsets of V for which $n(S) \geq 3$ for all $S \in R$. Then $|R| \leq (|V| - 1)/2$.

Proof. Let $S \in R$. If $|V_S| \geq 2$, then let S' be any two members of V_S. If $|V_S| \leq 1$ then since $n(S) \geq 3$, $|R_S| \geq 2$. In this case let S' be the union of any two members of R_S. Let $R' = R \cup \{S': S \in R\}$. Then $|R'| = 2|R|$. Moreover R' is a nested family containing no singletons so $|R'| \leq |V| - 1$ by (3.2.4). Therefore $|R| \leq 1/2(|V| - 1)$ and the proof is complete.□

3.3 Blossoms, Shrinking and Shrinkable Families

One feature of the blossom algorithm is the way it "shrinks" certain subgraphs of a graph to effectively reduce the size of the problem. In this section we define shrinking and describe the sorts of subgraphs which will be shrunk. We also prove some fundamental results concerning shrinkable graphs. The definitions and results of this section are also used in Chapter 4 where we show the close relationship between shrinkable graphs and facets of the matching polyhedron.

The basic structure used in defining shrinkable graphs is the blossom (the christening feature of the blossom algorithm) which is defined as follows.

A <u>blossom</u> is a connected graph B containing no even polygons, exactly one odd polygon P and for which the degree constraints satisfy the following conditions. Let $v \in V(P)$. By (3.1.16) there is a unique $x \in \mathbb{R}^J$ such that

(3.3.1) $x(\delta_B(i)) = b_i$ for all $i \in V(B) - \{v\}$

(3.3.2) $x(\delta_B(v)) = b_v - 1$.

In order that B be a blossom we require

(3.3.3) x_j be a nonnegative integer for all $j \in E(B)$,

(3.3.4) $x_j \geq 1$ for all $j \in E(B) - E(P)$

(3.3.5) $x_j \geq 1$ for each $j \in E(P)$ such that j is the first edge in the even length path in P from a node $i \in V(P) - \{v\}$ to v.

The choice of v is in fact arbitrary, we will show in (3.3.12) that if (3.3.1)-(3.3.5) hold for some $v \in V(P)$ then they also hold for any other choice of $v \in V(P)$.

In order that (3.3.1)-(3.3.3) hold we require

(3.3.6) $b(V(B))$ is odd for any blossom B.

Since we obtain a tree if we delete any $j \in E(P)$ from B, we have using (1.3.13) that

(3.3.7) $|V(B)| = |E(B)|$ for any blossom B.

The graph obtained from B by deleting all edges of P is a forest, each $v \in V(P)$ belongs to a unique (possibly trivial) tree T_v of the forest. These trees are called the petals of the blossom, T_v is the petal rooted at v. The edges belonging to $E(B) - E(P)$ are called the petal edges of the blossom.

(3.3.8) If $v \in V(B)$ has valence 1, or has valence 2 and belongs to $V(P)$ then V is called a terminal node of B.

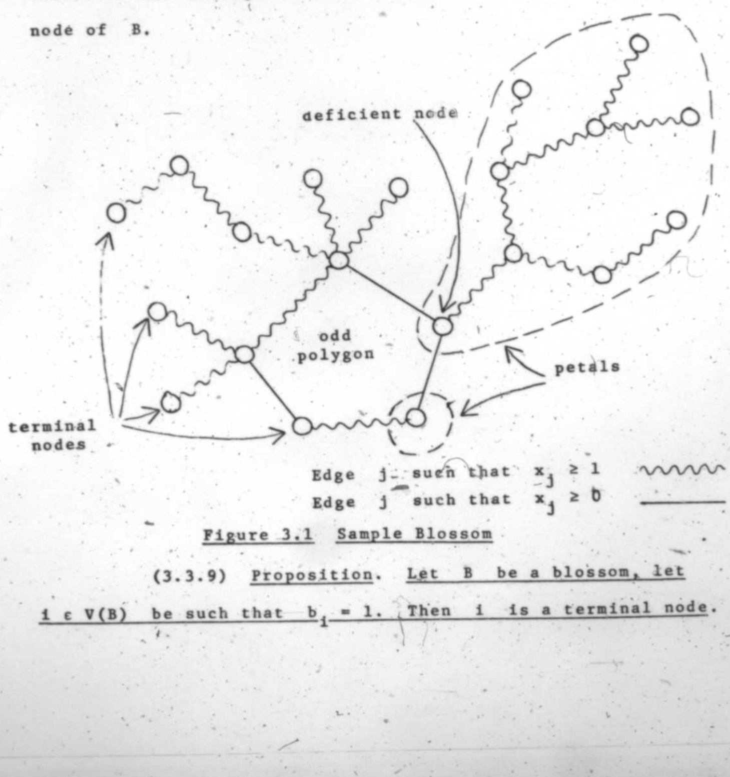


Figure 3.1 Sample Blossom

(3.3.9) Proposition. Let B be a blossom, let $i \in V(B)$ be such that $b_i = 1$. Then i is a terminal node.

Proof. If $i \in V(P)$ then by (3.3.1)-(3.3.5) we must have $b_i \geq 1 + |\delta_B(i) \cap E(T_i)|$. Hence $b_i = 1$ implies $E(T_i) = \phi$ and so i is a terminal node. If $i \in V(B) - V(P)$ then by (3.3.4) $b_i \geq |\delta_B(i)|$ so $|\delta_B(i)| = 1$ and i is a terminal node.□

(3.3.10) Proposition. If $b_i = 1$ for every $i \in V(B)$ then B is a blossom if and only if B is an odd polygon.

Proof. First suppose that B is a blossom. By (3.3.9) every node of B is a terminal node, if any petal T_v contained an edge then v could not be a terminal node, a contradiction. Hence all petals are single nodes and B is an odd polygon.

If B is an odd polygon let $v \in V(B)$ and let τ be a shortest odd length track in B from v to v. If we define $x_j = 0$ for every odd edge of τ and $x_j = 1$ for every even edge of τ then x satisfies (3.3.1)-(3.3.5) so B is a blossom.□

If B is a graph such that $b(V(B))$ is odd then clearly B can have no perfect matching.

(3.3.11) We define a near perfect matching (abbreviated by np matching) to be a matching x of B such that, for some $v \in V(B)$

$$x(\delta_B(i)) = b_i \quad \text{for all } i \in V(B) - \{v\},$$

$$x(\delta_B(v)) = b_v - 1.$$

(3.3.12) Proposition. Let B be a blossom containing the odd polygon P. Then for any $i \in V(B)$ there is a np matching x' of B deficient at i. Moreover if $i \in V(P)$ then x' satisfies (3.3.4), (3.3.5) (with x' substituted for x).

Proof. The proof of this proposition actually consists of an algorithm for obtaining such a matching, starting with a np matching x deficient at $v \in V(P)$ satisfying (3.3.1)-(3.3.5).

If $i = v$ then x is the matching we require and we are finished. Otherwise let τ be the shortest track from v to i having even length. Now define x' by

$$(3.3.13)\quad x'_j = \begin{cases} x_j + 1 & \text{if } j \text{ is an odd edge of } \tau, \\ x_j - 1 & \text{if } j \text{ is an even edge of } \tau, \\ x_j & \text{if } j \in E(B) - E(\tau). \end{cases}$$

Clearly x' is integer valued, $x'(\delta_B(s)) = b_S$ for all $s \in V(B) - \{i\}$ and $x'(\delta_B(i)) = b_i - 1$. Moreover if $j \in E(P)$ is an even edge of τ then j is the first edge in an even length path in P from some $w \in V(P) - \{v\}$ to v so by (3.3.5) $x_j \geq 1$ and $x'_j \geq 0$. If $j \in E(B) - E(P)$ is an even edge of τ then by (3.3.4) $x_j \geq 1$ so $x'_j \geq 0$. For any $j \in E(B)$ which is not an even edge of τ we have $x'_j \geq x_j \geq 0$ so $x' \geq 0$ and hence is a np matching deficient at i.

Now suppose $i \in V(P) - \{v\}$. First observe that each $j \in E(P)$ is the first edge in exactly two paths from nodes of P to i and since P is an odd polygon, both these paths have the same parity. If $j \in E(P) \cap E(\tau)$ then j is the first edge in an even path to i if and only if j is an odd edge of τ so by (3.3.13) $x'_j \geq 1$. If $j \in E(P) - E(\tau)$ is the first edge in an even path to i then it is easily seen that j is the first edge in an even path to v so by (3.3.5) $x'_j = x_j \geq 1$.

Since $i \in V(P)$ implies $E(\tau) \subseteq E(P)$, (3.3.13) and (3.3.4) ensure that $x'_j = x_j \geq 1$ for all $j \in E(B) - E(P)$ and the proof is complete.□

We now define <u>shrinking</u>. Let $G = (V, E, \psi)$ be a graph let $S \subseteq V$. We say that $\bar{G} = (\bar{V}, \bar{E}, \bar{\psi})$ is the <u>graph obtained from G by shrinking S</u> if

$$\bar{V} = V - S \cup \{S\},$$

$$\bar{E} = E - \gamma(S)$$

$$\bar{\psi}(j) = \begin{cases} \psi(j) & \text{if } j \in \bar{E} - \delta(S) \\ \psi(j) - S \cup \{S\} & \text{if } j \in \delta(S). \end{cases}$$

In other words, $\bar{G}$ is the graph obtained from G by contracting all edges of G which have both ends in S and calling the resulting node "S". We denote $\bar{G}$ by $G \times S$ and call S a pseudonode of $\bar{G}$ (with respect to G). We define the degree constraint $b_S \equiv 1$ for any pseudonode S. We also define

$$(3.3.14) \quad \bar{V}^{=} \equiv \begin{cases} V^{=} - S \cup \{S\} & \text{if } S \subseteq V^{=}, \\ V^{=} - S & \text{if } S \not\subseteq V^{=}. \end{cases}$$

$$\bar{V}^{\leq} \equiv \bar{V} - \bar{V}^{=}.$$

Let R be a nested family of subsets of V (see Section 3.2). For any $S \in R$ we define

(3.3.15) $R[S] \equiv \{T \in R: T \subset S\}$.

If $\{S_1, S_2, \ldots, S_k\}$ is the set of maximal members of R then we let $G \times R$ denote

(3.3.15a) $(\ldots((G \times S_1) \times S_2) \times \ldots) \times S_K$.

It is easily seen that the ordering of the sets $S_1, S_2, \ldots, S_k$ has no effect on $G \times R$.

We say that $G = (V, E, \psi)$ is <u>shrinkable</u> if there is a possible empty nested family R of subsets of V such that

(3.3.16) for every $S \in R$, $G[S] \times R[S]$ is spanned by a blossom B_S,

(3.3.17) $b(V(G \times R)) = 1$.

It is easy to see that

(3.3.18) $V \in R$ is equivalent to (3.3.17) if $R \neq \phi$.

We call R a <u>shrinking family</u> of G. Note that in particular any graph spanned by a blossom is shrinkable. For any $S \subseteq V$ we say that S is <u>shrinkable</u> if $G[S]$ is shrinkable.

If $R = \phi$ is a shrinking family of $G = (V, E, \psi)$ then $|V| = 1$, $|E| = \phi$ and $b(V) = 1$. We call such a graph <u>degenerate</u>, all other shrinkable graphs are called <u>nondegenerate</u>.

(3.3.19) <u>Proposition</u>. <u>If $G = (V, E, \psi)$ is shrinkable, then $b(V)$ is odd</u>.

<u>Proof</u>. Let R be a shrinking family of G, we prove by induction on $|R|$. If $|R| = 0$ then G is degenerate and the result is trivial. Suppose (3.3.19) holds when G has a shrinking family of fewer than k sets for some $k \geq 1$ and assume $|R| = k$. By (3.3.16) there is a blossom B_V spanning $G \times R[V]$ and by (3.3.6),

(3.3.19a) $b(V(B_V))$ is odd.

Let S be any maximal member of $R[V]$ and hence a pseudonode of $G \times R[V]$. Then $R[S] \cup \{S\}$ is a shrinking family of $G[S]$ and since $|R[S] \cup \{S\}| < |R|$ we have by induction

(3.3.19b) $b(S)$ is odd.

If W is the set of pseudonodes of $G \times R[V]$ then

$$b(V) = b(V(G \times R[V])) + \Sigma(\{b(S): S \in W\} - 1)$$

so since $V(G \times R[V]) = V(B_V)$ we have by (3.3.19a) and (3.3.19b) that $b(V)$ is odd as asserted.□

If R is a shrinking family of G then for any $S \in R$, $R[S] \cup \{S\}$ is a shrinking family of $G[S]$. Hence we have the following corollary of (3.3.19).

(3.3.20) <u>Corollary</u>. <u>If R is a shrinking family for G then $b(S)$ is odd for all $S \in R$.</u>

(3.3.21) <u>Proposition</u>. <u>Let $G = (V, E, \psi)$ be shrinkable and let R be a shrinking family of G. Then for any $v \in V$ there is a np matching x of G deficient</u>

<u>at v and which satisfies</u>

(3.3.22) <u>$x|\gamma(S)$ is a np matching of $G[S]$ for all $S \in R$.</u>

<u>Proof</u>. We prove by induction on $|R|$. If $|R| = 0$ then G is degenerate and the result is trivial. Assume the result true for graphs having a shrinking family consisting of fewer than k sets for $k \geq 1$ and suppose $|R| = k$. Let v be any node of G. Every maximal $S \in R[V]$ is a pseudonode of the blossom B_V which spans $G \times R[V]$. Let $p \equiv v$ if $v \in V(B_V)$, let $p \equiv S$ if $v \in S$ for some pseudonode S of B_V. By (3.3.12) there is a np matching $\tilde{x}$ of B_V deficient at p. For every pseudonode $T \in V(B_V)$ there is at most one node of T incident with some $j \in E(B_V)$ for which $\tilde{x}_j = 1$ since $b_T = 1$. If such a node $w(T)$ exists, let $\bar{x}^T$ be a np matching of $G[T]$ deficient at $w(T)$ satisfying

(3.3.23) $\bar{x}^T|\gamma(Z)$ is a np matching of $G[Z]$ for every $Z \in R[T]$,

which exists by our induction hypothesis. If no such $w(T)$ exists, then $v \in T$ and we let $\bar{x}^T$ be a np matching of $G[T]$ deficient at v which satisfies (3.3.23). Now define x by

$$x_j = \begin{cases} \tilde{x}_j & \text{for } j \in E(B_V) \\ 0 & \text{for } j \in E(G \times R[V]) - E(B_V) \\ \bar{x}^T & \text{for } j \in \gamma(T), \text{ for } T \in R[V]. \end{cases}$$

x_j is easily seen to be a np matching of G deficient at v and satisfying (3.3.22), thereby completing the proof.□

We close this section by noting the following basic property of matchings.

(3.3.24) Proposition. A matching x is a np matching of $G = (V, E, \psi)$ if and only if $x(E) = q_V (=1/2(b(V)-1))$ and $x(\delta(i)) \le b_i$ for all $i \in V$.

Proof. For any matching x of G,

$$2x(E) = b(V) - \Sigma(b_i - x(\delta(i)): i \in V).$$

Thus any np matching x of G satisfies $x(E) = 1/2(b(V)-1)$ (and trivially $x(\delta(i)) \le b_i$ for all $i \in V$). Conversely any matching x which satisfies $x(E) = q_V$ and $x(\delta(i)) \le b_i$ for all $i \in V$ must satisfy $x(\delta(i)) = b_i$ for all $i \in V-\{v\}$ and $x(\delta(v)) = b_v - 1$ for some $v \in V$. Thus x is a np matching of G and the result follows.□

3.4 The Matching Polyhedron.

The matching polyhedron $P(G, b)$ is defined to be the bounded polyhedron in $\mathbb{R}^E$ containing all matchings x of $G = (V, E, \psi)$ which satisfy

(3.4.1) $\quad x(\delta(i)) \le b_i$ for all $i \in V$

and for which every vertex is such a matching. (Equivalently, $P(G, b)$ is the convex hull of the set of matchings of G which satisfy (3.4.1).)

Let $Q = \{S \subseteq V: |S| \ge 3$ and $b(S)$ is odd$\}$. Edmonds [E3] has shown that $P(G, b) = \{x \in \mathbb{R}^E$:

(3.4.2) $x_j \geq 0$ for all $j \in E$,

(3.4.3) $x(\delta(i)) \leq b_i$ for all $i \in V$

(3.4.4) $x(\gamma(S)) \leq q_S$ for all $S \in Q\}$.

The proof is a consequence of a blossom algorithm similar to the version we are developing here in the following way. The algorithm shows that for any $c \in \mathbb{R}^E$, $c \cdot x$ is maximized over all x (not necessarily integer valued) satisfying (3.4.2)-(3.4.4) by a matching of G which satisfies (3.4.1). It is implicit in the algorithm that Q can be replaced in (3.4.4) by a subset of itself which is generally much smaller than Q.

Let $Q^0 \equiv \{S \subseteq V\colon |S| \geq 3$ and S is a shrinkable subset of $V\}$. (By (3.3.19) $b(S)$ is odd for each $S \in Q$).

(3.4.5) <u>Theorem.</u>
<u>$P(G, b) = P \equiv \{x \in \mathbb{R}^E$:</u>

(3.4.6) <u>$x_j \geq 0$ for all $j \in E$,</u>

(3.4.7) <u>$x(\delta(i)) \leq b_i$ for all $i \in V$</u>

(3.4.8) <u>$x(\gamma(S)) \leq q_S$ for every $S \in Q^0\}$.</u>

<u>Proof.</u> It is easily seen that any matching x of G which satisfies (3.4.1) belongs to P, for it satisfies (3.4.6), (3.4.7) by definition and it satisfies (3.4.8) by (3.1.7) and (3.3.19).

We will show by means of the blossom algorithm that for any $c \in \mathbb{R}^E$, there is a matching x^0 of G satisfying (3.4.1) which maximizes cx over $x \in P$. By (2.4.1) for each vertex v of P there is a vector $c \in \mathbb{R}^E$ such that cx is maximized over $x \in P$ only by v. Hence all vertices of P are matchings satisfying (3.4.1).

(We saw in (2.4.10) that every bounded polyhedron is the convex hull of its set of vertices. Since P contains all matchings of G satisfying (3.4.1) and since all vertices of P are such matchings it follows that P is the convex hull of the matchings of G which satisfy (3.4.1).)□

When we require matchings satisfying

$$(3.4.9) \quad x(\delta(i)) = b_i \text{ for } i \in V^= \subseteq V$$

then we are in fact considering a face F of $P(G, b)$. Thus the blossom algorithm presented in this chapter will find (if one exists) a matching $x^0 \in F \subseteq P(G, b)$ maximizing $c \cdot x^0$ over F where F is a face of $P(G, b)$ obtained by requiring (3.4.9) hold. (If $V^= = \phi$ then $F = P(G, b)$). In chapter 5 we study the more general problem of maximizing $c \cdot x$ over any face F of $P(G, b)$.

3.5 Linear Programming Formulation

The following linear program is equivalent to the problem of maximizing cx for $x \in F \subseteq P$ where F is the face of P (defined in (3.4.5)) obtained by requiring (3.4.9) hold.

$$(3.5.1) \quad \text{Maximize } c \cdot x$$

over $x \in \mathbb{R}^E$ which satisfy

(3.5.2) $x_j \geq 0$ for all $j \in E$,

(3.5.3) $x(\delta(i)) \leq b_i$ for all $i \in V^{\leq} = V - V^{=}$,

(3.5.4) $x(\delta(i)) = b_i$ for all $i \in V^{=}$,

(3.5.5) $x(\gamma(S)) \leq q_S$ for all $S \in Q^0$.

For any $j \in E$ let $Q^0(j) = \{S \in Q^0: j \in \gamma(S)\}$. The dual linear program is

(3.5.6) minimize $\Sigma(b_i y_i: \quad i \in V) + \Sigma(q_S y_S: S \in Q^0)$

over $y \in \mathbb{R}^{V \cup Q^0}$ which satisfy

(3.5.7) $y_S \geq 0$ for all $S \in Q^0$,

(3.5.8) $y_i \geq 0$ for all $i \in V^{\leq}$

(3.5.9) $y(\psi(j)) + y(Q^0(j)) \geq c_j$ for all $j \in E$.

By complementary slackness (1.5.16) x^0 satisfying (3.5.2)-(3.5.5) and y^0 satisfying (3.5.7)-(3.5.9) are optimal if and only if

(3.5.10) $x_j^0 > 0$ implies $y^0(\psi(j)) + y^0(Q^0(j)) = c_j$ for any $j \in E$,

(3.5.11) $y_i^0 > 0$ implies $x^0(\delta(i)) = b_i$ for all $i \in V^{\leq}$,

(3.5.12) $y_S^0 > 0$ implies $x^0(\gamma(S)) = q_S$ for all $S \in Q^0$.

The blossom algorithm will actually find a feasible matching x and a dual solution y satisfying (3.5.7)-(3.5.9) such that x and y satisfy (3.5.10)-(3.5.12) or else will show that no feasible matching exists in a manner described in section 3.7.

We call y a <u>dual solution</u> to the matching problem (3.1.2)-(3.1.6) if y satisfies (3.5.7)-(3.5.9) and an optimal dual solution if y minimizes $\Sigma(b_i y_i: i \in V) + \Sigma(q_S y_S: S \in Q^0)$ over all dual solutions.

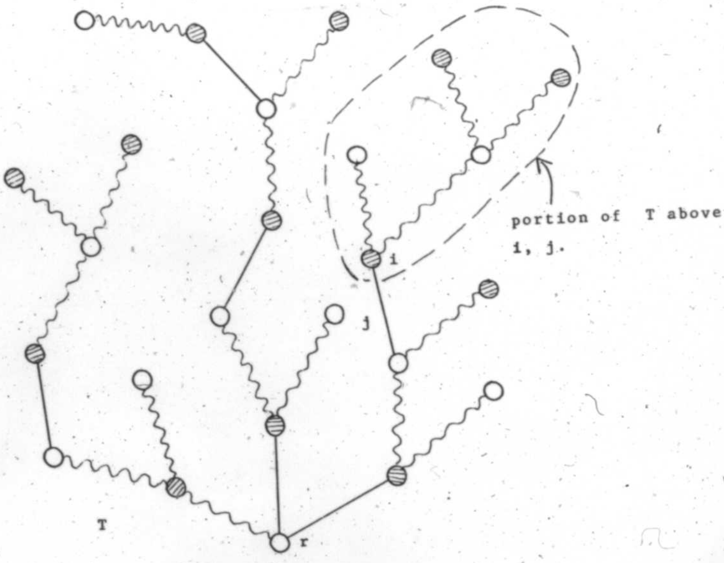
3.6 Alternating Forests

During the course of the blossom algorithm we construct forests having special properties with respect to a matching. Let T be a tree contained in $G = (V, E, \psi)$, let $r \in V(T)$ be designated as the <u>root</u> of T. There is a unique path $\pi(i)$ in T from r to each $i \in V(T)$. We call i an <u>even node</u> or an <u>odd node</u> of T according as the length of $\pi(i)$ is even or odd. In particular, r is an even node of T. We call $j \in E(T)$ <u>even</u> or <u>odd</u> according as j is the last edge of a path $\pi(i)$ in T to an even or odd node of T (or equivalently, according as j is an even edge or odd edge of any path $\pi(i)$ in T from r to some node $i \in V(T) - \{r\}$ such that $j \in E(\pi(i))$).

Let x be a matching of G. We call T an <u>alternating tree</u> with respect to x (see Figure 3.2) if

(3.6.1) $x(\delta(r)) < b_r$,

(3.6.2) $x(\delta(i)) = b_i$ for all $i \in V(T) - \{r\}$,



<u>Figure 3.2</u> Alternating tree

even nodes ○

odd nodes

edge j such that $x_j \geq 1$

edge j such that $x_j \geq 0$

(3.6.3) if $x_j > 0$ and $\psi(j) \cap V(T) \neq \phi$ then $j \in E(T)$,

(3.6.4) $x_j > 0$ for every even edge j of T.

If we are considering 1-matchings then (3.6.1)-(3.6.4) imply that every even edge j of T has $x_j = 1$ and every odd edge j has $x_j = 0$.

Note that for any $i \in V$, $\{i\}$ is the node set of an alternating tree if $x(\delta(i)) = 0$. We call a nonempty collection of alternating trees an <u>alternating forest</u>.

Let j be an edge of a tree T with root r. If we delete j from T then the resulting graph will consist of two trees, one of which, T', will not contain r. We call T' the <u>portion of T above j</u>.

Let i be any node of T. If $i = r$ then we say that T is the <u>portion of T above i</u>. Otherwise let k be the first edge of the path in T from i to r and let T' be the portion of T above k. We say that T' is the <u>portion of T above i</u>.

(3.6.5) <u>Proposition</u>. <u>Let T be an alternating tree with respect to the matching x. Let r be the root of T, let I be the set of odd nodes of T and let W be the set of even nodes of T. Then</u>

$$\underline{b(W) - (b_r - x(\delta(r))) = b(I)} .$$

<u>Proof</u>. By (3.6.1) and (3.6.2)

$$(3.6.6) \quad b(W) = \Sigma(x(\delta(i)) : i \in W) + b_r - x(\delta(r)).$$

Since no edge of T can join two even nodes and by (3.6.3),

$$(3.6.7) \quad \Sigma(x(\delta(i)): i \in W) = x(\delta(W) \cap E(T)).$$

By (3.6.2)

$$(3.6.8) \quad b(I) = \Sigma(x(\delta(i)): i \in I).$$

Since no edge of T can join two odd nodes and by (3.6.3),

$$(3.6.9) \quad \Sigma(x(\delta(i)): i \in I) = x(\delta(I) \cap E(T)).$$

But for any $j \in E(T)$, $j \in \delta(I)$ and $j \in \delta(W)$ so

$$(3.6.10) \quad \delta(I) \cap E(T) = \delta(W) \cap E(T).$$

By (3.6.10), (3.6.9) and (3.6.7) we have

$$\Sigma(x(\delta(i)): i \in W) = \Sigma(x(\delta(i)): i \in I).$$

Hence (3.6.6) and (3.6.8) combine to give the result.□

(3.6.11) <u>Corollary</u>. <u>Let F be an alternating forest with respect to the matching x, let K be the set of roots of the trees of F. Let W and I be the sets of even nodes and odd nodes of F respectively.</u>

<u>Then</u>

$$\underline{b(W) - \Sigma(b_r - x(\delta(r)): r \in K) = b(I)}.$$

Note that (3.6.1) implies therefore the following.

(3.6.12) <u>Corollary</u>. <u>If W and I are the sets of even and odd nodes of an alternating forest F then</u>

$$\underline{b(W) > b(I)} .$$

3.7. Hungarian Forests.

Let $\bar{G} = (\bar{V}, \bar{E}, \bar{\psi})$ be the graph obtained from $G = (V, E, \psi)$ by shrinking a (possibly empty) family R of disjoint shrinkable subsets of V. We define

$$(3.7.1) \quad \bar{V}^{=} \equiv (V^{=} \cap \bar{V}) \cup \{S \in R: S \subseteq V^{=}\} .$$

Let F be an alternating forest contained in $\bar{G}$ with respect to a matching $\bar{x}$ of $\bar{G}$ which satisfies

$$(3.7.2) \quad \bar{x}(\delta_{\bar{G}}(i)) \le b_i \quad \text{for all} \quad i \in \bar{V}.$$

We call F Hungarian in $\bar{G}$ with respect to $\bar{x}$ if

(3.7.3) no edge of $\bar{G}$ joins two even nodes of F,

(3.7.4) no edge of $\bar{G}$ joins an even node of F to a node not in F,

(3.7.5) every odd node of F is a node of G, that is, not a pseudonode of $\bar{G}$,

(3.7.6) if $v \in \bar{V}$ is an even node of F then $v \in \bar{V}^{=}$,

(3.7.7) for any $i \in \bar{V}^{=}$, if $\bar{x}(\delta_{\bar{G}}(i)) < b_i$ then i is the root of a tree in F.

Let x be any matching of G which satisfies

$$(3.7.8) \quad x(\delta(i)) \le b_i \quad \text{for all} \quad i \in V.$$

We define

$$(3.7.9)\quad d(G, V^{=}; x) \equiv \Sigma(b_i - x(\delta(i)) : i \in V^{=}).$$

If M is the set of all matchings of G which satisfy (3.7.8) then we let

$$(3.7.10)\quad D(G, V^{=}) \equiv \min\{d(G, V^{=}; x) : x \in M\}.$$

Thus $d(G, V^{=}; x)$ is a measure of the amount by which x fails to be a feasible matching of G and $D(G, V^{=})$ measures how closely we can come to obtaining a feasible matching of G. Clearly

(3.7.11) G has a feasible matching if and only if $D(G, V^{=}) = 0$.

Later in this section we show the connection between Hungarian forests and the value of $D(G, V^{=})$. We also show in (3.7.36) that knowledge of a Hungarian forest of G enables us to characterize those matchings x of G for which $d(G, V^{=}; x) = D(G, V^{=})$.

First we prove the following basic result which also indicates the importance of shrinkable sets in the blossom algorithm.

(3.7.12) <u>Proposition</u>. <u>Let R be a family of disjoint shrinkable subsets of V and let $\bar{G} = (\bar{V}, \bar{E}, \bar{\psi})$ be the graph obtained from $G = (V, E, \psi)$ by shrinking the members of R. Let $\bar{V}^{=}$ be defined as in (3.7.1). Then any matching $\bar{x}$ of $\bar{G}$ satisfying (3.7.2) can be extended to a matching x of G satisfying (3.7.8) such that</u>

$$(3.7.13)\quad \underline{d(G, V^{=}; x) = d(\bar{G}, \bar{V}^{=}; \bar{x})}.$$

Proof. For each $S \in R$ we define a node $i(S)$ as follows. If there is some $j \in \delta_{\bar{G}}(S)$ such that $\bar{x}_j = 1$ then let $\{i(S)\} \equiv \psi(j) \cap S$. Otherwise if $S - V^{=} \neq \phi$, let $i(S)$ be any member of $S - V^{=}$. Otherwise let $i(S)$ be any node of S. By (3.3.21) there is a np matching x^S of $G[S]$ deficient at $i(S)$ for every $S \in R$. We define x by

$$x_j \equiv \begin{cases} \bar{x}_j & \text{for } j \in \bar{E}, \\ x_j^S & \text{for } j \in \gamma(S) \text{ for all } S \in R. \end{cases}$$

x is easily seen to satisfy (3.7.8).

For any $v \in \bar{V} - R$ we have $\delta_{\bar{G}}(v) = \delta(v)$ so

(3.7.14) $b_v - x(\delta(v)) = b_v - \bar{x}(\delta_{\bar{G}}(v))$ for all $v \in \bar{V} - R$.

Let $S \in \bar{V}^{=} \cap R$. Then $S \subseteq V^{=}$ so

$$\begin{aligned} &\Sigma(b_i - x(\delta(i)) : i \in S) \\ &\quad = \Sigma(b_i - x(\delta(i)) : i \in S - \{i(S)\}) + b_{i(S)} - x(\delta(i(S))) \\ &\quad = 0 + (b_{i(S)} - x^S(\delta_{G[S]}(i(S)))) - \bar{x}(\delta_{\bar{G}}(S)) \\ &\quad = 1 - \bar{x}(\delta_{\bar{G}}(S)). \end{aligned}$$

Therefore

(3.7.15) $\Sigma(b_i - x(\delta(i)) : i \in S \cap V^{=}) = b_S - \bar{x}(\delta_{\bar{G}}(S))$ for all $S \in \bar{V}^{=} \cap R$.

Let $S \in R - \bar{V}^{=}$. If $i(S) \in V - V^{=}$ then

$$\Sigma(b_i - x(\delta(i)): i \in S \cap V^=) = 0.$$

If $i(S) \in V^=$ then there is $j \in \delta(i(S)) \cap \delta(S)$ such that $\bar{x}_j = 1$. Therefore

$$\begin{aligned}\Sigma(b_i - x(\delta(i)): i \in S \cap V^=) \\ &= b_{i(S)} - x^S(\delta_{G[S]}(i(S))) - \bar{x}_j \\ &= 1 - 1 = 0.\end{aligned}$$

Hence

(3.7.16) $\Sigma(b_i - x(\delta(i)): i \in S \cap V^=) = 0$ for all $S \in R - \bar{V}^=$.

Combining (3.7.14)-(3.7.16) gives (3.7.13).□

(3.7.17) <u>Theorem. Let $\bar{G} = (\bar{V}, \bar{E}, \bar{\psi})$, $G = (V, E, \psi)$ R, $V^=$ and $\bar{V}^=$ be as in (3.7.12). Let F be a Hungarian forest in $\bar{G}$ with respect to a matching $\bar{x}$. Let $K \subseteq \bar{V}$ be the set of roots of trees of F. Then</u>

(3.7.18) <u>$D(G, V^=) = \Sigma(b_i - \bar{x}(\delta_{\bar{G}}(i)): i \in K)$</u>.

<u>Proof</u>. By (3.7.6) and (3.7.7) $d(\bar{G}, \bar{V}^=; \bar{x}) = \Sigma(b_i - \bar{x}(\delta_{\bar{G}}(i)): i \in K)$. By (3.7.12) $\bar{x}$ can be extended to a matching x^0 of G for which $d(G, V^=; x) = d(\bar{G}, \bar{V}^=; x^0)$ so

(3.7.19) $D(G, V^=) \le \Sigma(b_i - \bar{x}(\delta_{\bar{G}}(i)): i \in K)$.

Now consider the linear program

$$(3.7.20) \quad \text{maximize} \quad 2x(\gamma(V^=)) + x(\delta(V^=))$$

over $x \in \mathbb{R}^E$ satisfying

$$x \geq 0,$$
$$(3.7.21) \quad x(\delta(i)) \leq b_i \quad \text{for all} \quad i \in V,$$
$$x(\gamma(S)) \leq q_S \quad \text{for all} \quad S \in Q^0.$$

By (3.1.7) any matching x of G satisfying (3.7.8) is a feasible solution to this linear program.

The dual linear program is

$$(3.7.22) \quad \text{minimize} \quad \Sigma(b_i y_i : i \in V) + \Sigma(q_S y_S : S \in Q^0)$$

for $y \in \mathbb{R}^{V \cup Q^0}$ satisfying

$$(3.7.23) \quad y_i \geq 0 \quad \text{for all} \quad i \in V \cup Q^0,$$

$$(3.7.24) \quad y(\psi(j)) + y(Q^0(j)) \geq |\psi(j) \cap V^=|$$

for all $j \in E$.

We define a vector y^0 as follows. Let I and W be the sets of odd and even nodes of F respectively.

$$(3.7.25) \quad y_i^0 \equiv \begin{cases} 2 & \text{if} \quad i \in I \cap V^= \\ 1 & \text{if} \quad i \in I - V^= \text{ or if} \\ & \quad i \in V^= - V(F) - \cup(R \cap V(F)), \\ 0 & \text{if} \quad i \in V - V^= - I; \end{cases}$$

$$(3.7.26) \quad y_S^0 \equiv \begin{cases} 2 & \text{if} \quad S \in R \cap W \\ 0 & \text{if} \quad S \in Q^0 - (R \cap W). \end{cases}$$

Now we show that

(3.7.27) y^0 is dual feasible.

If neither end of j is in F or is contained in a pseudonode of F then

$$y^0(\psi(j)) = |\psi(j) \cap V^=|$$

so (3.7.24) is satisfied.

(3.7.28) If exactly one end of j is in F or is contained in a pseudonode of F then by (3.7.4) j must meet an odd node of F so

$$y^0(\psi(j)) = |\psi(j) \cap V^=| + 1$$

and (3.7.24) is satisfied.

If $j \in \gamma(S)$ for some pseudonode S of F then

$$y^0(Q^0(j)) = 2 = |\psi(j) \cap V^=|$$

since by (3.7.6) and (3.7.1), $S \subseteq V^=$. Hence (3.7.24) is satisfied.

If $|\bar{\psi}(j) \cap I| = |\bar{\psi}(j) \cap W| = 1$ then since by (3.7.6) and (3.7.1) $\psi(j) - I \subseteq V^=$ it follows that

$$y^0(\psi(j)) = |\psi(j) \cap V^=|,$$

Thus (3.7.24) is satisfied.

(3.7.29) if $|\bar{\psi}(j) \cap I| = 2$ then $y^0(\psi(j)) = |\psi(j) \cap V^=| + 2$ so (3.7.24) is satisfied.

By (3.7.3) this exhausts all cases, so since $y^0 \geq 0$ we have proved (3.7.27).

Now we evaluate the dual objective function for y^0.

$$(3.7.30)\quad \Sigma(b_i y_i^0 : i \in V) + \Sigma(q_S y_S : S \in Q^0)$$
$$= b(V^= - V(F) - \cup(R \cap V(F))) + b(I - V^=)$$
$$+ 2b(I \cap V^=) + 2\Sigma(q_S : S \in R \cap W).$$

By (3.6.5),

$$(3.7.31)\quad b(I - V^=) + 2b(I \cap V^=)$$
$$= b(W) + b(I \cap V^=) - \Sigma(b_i - \bar{x}(\delta_{\bar{G}}(i)) : i \in K).$$

By (3.1.10)

$$(3.7.32)\quad 2\Sigma(q_S : S \in R \cap W) = \Sigma(b(S) : S \in R \cap W) - b(R \cap W).$$

Substituting (3.7.31) and (3.7.32) into (3.7.30) and simplifying we obtain

$$\Sigma(b_i y_i^0 : i \in V) + \Sigma(q_S y_S : S \in Q^0)$$
$$= b(V^=) - \Sigma(b_i - \bar{x}(\delta_{\bar{G}}(i)) : i \in K).$$

It therefore follows from the weak L.P. duality theorem (1.5.12) that

$$(3.7.33)\quad 2x(\gamma(V^=)) + x(\delta(V^=)) \le$$
$$b(V^=) - \Sigma(b_i - \bar{x}(\delta_{\bar{G}}(i)) : i \in K)$$

for any feasible solution x to the primal linear program (3.7.21). Since every matching of G which satisfies (3.7.8) is such a feasible solution, and since

$$(3.7.34)\quad \Sigma(b_i - x(\delta(i)) : i \in V^=) =$$
$$b(V^=) - (2x(\gamma(V^=)) + x(\delta(V^=))$$

it follows that

$$(3.7.35)\quad D(G, V^=) \geq \Sigma(b_i - \bar{x}(\delta_{\bar{G}}(i)): i \in K).$$

Combining (3.7.19) and (3.7.35) proves the theorem.□

By using the complementary slackness principle of linear programming we obtain the following characterization of matchings x which minimize $d(G, V^=; x)$.

(3.7.36) Theorem. Let $\bar{G} = (\bar{V}, \bar{E}, \bar{\psi})$, $G = (V, E, \psi)$, R, $V^=$ and $\bar{V}^=$ be as in (3.7.12). Then for any matching x of G satisfying (3.7.8) we have $D(G, V^=) = d(G, V^=; x)$ if and only if the following conditions are satisfied.

(3.7.37) $x(\gamma(S)) = q_S$ for all $S \in R \cap V(F)$.

(3.7.38) $x(\delta(i)) = b_i$ for every odd node i of F and for every $i \in V^= - V(F) - \cup(R \cap V(F))$.

(3.7.39) If I and W are the sets of odd and even nodes of F respectively, then $x_j = 0$ for all $j \in \bigcup_{i \in I} \delta(i) - \delta_{\bar{G}}(W)$.

Proof. In the proof of (3.7.17) we displayed a matching x^0 satisfying (3.7.8) and a dual solution y^0 such that $2x^0(\gamma(V^=)) + x^0(\delta(V^=)) = \Sigma(b_i y_i: i \in V) + \Sigma(q_S y_S: S \in Q^0)$. Thus y^0 is an optimal solution to the dual linear program (3.7.22)-(3.7.24) so every optimal solution $\hat{x}$ to the primal linear program (3.7.20), (3.7.21) must satisfy the complementary slackness conditions (see (1.5.16)) with respect to y^0.

Thus by (3.7.25) we must have (3.7.38); by (3.7.26) we require (3.7.37); by (3.7.28) and (3.7.29) we require (3.7.39). Since by (3.7.34) $\hat{x}$ maximizes $2x(\gamma(V^=)) + x(\delta(V^=))$ for x satisfying (3.7.21) if and only if $\hat{x}$ minimizes $d(G, V^=; x)$ for x satisfying (3.7.21) and since we have exhibited a matching x^0 for which $d(G, V^=; x^0) = D(G, V^=)$ the result now follows.□

If we are considering a matching problem in which $V^= = \phi$ then by (3.7.1) and (3.7.6) there could be no even nodes in a Hungarian forest F in a graph $\bar{G}$ obtained from $G = (V, E, \psi)$ by shrinking some disjoint shrinkable subsets of V. But since every Hungarian forest contains at least one tree rooted at an even node, this means that no Hungarian forest can exist. In other words, Hungarian forests are structures which can arise only when dealing with matching problems in which $V^= \neq \phi$.

The following corollary of (3.7.17) is a necessary condition for a graph G to have a feasible matching.

(3.7.40) Corollary. If G has a feasible matching then no graph $\bar{G}$ obtained from G by shrinking a collection of disjoint shrinkable subsets of V can contain a Hungarian forest.

Proof. If $\bar{G}$ contains a Hungarian forest F with respect to a matching $\bar{x}$ then if K is the set of roots of trees of F, we have

$$\Sigma(b_i - \bar{x}(\delta_{\bar{G}}(i)): i \in K) > 0.$$

Therefore by (3.7.17), $D(G, V^=) > 0$. Therefore by (3.7.11) G has no feasible matching.□

In fact, the converse of this corollary is true and will be proved by the blossom algorithm for it will always terminate with either an optimum feasible matching or else with a Hungarian forest.

3.8 The Blossom Algorithm

In this section we describe the blossom algorithm which solves the problem (3.1.2)-(3.1.6). This algorithm is also used in later chapters when we consider more general problems. In Section 3.9 we derive a bound on the amount of work performed by the blossom algorithm in solving a matching problem.

At each stage of the algorithm we have the following things.

(3.8.1) a matching $x = (x_j : j \in E)$,

(3.8.2) a dual solution $y = (y_i : i \in V \cup Q^0)$ which satisfies (3.5.7)-(3.5.9).

Let $G^= = (V, E^=, \psi|E^=)$ be the spanning subgraph of G whose edge set consists of all those $j \in E$ satisfying

$$(3.8.3) \quad y(\psi(j)) + y(Q^0(j)) = c_j .$$

$G^=$ is called the equality subgraph. The complementary slackness condition (3.5.10) is satisfied by x and y, that is

(3.8.4) $x_j > 0$ only if $j \in E^=$.

We also have a nested subfamily R of Q such that

(3.8.5) for each $S \in R$, $H(S) \equiv G^=[S] \times R[S]$

is spanned by a blossom $B(S)$. (See (3.3.15), (3.3.15a) for the definition of $G^=[S] \times R[S]$.)

Moreover

(3.8.6) $x|E(H(S))$ is a np matching of $H(S)$

deficient at some $i(S)$ belonging to the odd polygon of $B(S)$ and

(3.8.7) $x_j = 0$ for all $j \in E(H(S)) - E(B(S))$.

As a result of (3.3.24) a simple induction shows

(3.8.8) $x(\gamma(S)) = q_S$ for all $S \in R$.

The dual solution y has the property that

(3.8.9) $y_S > 0$ for $S \in Q^0$ only if $S \in R$.

Thus x and y satisfy the complementary slackness condition (3.5.12).

Let $\bar{G} = (\bar{V}, \bar{E}, \bar{\psi})$ be the graph $G^= \times R$. The matching x satisfies

(3.8.10) $x(\delta(i)) \le b_i$ for all $i \in \bar{V}$.

(Note that for any $i \in \bar{V}$, $\delta(i) = \delta_{\bar{G}}(i)$.)

We define subsets $\bar{V}^=$ and $\bar{V}^{\le}$ of $\bar{V}$ by

$$(3.8.11) \quad \bar{V}^{=} \equiv (V^{=} \cap \bar{V}) \cup \{S \in \bar{V}: S \subseteq V^{=}\},$$

$$(3.8.12) \quad \bar{V}^{\leq} \equiv \bar{V} - \bar{V}^{=}.$$

The matching x also has the following property. Let $G^+(x) = (\bar{V}, E^+(x), \bar{\psi}|E^+(x))$ be the spanning subgraph of $\bar{G}$ whose edges are all those edges of $\bar{G}$ such that $x_j > 0$. Thus $E^+(x) = \{j \in \bar{E}: x_j > 0\}$. Let H be any component of $G^+(x)$. Then

(3.8.13) H contains no even polygon;

(3.8.14) H contains at most one odd polygon;

(3.8.15) if H contains an odd polygon then $x(\delta(i)) = b_i$ for all $i \in V(H)$;

(3.8.16) if H contains no polygons then H has at most one node i for which $x(\delta(i)) < b_i$.

We also have an alternating forest F contained in $\bar{G}$.

(3.8.17) Each $i \in \bar{V}$ such that $x(\delta(i)) < b_i$ is the root of a tree in F.

F is partitioned into two subforests F^0 and F^1. F^0 consists of all those trees in F such that the root r belongs to $\bar{V}^{\leq}$ and $y_r = 0$ if $r \in V$ or $y_i = 0$ for some $i \in r$ if $r \in R$. F^1 consists of all other trees of F. It will be seen in the description of the algorithm that as long as there are nodes in F^1, we do not have the optimum feasible matching we seek and as soon as $V(F^1) = \phi$, we implicitly have an optimal solution.

In order that x and y be the optimal solutions we seek, all we need is that they satisfy (3.5.3), (3.5.4) and (3.5.11) for as we showed in (3.1.7), this together with the fact that x is a matching will ensure (3.5.5) is satisfied. We will show in the algorithm that if x and y satisfy the following analogues of (3.5.3), (3.5.4) and (3.5.11) then the required x can be obtained in a straightforward fashion.

(3.8.18) $x(\delta(i)) \le b_i$ for all $i \in \bar{V}^{\le}$,

(3.8.19) $x(\delta(i)) = b_i$ for all $i \in \bar{V}^{=}$,

(3.8.20) $y_i > 0$ for any $i \in V \cap \bar{V}^{\le}$ implies $x(\delta(i)) = b_i$,

(3.8.21) $y_i > 0$ for all $i \in S \cap V^{\le}$ for any $S \in R \cap \bar{V}^{\le}$ implies $x(\delta(i)) = b_i$.

We now define a measure of the amount by which (3.8.18)-(3.8.21) are violated. Let

(3.8.22) $\Delta(\bar{G}; x, y) \equiv \Sigma(b_i - x(\delta(i))$: $i \in \bar{V}^{=}$ or ($i \in \bar{V}^{\le} \cap V$ and $y_i > 0$) or ($i \in \bar{V}^{\le} \cap R$ and $y_v > 0$ for all $v \in i \cap V^{\le}$)).

It follows from the definition of F^1 that

(3.8.23) $\Delta(\bar{G}; x, y) = \Sigma(b_i - x(\delta(i))$: i is the root of a tree of F^1).

Clearly $\Delta(\bar{G}; x, y) \ge 0$ for any x satisfying (3.8.10) and $\Delta(\bar{G}; x, y) = 0$ if and only if x and y satisfy (3.8.18)-(3.8.21). In general, one "cycle" of the blossom algorithm will involve finding a new x' and y' and possibly a new graph $\bar{G}'$ such that $\Delta(\bar{G}'; x', y') \le \Delta(\bar{G}; x, y) - 1$.

(3.8.24) Initially we may take $x_j \equiv 0$ for all $j \in E$, $y_i \equiv \bar{c} \equiv 1/2 \max\{c_j : j \in E\}$ for $i \in V^=$, $y_i \equiv \max\{0, \bar{c}\}$ for $i \in V^{\le}$ and $R \equiv \phi$. Then it is easily seen that all our conditions are satisfied. F will be the spanning forest of G in which every tree consists of a single node.

We now describe the algorithm itself.

Step 1: Scan $\bar{E}$ to find an edge j joining an even node v_1 of F^1 to something other than an odd node of F^1. If no such edge exists go to Step 8. Otherwise go to Step 2.

Step 2: Let $\{v_2\} \equiv \bar{\psi}(j) - \{v_1\}$.

If v_2 belongs to a component of $G^+(x)$ which is not contained in F then go to Step 3.

If v_2 is an even node of a tree in F which is different from the tree containing v_1 then go to Step 4.

If v_1 and v_2 belong to the same tree of F then go to Step 5.

If v_2 is an odd node of a component of F^0 then go to Step 7.

This exhausts all possibilities for v_2.

Step 3: Grow Forest F. Let K be the component of $G^+(x)$ containing v_2. If K contains a polygon then go to Step 3c.

Step 3a (see Figure 3.3): If K contains no polygon,

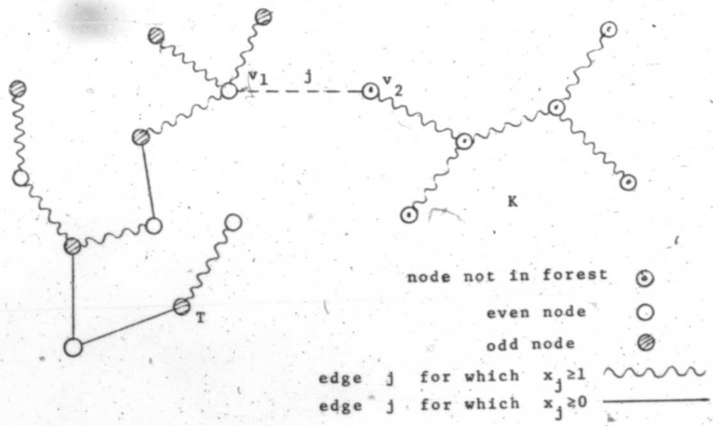


Figure 3.3 Forest Growth

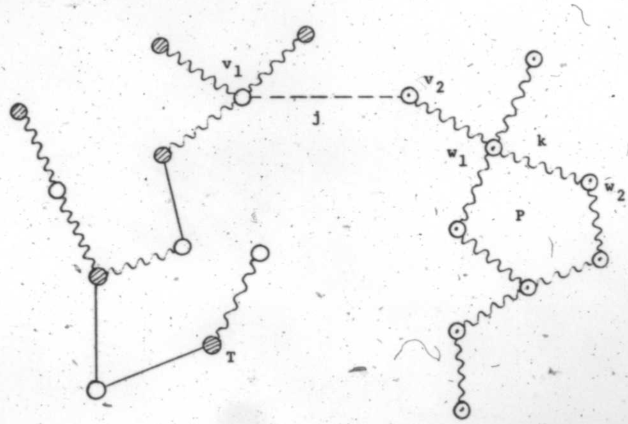


Figure 3.4 Addition of Polygon to Forest

that is, if K is a tree then we grow the alternating tree T containing v_1 by attaching v_2 and K to v_1 by means of the edge j. Since j becomes an odd edge of the new forest F' thereby obtained and by (3.8.17) it is easily seen that (3.6.1)-(3.6.4) are satisfied for F'.

<u>Step 3b</u>: Replace F by F' and go to Step 1.

<u>Step 3c</u> (see Figure 3.4): K contains an odd polygon P. Let w_1 be a node of P which is an odd distance from v_2 in K and for which this distance is as short as possible. Let w_2 be a node of P adjacent to w_1 in P which is no closer to v_2 in P then w_1. Let k be the edge of P joining w_1 and w_2. Let K' be the tree obtained from K by removing the edge k. Add K' to the forest by using j as described in Step 3a, thereby obtaining a larger forest F'. Edge k now joins two even nodes of some tree in F'. Replace v_1, v_2, j and F by w_1, w_2, k and F' respectively and go to Step 5.

<u>Step 4</u>: <u>Augmentation (Two trees)</u> (see Figure 3.5).

<u>Step 4a</u>: <u>Calculation of σ</u>. Let r_2 be the root of the tree T_1 of F^1 containing v_1 and let r_2 be the root of the tree T_2 of F containing v_2. Let $\sigma_1 \equiv \min\{x_k\}$ where k is an even edge of the path π_1 in T_1 from r_1 to v_1 or let $\sigma_1 \equiv \infty$ if no such edge exists. Let σ_2 and π_2 be analogously defined for T_2, v_2 and r_2. By (3.6.4), $\sigma_1, \sigma_2 \geq 1$. Let $\sigma \equiv \min\{\sigma_1, \sigma_2, b_{r_1} - x(\delta(r_1)), b_{r_2} - x(\delta(r_2))\}$. By (3.6.1), $\sigma \geq 1$.

<u>Step 4b</u>: <u>Augumentation</u>. Define x' by

$$x'_k \equiv \begin{cases} x_k - \sigma \text{ if } k \text{ is an even edge of } \pi_1 \text{ or } \pi_2 \\ x_k + \sigma \text{ if } k \text{ is an odd edge of } \pi_1 \text{ or } \pi_2, \\ \qquad \text{or if } k = j \\ x_k \text{ for all } k \in E - (E(\pi_1) \cup E(\pi_2) \cup \{j\}). \end{cases}$$

Now x' is a matching satisfying (3.8.4), (3.8.6)-(3.8.8), and (3.8.10) and $\Delta(\bar{G}; x', y) \leq \Delta(\bar{G}; x, y) - 1$ since $b_{r_1} - x'(\delta(r_1)) \leq b_{r_2} - x(\delta(r_1)) - 1$.

<u>Step 4c</u>. <u>Computation of new F</u>. We obtain a new alternating forest in the following way. If $x'(\delta(r_1)) = b_{r_1}$ then we remove T_1 from F. Similarly if $x'(\delta(r_2)) = b_{r_2}$ then we remove T_2 from F. If k is an even edge of π_1 or π_2 for which $x'_k = 0$ then we remove k and the portion of the tree above it from F. By our choice of σ, at least one of these things must occur. Thus at most one of v_1 and v_2 can be in the new forest F'. If neither are in F' then replace x by x', F by F' and go to Step 1. If one, say v_1, belongs to F then perform Step 3a to add the component K of $G^+(x')$ containing v_2 to F'' using the edge j, let F'' be the forest thereby obtained. Replace x by x', F by F'' and return to Step 1.

<u>Step 5</u>: <u>Augmentation (One tree)</u> (see Figure 3.6)

<u>Step 5a</u>: <u>Calculation of σ and Blossom Test</u>. Let r be the root of the tree T of F^1 containing v_1 and v_2. Let π_1 be the path in T from r to v_1 and let π_2 be

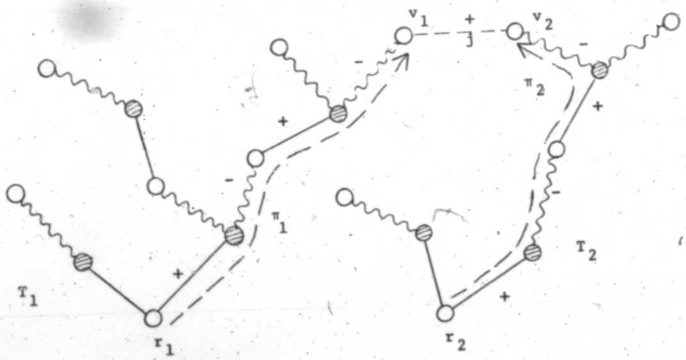


Figure 3.5 Two Tree Augmentation

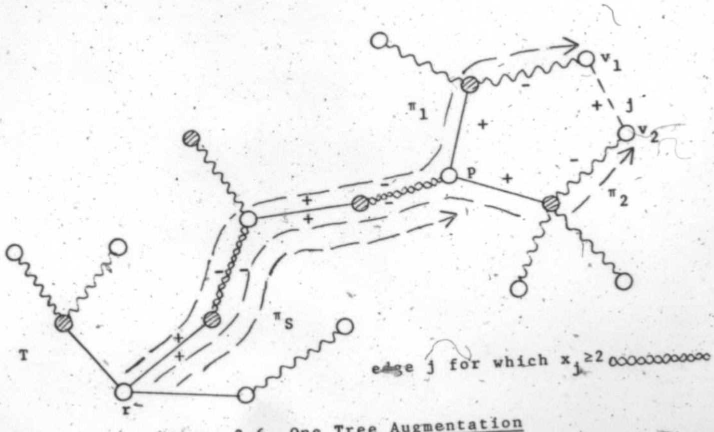


Figure 3.6 One Tree Augmentation

the path in T from r to v_2. Let π_s be the common position of π_1 and π_2. π_s is the path in T from r to some node p. (Of course, p may equal r in which case π_s is an empty sequence). Then $E(\pi_1) \cup E(\pi_2) \cup \{j\} - E(\pi_s)$ are the edges of an odd polygon P containing p. ($|E(P)|$ is odd because j joins two even nodes of T.)

Let $\sigma_0 \equiv \min\{x_k: k$ is an even edge of $\pi_s\}$, or let $\sigma_0 \equiv \infty$ if no such edge exists. Let $\sigma_1 \equiv \min\{x_k: k$ is an even edge of π_1 and $k \notin E(\pi_s)\}$. or let $\sigma_1 \equiv \infty$ if no such edge exists. Let σ_2 be defined analogously for π_2. By (3.6.4), $\sigma_0, \sigma_1, \sigma_2 \geq 1$. Let

$$\sigma \equiv \min\{[1/2\ \sigma_0], \sigma_1, \sigma_2, [1/2(b_r - x(\delta(r))]\}$$

(where for any $\alpha \in \mathbb{R}$, $[\alpha]$ is the largest integer no greater than α). If $\sigma \geq 1$ then go to Step 5b where we augment. Otherwise go to Step 6 where we shrink a portion of $\bar{G}$.

<u>Step 5b</u>: <u>Augmentation</u>. Define x' as follows.

$$x'_k \equiv \begin{cases} x_k - 2\sigma & \text{if } k \text{ is an even edge of } \pi_s, \\ x_k + 2\sigma & \text{if } k \text{ is an odd edge of } \pi_s, \\ x_k - \sigma & \text{if } k \text{ is an even edge of } \pi_1 \text{ or } \pi_2 \\ & \text{not belonging to } \pi_s, \\ x_k + \sigma & \text{if } k = j \text{ or if } k \text{ is an odd edge} \\ & \text{of } \pi_1 \text{ or } \pi_2 \text{ not belonging to } \pi_s, \\ x_k & \text{for all } k \in E - (E(\pi_1) \cup E(\pi_2) \cup \{j\}). \end{cases}$$

We can see by our choice of σ that x' is a matching satisfying (3.8.4), (3.8.6)-(3.8.8), (3.8.10) and $\Delta(\bar{G}; x', y) \leq \Delta(\bar{G}; x, y) - 2$ since $b_r - x'(\delta(r)) \leq b_r - x(\delta(r)) - 2$

and $b_i - x'(\delta(i)) = b_i - x(\delta(i))$ for all $i \in V - \{r\}$.

Step 5c: Computation of new F. Each component H of $G^+(x')$ will satisfy (3.8.13), (3.8.14) and (3.8.16) but need not satisfy (3.8.15). That is there may be a component of $G^+(x')$ containing both a deficient node and an odd polygon. We now analyze the various possibilities.

If $x'(\delta(r)) = b_r$ then let F' be the forest obtained from F by removing T. Since $x'(\delta(i)) = b_i$ for all $i \in V(T)$, each component H of $G^+(x')$ satisfies (3.8.15). F' is an alternating forest. Replace x and F by x' and F' respectively and go to Step 1.

If $x'(\delta(r)) < b_r$ but there are $\ell \in E(\pi_s)$ such that $x'_\ell = 0$, let k be the first such edge in π_s. Let T' be the portion of T above k. Remove T' and k from F thereby obtaining a new alternating forest F'. Since $x'(\delta(i)) = b_i$ for all $i \in V(T')$, each component H of $G^+(x')$ satisfies (3.8.15). Replace x and F by x' and F' and go to Step 1.

If $x'(\delta(r)) < b_r$, $x'_\ell > 0$ for all $\ell \in E(\pi_s)$ but $x'_k = 0$ for some edge k of P, then we remove all such edges k from F thereby obtaining a forest F'. If one end of j, say v_1, is in F' then the other end v_2 cannot be in F', adjoin the component H of $G^+(x')$ containing v_2 to F' by means of j, thereby obtaining a new alternating forest F''. Each component H of $G^+(x')$ satisfies (3.8.15). Replace x and F by x' and F'' and go to Step 1.

Finally, if $x'(\delta(r)) < b_r$ and $x'_\ell > 0$ for all $\ell \in E(\pi_1) \cup E(\pi_2) \cup \{j\}$ then by our choice of σ there is an even edge k of π_s for which $x'_k = 1$ or $x'(\delta(r)) = b_r - 1$. Replace x by x' and go to Step 6. Note that this is the one case in which there is a component H of $G^+(x')$ violating (3.8.15). This is handled in Step 6.

Step 6: Shrinking Step (see Figure 3.7). We now identify a blossom in $\bar{G}$. T is the tree of F^1 containing v_1 and v_2, π_s is the path in T from its root r to the nearest node p of P, the odd polygon formed by adding j to T. Let w be the first even node of π_s such that the path π' in T from w to p contains no even edge k for which $x_k = 1$. (Thus $x_k \geq 2$ for every even edge of π'.) The blossom B consists of P, the subgraph of T induced by π' and any component H of G^+, such that $V(H) \cap V(\pi') \neq \phi$ or $V(H) \cap V(P) \neq \phi$ except for the even edge of T incident with w if it exists. Let S be the set of all those nodes of G which either belong to $V(B)$ or are contained in pseudonodes of B.

We see that $x(\delta_B(i)) = b_i$ for all $i \in V(B) - \{w\}$ and $x(\delta_B(w)) = b_w - 1$. Thus $x|E(B)$ is a np matching of $\bar{G}[V(B)]$ deficient at w. If $w \notin V(P)$ then we modify our matching so that it will be deficient at a node of P, as this simplifies later discussions. Define x' by

$$x'_k \equiv \begin{cases} x_k + 1 & \text{for every odd edge of } \pi'. \\ x_k - 1 & \text{for every even edge of } \pi'. \end{cases}$$

If p is an even node of F then let $i(S) \equiv p$. If p is an odd node, let $i(S)$ be an even node of P adjacent to p. Where ℓ is the edge of B joining $i(S)$ and p let

$$x'_\ell \equiv x_\ell - 1.$$

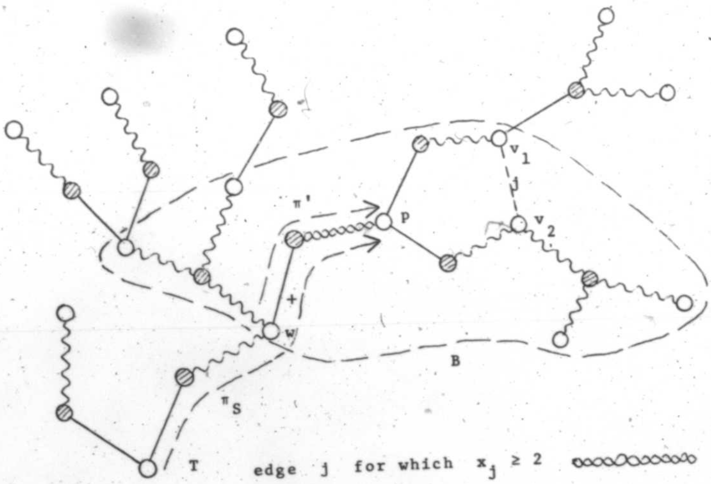


Figure 3.7 Shrinking Step

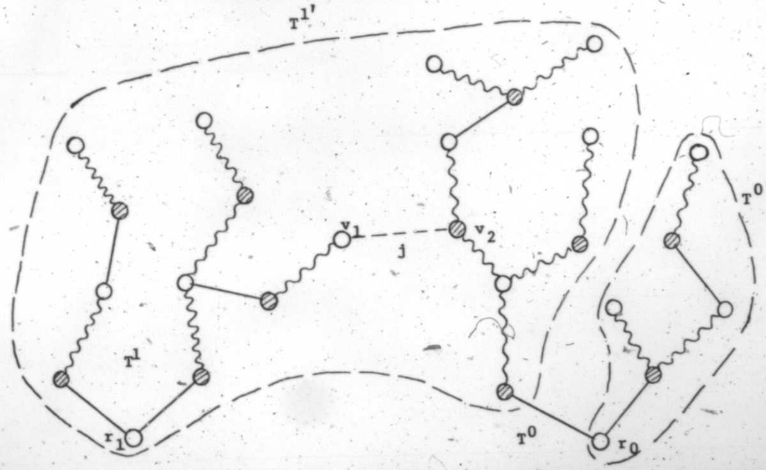


Figure 3.8 Pseudo Forest Growth

For all other edges k of G let $x'_k \equiv x_k$. Now replace x by x'.

$x|E(B)$ satisfies (3.3.1)-(3.3.5) taking $v \equiv i(S)$. Let $B(S) \equiv B$. Now if we let $R' \equiv R \cup \{S\}$ we see that R' so defined satisfies (3.8.5)-(3.8.7).

Let $\bar{G}' \equiv G^= \times R'$. Let F' be the forest in $\bar{G}'$ with node set equal to $V(F) \cap V(\bar{G}') \cup \{S\}$ and edge set equal to $E(F) \cap E(\bar{G}')$. Then F' is an alternating forest in $\bar{G}'$ and

(3.8.25) S is an even node of F'.

Let $G'^+(x)$ be defined for $\bar{G}'$ in the same way that $G^+(x)$ was defined for $\bar{G}$. It is easily seen that every component H of $G'^+(x)$ satisfies (3.8.13)-(3.8.16) since the only component of $G^+(x)$ which could have violated these conditions was the one containing the polygon P and it has been shrunk away.

Note also that $\Delta(\bar{G}'; x, y) \leq \Delta(\bar{G}; x, y)$. Replace $\bar{G}$, R and F by $\bar{G}'$, R' and F' respectively and go to Step 1.

Step 7. Grow forest F^1 (Pseudo forest growth). (see Figure 3.8.)

Edge j joins an even node v_1 of a tree T^1 in F^1 to an odd node v_2 of a tree T^0 in F^0. Let r_0 and r_1 be the roots of T^0 and T^1 respectively. Let $\bar{T}$ be the portion of T^0 above v_2. We adjoin $\bar{T}$ and the component H of $G^+(x)$ containing x_2 to v_1 by means of the edge j thereby obtaining a larger tree $T^{1'}$. (H is a subgraph of T^0 by (3.6.3).)

If $r_0 \notin V(T^{1'})$ then replace T^1 by $T^{1'}$ in F thereby obtaining a larger forest $F^{1'}$. Remove $\bar{T}$, H and any edge of T^0 incident with a node of $\bar{T}$ or H from T^0, thereby obtaining a smaller tree $T^{0'}$ and a smaller forest $F^{0'}$. Replace F^0, F^1 by $F^{0'}$, $F^{1'}$ and go to Step 1.

If $r_0 \in V(T^{1'})$ then remove T^0 from F^0, let T denote $T^{1'}$ and perform the following step.

Step 7a. (Pseudo Augmentation). Let π be the path in T from r_1 to r_0. Observe that both r_0 and r_1 are even nodes of T. Let $\sigma_1 \equiv \min\{x_j : j$ is an even edge of $\pi\}$. Let $\sigma \equiv \min\{\sigma_1, b_{r_1} - x(\delta(r_1))\}$. Then $\sigma \geq 1$. Let x' be defined by

$$x'_k \equiv \begin{cases} x_k - \sigma & \text{if } k \text{ is an even edge of } \pi \\ x_k + \sigma & \text{if } k \text{ is an odd edge of } \pi \\ x_k & \text{if } k \notin E(\pi). \end{cases}$$

Since $b_{r_1} - x'(\delta(r_1)) = b_{r_1} - x(\delta(r_1)) - \sigma$ and $\sigma \geq 1$ it follows from (3.8.23) that $\Delta(\bar{G}; x', y) \leq \Delta(\bar{G}; x, y) - 1$.

If $x'(\delta(r_1)) = b_{r_1}$ then remove T from F^1 thereby obtaining a new forest $F^{1'}$. Reroot T at r_0 and add T to F^0 thereby obtaining a new forest $F^{0'}$. It is easily checked that T rooted at r_0 satisfies (3.6.1)-(3.6.4) with respect to x'.

If $x'(\delta(r_1)) < b_{r_1}$ then by our choice of σ we must have $x'_\ell = 0$ for some even edge ℓ of π; let k be the first such edge of π. Let $\bar{T}$ be the portion of T above k. Remove $\bar{T}$ and k from T thereby obtaining a new forest $F^{1'}$. Reroot $\bar{T}$ at r_0 and add it to F^0 thereby obtaining a new forest $F^{0'}$. Again it is easily checked that $\bar{T}$ rooted at r_0 satisfies (3.6.1)-(3.6.4) relative to x'.

Replace x, F^0 and F^1 by x', $F^{0'}$ and $F^{1'}$ respectively and go to Step 1.

<u>Step 8</u>: <u>Termination Test</u>. We now decide whether or not we are ready to go to the final stage of the algorithm. If $V(F^1) = \phi$ then by (3.8.23) $\Delta(\bar{G}; x, y) = 0$ and we go to Step 11, the termination step. Otherwise we go to Step 9 where we will attempt to make a change in the dual variables which will enable further progress.

<u>Step 9</u>: <u>Dual Variable Change</u>.

<u>Step 9a</u>: <u>Calculation of ϵ</u>. Let $E_1 \equiv \{j \in \bar{E}$: one member of $\bar{\psi}(j)$ is an even node of F^1 and the other member of $\bar{\psi}(j)$ is not a node of $F^1\}$. If $E_1 = \phi$ then let $\epsilon_1 \equiv \infty$, otherwise let

$$\epsilon_1 \equiv \min\{y(\psi(j)) + y(R(j)) - c_j\colon j \in E_1\},$$ where $R(j) \equiv \{S \in R\colon j \in \gamma(S)\}$.

Let $E_2 \equiv \{j \in E$: both members of $\bar{\psi}(j)$ are even nodes of $F^1\}$. If $E_2 = \phi$ then let $\epsilon_2 \equiv \infty$, otherwise let

$$\epsilon_2 \equiv 1/2 \min\{y(\psi(j)) + y(R(j)) - c_j\colon j \in E_2\}.$$

Let $P \equiv \{S \in R\colon S$ is an odd node of $F^1\}$. If $P = \phi$ then let $\epsilon_3 \equiv \infty$, otherwise let

$$\epsilon_3 = 1/2 \min\{y_S\colon S \in P\}.$$

Let $Y \equiv \{i \in V^{\leq}$: i is an even node of F^1 or $i \in S \in R$

and S is an even node of $F^1\}$. If $Y = \phi$ then let $\epsilon_4 \equiv \infty$, otherwise let

$$\epsilon_4 \equiv \min\{y_i : i \in Y\}.$$

Let $\epsilon \equiv \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$. If $\epsilon = \infty$ then go to Step 10 where we show that there exists no feasible matching. If $\epsilon = 0$ then no dual variable change is necessary so go to Step 9c. Otherwise go to Step 9b where the dual variables are changed.

<u>Step 9b</u>: <u>Change of Dual Variables</u>. Define a new dual solution y' as follows. Let

$$y'_i \equiv \begin{cases} y_i + \epsilon & \text{if } i \in V \text{ is an odd node of } F^1 \text{ or} \\ & \quad\text{belongs to an odd pseudonode of } F^1, \\ y_i - \epsilon & \text{if } i \in V \text{ is an even node of } F^1 \text{ or} \\ & \quad\text{belongs to an even pseudonode of } F^1, \\ y_i & \text{if } i \in V - V(F^1) - \cup(R \cap V(F^1)). \end{cases}$$

$$y'_S \equiv \begin{cases} y_S + 2\epsilon & \text{if } S \in R \text{ is an even node of } F^1, \\ y_S - 2\epsilon & \text{if } S \in R \text{ is an odd node of } F^1, \\ y_S & \text{if } S \in Q^0 - (R \cap V(F^1)). \end{cases}$$

Because of our choice of ϵ, y' is a feasible dual solution, that is, it satisfies (3.5.7)-(3.5.9). y' also satisfies (3.8.9). Moreover

$$(3.8.26)\quad y'(\psi(j)) + y'(Q^0(j)) = y(\psi(j)) + y(Q(j))$$

for all $j \in E(G^+) \cup E(F) \cup \bigcup_{S \in R} \gamma(S)$.

Let $G^{'=}$ be the spanning subgraph of G whose edges are all those $j \in E$ such that $y'(\psi(j)) + y'(Q^0(j)) = c_j$. Let $\bar{G}' \equiv G^{'=} \times R$. F is now an alternating forest in $\bar{G}'$. By (3.8.26) for each $S \in R$, $H(S) = G^{'=}[S] \times R[S]$ and $B(S)$ is a blossom spanning $H(S)$ (where $H(S)$ and $B(S)$ are as defined in (3.8.5)).

Step 9c: If $\epsilon \in \{\epsilon_1, \epsilon_2\}$ then there is an edge $j \in E(\bar{G}') - E(\bar{G})$ of the sort we sought in Step 1. Replace y, $\bar{G}$ and $G^=$ by y', $\bar{G}'$ and $G^{'=}$ respectively and go to Step 1 and from there as directed.

Step 9d. If $\epsilon = \epsilon_4$ then let I be the set of nodes $i \in V^{\le}$ such that $y_i' = 0$ and i is either an even node of F^1 or is contained in an even pseudonode of F^1. Since $\epsilon = \epsilon_4$, $I \neq \phi$. For each $i \in I$ let $r(i) \equiv i$ if $i \in V(F^1)$, let $r(i) \equiv S$ if $i \in S \in R \cap V(F^1)$.

For each $i \in I$ such that $r(i)$ is the root of a tree T_i in F^1, remove T_i from F^1 and add it to F^0. If any such i exists then we have by (3.8.23) that $\Delta(\bar{G}; x, y') \le \Delta(\bar{G}; x, y) - 1$; replace y, $\bar{G}$ and $G^=$ by y', $\bar{G}'$ and $G^{'=}$ respectively and go to Step 1.

If there is no $i \in I$ such that $r(i)$ is the root of a tree in F^1, then choose any $i_0 \in I$, let $r_0 \equiv r(i)$, let T be the tree of F^1 containing r_0 and let r_1 be the root of T. Replace y, $\bar{G}$, $G^=$ by y', $\bar{G}'$, $G^{'=}$ respectively and go to Step 7a.

Step 9e. If $\epsilon = \epsilon_3$ then we must expand an odd pseudonode S of F^1 for which $y_S' = 0$. Since $b_S = 1$, by (3.6.2) there is an edge $j \in \delta_{F^1}(S)$ such that $x_j = 1$. Let $H(S)$ and $B(S)$ be as defined in (3.8.5). Let v be the node of $B(S)$ incident with j. By (3.8.6) we can apply the procedure described in the proof of (3.3.12) to $x|E(B(S))$ and thereby obtain a np matching $\bar{x}$ of $B(S)$ deficient at v.

Let $R' \equiv R - \{S\}$. Since S is a maximal member of R, (3.8.5) is satisfied by R'. Let $\bar{G}" \equiv G'^{=} \times R'$. $B(S)$ is a subgraph of $\bar{G}"$. Define x' by

$$x'_k \equiv \begin{cases} \bar{x}_k & \text{if } k \in E(B(S)), \\ x_k & \text{if } k \in E - E(B(S)). \end{cases}$$

x' is easily seen to satisfy

$$x'(\underline{\delta}(i)) \le b_i \quad \text{for all } i \in V(\bar{G}").$$

Moreover $\Delta(\bar{G}"; x', y') = \Delta(\bar{G}; x, y)$. Replace j, $\bar{G}$, $G^{=}$, and R by y', $\bar{G}'$, $G'^{=}$ and R' respectively and go to Step 9f where we determine a new forest F.

Step 9f: If j is an odd edge of F then since by (3.8.4) we have $x_k > 0$ for any even edge of F and since $b_S = 1$ it follows that j is the only edge of F incident with S. Let F' be the subgraph of $\bar{G}$ obtained by replacing the pseudonode S in F with the component K of $G^+(x')$ containing v. Go to Step 9g.

If j is an even edge of F then let ℓ be the unique odd edge of F incident with S. Since S is an odd node of F and $b_S = 1$ these are the only two edges of F incident with S. Let w be the node of $B(S)$ met by ℓ and let π be a track in $B(S)$ from v to w having even length and for which this length is as small as possible. Let $\bar{G}(\pi) \equiv (V(\pi), E(\pi), \bar{\psi}|E(\pi))$. Let F' be the subgraph of $\bar{G}$ obtained by replacing the pseudonode S in F with $\bar{G}(\pi)$ and any component of $G^+(x')$ which contains a node of π.

<u>Step 9g</u>. If F' contains no polygon then it is easily seen that F' is an alternating forest in $\bar{G}$; replace x and F by x' and F' respectively and go to Step 1.

If F' contains a polygon P, then P is the odd polygon of the blossom $B(S)$. Let v_1 be a node of P which is an odd distance from w in $B(S)$ and for which this distance is as small as possible. Let v_2 be a node of P adjacent to v_1 in P and not belonging to the path in P joining w and v_1. Let j' be the edge of P joining v_1 and v_2. Remove j' from F', let F'' be the forest thereby obtained. Now j' joins two even nodes of F''. Replace F and j by F'' and j' respectively and go to Step 5. At this point F fails to be an alternating forest because j violates (3.6.3) and the component H of $G^+(x)$ containing v_1 may not satisfy (3.8.15). However these situations are automatically corrected in Steps 5 and 6.

<u>Step 10</u>: <u>Hungarian Forest</u>. Since $\epsilon = \infty$ we observe the following. $\epsilon_1 = \infty$ is equivalent to F^1 satisfying (3.7.4). $\epsilon_2 = \infty$ is equivalent to (3.7.3) for F^1. $\epsilon_3 = \infty$ is equivalent to (3.7.5) for F^1 and $\epsilon_4 = \infty$ is equivalent to (3.7.6) for F^1. Therefore F^1 is a Hungarian forest so by (3.7.40), G has no feasible matching. By (3.7.17) and (3.8.23),

$D(G, V^=) = \Delta(\bar{G}; x, y)$. If desired, perform Step 12 so as to "correct" the matching x for edges $j \in \gamma(S)$ for $S \in R$ so that the resulting matching x' will satisfy $x'(\delta(i)) \le b_i$ for all $i \in V$ and $d(G, V^=; x') = D(G, V^=)$. We do not bother performing Step 12 in the applications we make of this algorithm in later chapters.

(3.8.27) Finally note that if F_1 is a Hungarian forest, then for any $\epsilon \in \mathbb{R}$ such that $\epsilon \ge 0$ we have that y' as defined in Step 9b is a feasible dual solution also satisfying (3.8.26).

Step 11: Termination with Optimal Solution. Apply Step 12 to "correct" the matching x and then stop, the resulting matching x is the optimal feasible matching we seek and y is an optimum dual solution. Since Step 12 ensured that $\Delta(G; x, y) = 0$ and $x(\delta(i)) \le b_i$ for all $i \in V$ it follows that (3.5.3) and (3.5.4) and x and y satisfy (3.5.11). Since x is a matching satisfying (3.5.3) and (3.5.4), we also have (3.5.2) and (3.5.5) satisfied. By (3.8.8) and (3.8.9) we know that (3.5.12) is satisfied. By (3.8.2) y satisfies (3.5.7)-(3.5.9). Therefore x is the optimal matching we require and y is an optimal dual solution.

Step 12: Pseudonode Matching Correction. Let $D \equiv \phi$. D is the set of members of R for which the matching has been corrected.

Step 12a: If $R = D$ then return to Step 10 or 11 from whence we come.

Step 12b: Let S be a maximal member of $R - D$ and let $D' \equiv D \cup \{S\}$. Let $G' \equiv G^{=} \times (R - D')$. Then $B(S)$ (as defined in (3.8.5)) is a blossom contained in G'. If $x(\delta(S)) = 0$ then go to Step 12d.

Step 12c: Let $j \in \delta(S)$ be such that $x_j = 1$, let v be the node of $B(S)$ met by j in G'. Apply the procedure described in the proof of (3.3.12) to obtain a np matching $\hat{x}$ of $B(S)$ deficient at v. Let x' be defined by

$$(3.8.28)\quad x'_k \equiv \begin{cases} x_k & \text{for } k \in E - E(B(S)), \\ \hat{x}_k & \text{for } k \in E(B(S)). \end{cases}$$

Then we have

$$(3.8.29)\quad x'(\delta(i)) \le b_i \quad \text{for all } i \in V(G'),$$

$$(3.8.30)\quad x'(\gamma(T)) = q_T \quad \text{for all } T \in R,$$

$$(3.8.31)\quad \Delta(G'; x', y) = \Delta(\bar{G}; x, y).$$

Replace x, D and $\bar{G}$ by x', D' and G' respectively. Return to Step 12a.

Step 12d. Let $v \in S \cap V^{\le}$ be such that $y_v = 0$ if such a node exists, otherwise go to Step 12e. Let $r \equiv v$ if $v \in V(B(S))$, let $r \equiv T$ if $v \in T \in R \cap V(B(S))$. As in Step 12c we apply the procedure described in the proof of (3.3.12) to obtain a np matching $\hat{x}$ of $B(S)$ deficient at r. Let x' be defined as in (3.8.28). Again (3.8.29) and (3.8.30) are immediate and since the only new deficiency we created was at r and since $y_v = 0$, (3.8.31) is satisfied.

Replace x, D and $\bar{G}$ by x', D' and G' respectively. Return to Step 12a.

<u>Step 12e</u>: (This step can only be performed if we terminated in Step 10.) In this case $S \in \bar{V}^{=}$ since S must have been an even node or contained in an even pseudonode of the Hungarian forest F^1. Therefore the term corresponding to S contributes 1 to $\Delta(\bar{G}; x, y)$. Let v be the node of $B(S)$ at which $x|E(B(S))$ is deficient. If we let $x' \equiv x$ then (3.8.29) and (3.8.30) are satisfied and since the term corresponding to v contributes 1 to $\Delta(G'; x', y)$ we have (3.8.31) satisfied. Go to Step 12a.

3.9 <u>Efficiency of the Blossom Algorithm</u>.

In this section we derive an upper bound on the amount of work done by the blossom algorithm in solving a matching problem. We make a fixed word assumption, that the amount of work required to perform arithmetic (addition, subtraction, division by two) on any numbers encountered in the algorithm is independent of the number of significant digits. Since this is the way in which most large scale computers operate (for reasonably sized numbers) this is a realistic assumption.

(3.9.1) <u>Theorem</u>. <u>An upper bound on the amount of work required by the blossom algorithm to solve a matching problem is of the order</u>

$$\underline{\Delta(G; x^0, y^0) \cdot |V| \cdot |E|}$$

<u>where x^0 and y^0 are the starting matching and dual solution.</u>

<u>Proof</u>. First we establish an upper bound on the amount of work that can be done by the algorithm without decreasing $\Delta(\bar{G}; x, y)$. Steps 4, 5, 7a and 9d all decrease $\Delta(\bar{G}; x, y)$ by at least one.

In Steps 3 and 7 we grow the forest F^1. Since $|V(F^1)|$ decreases only after performing one of Steps 4, 5, 7a or 9d, it follows that Steps 3 and 7 can be performed at most $|V|$ times without a decrease in $\Delta(\bar{G}; x, y)$.

In Step 6 we shrink. By (3.8.5) $n(S) \geq 3$ for every $S \in R$ (where $n(S)$ is as defined in (3.2.7)). Thus by (3.2.8) we must have $|R| \leq 1/2(|V| - 1)$ at any point in the algorithm. By (3.8.25) any new S added to R becomes an even node of F^1. We only expand odd nodes of F^1 (in Step 9e). Thus Step 6 can be performed at most $1/2(|V| - 1)$ times without a decrease in $\Delta(\bar{G}; x, y)$.

In Steps 9e-9g we expand an odd pseudonode of F^1. This pseudonode must have been in F^1 following the previous augmentation since any pseudonode formed since is an even node of F^1. Hence Steps 9e-9g can be performed at most $1/2(|V| - 1)$ times without making a change in $\Delta(\bar{G}; x, y)$.

Steps 10, 11, 12 are performed only once in the course of the algorithm. A bound on the amount of work required by these steps is of the order $|V| \cdot |E|$.

Steps 1, 2, 8, 9a, 9b, 9c are performed at most once for each performance of steps 3, 4, 5, 6, 7, 10, 11, 12. A bound on the amount of work performed by each of these can be seen to be of the order $|E|$. The only ones of these steps for which this bound is not obvious are 9a, 9b. However if we preserve the value of $y(\psi(j)) + y(Q^0(j)) - c_j$ for each $j \in E$ at all times, then it can be seen that this bound is satisfied for these steps.

Finally a bound on the amount of work required for each of Steps 4, 5, 7a or 9d is of the order $|E|$.

Thus a bound on the amount of work that can be done without decreasing $\Delta(\bar{G}; x, y)$ by at least one is of the order $|E| \cdot |V|$ and the theorem follows.□

(3.9.2) Corollary. If we start with the matching described in (3.8.24) then an upper bound on the amount of work done in solving a matching problem is of the order

$$b(V) \cdot |V| \cdot |E| .$$

Proof. This follows from the fact that if x and y are as defined in (3.8.24) then $\Delta(G; x, y) \le b(V)$.□

3.10 Min-Max Theorems and Discreteness of the Dual Solution

Whenever we know a set of linear inequalities sufficient to define a polyhedron P, linear programming duality gives us a min-max theorem concerning any subset of P that contains the vertices. Conversely, we used the blossom algorithm to prove the following min-max theorem which established Theorem (3.4.5).

(3.10.1) Theorem. Let $G = (V, E, \psi)$ be a graph, let $b = (b_i : i \in V)$ be a vector of positive integers and let $c = (c_j : j \in E)$ be an arbitrary real vector. Then the maximum value of $c \cdot x$ for any matching x of G which satisfies

(3.10.2) $x(\delta(i)) \le b_i$ for all $i \in V$

is equal to the minimum value of

(3.10.3) $\Sigma(b_i y_i : i \in V) + \Sigma(q_S y_S : S \in Q^0)$

for real $(y_i : i \in V)$ and $(y_S : S \in Q^0)$ satisfying

(3.10.4) $y_i \ge 0$ for all $i \in V$,

(3.10.5) $y_S \ge 0$ for all $S \in Q^0$

(3.10.6) $y(\psi(j)) + y(Q^0(j)) \ge c_j$ for all $j \in E$.

If the objective function c satisfies certain discreteness properties, then we are able to require certain discreteness properties of the dual variables.

(3.10.7) Theorem. If c_j is integer valued for all $j \in E$ then there is an optimal feasible solution y^0 to the problem of minimizing (3.10.3) subject to (3.10.4)-(3.10.6) which satisfies

(3.10.8) y_i is congruent with 0 (mod 1/2) for all $i \in V$,

(3.10.9) y_S is congruent with 0 (mod 1) for all $S \in Q^0$.

Proof. The problem of minimizing (3.10.3) subject to (3.10.4)-(3.10.6) is the dual linear program to the matching problem maximize cx for $x \in P(G, b)$. We will show that

(3.10.10) if the starting dual solution used by the blossom algorithm is integer valued, then at any point in the solution of this matching problem the dual solution y will satisfy (3.10), (3.10.9). This we prove by showing that at any point of the algorithm.

(3.10.11) the values of y_i for $i \in V$ belonging to F^1 or contained in a pseudonode of F^1 will be congruent modulo 1.

If the initial dual solution is integer valued, (3.10.8), (3.10.9) and (3.10.11) are obviously satisfied. Now observe that at no point of the algorithm do we add a new tree to F^1. Moreover at any time we grow a tree in F^1, all new edges j must belong to the equality subgraph so since c_j is integer valued for all such j, (3.10.8) and (3.10.9) ensure that (3.10.11) will continue to hold.

When computing ϵ so as to make a change of dual variables, (3.10.8), (3.10.9) and (3.10.11) ensure that any of ϵ_1, ϵ_2, ϵ_3, ϵ_4 which are finite will be congruent with $0 \pmod{1/2}$. Since $V^= = \phi$, we cannot obtain a Hungarian forest so ϵ is finite and congruent with $0 \pmod{1/2}$. Hence y' as defined in Step 9b also satisfies (3.10.8), (3.10.9) and (3.10.11). Thus (3.10.10) is proved and the theorem follows.□

The following is obtained by combining (3.10.1) and (3.10.7).

(3.10.12) <u>Theorem</u>. <u>If c is integer valued, then the maximum value of cx for any matching x of G satisfying (3.10.2) is equal to the minimum of (3.10.3) subject to (3.10.4)-(3.10.6) and an optimal y can be chosen so as to satisfy (3.10.8), (3.10.9).</u>

In the case that c is further restricted to being 0, 1 valued, we can obtain the following result.

(3.10.13) Theorem. If $c_j \in \{0, 1\}$ for all $j \in E$ then there is an optimal feasible solution y^0 to the problem of minimizing (3.10.3) subject to (3.10.4)-(3.10.6) which satisfies

(3.10.14) $y_i, y_S \in \{0, 1\}$ for all $i \in V$, for all $S \in Q^0$.

Proof. Let $G = (V, E, \psi)$ be a graph for which (3.10.13) fails for some b and such that $|V \cup E|$ is minimum. Clearly $|V| \geq 3$, and we must have $c_j = 1$ for all $j \in E$ since the graph obtained by deleting any edge k for which $c_k = 0$ would still violate (3.10.13).
Suppose G has a perfect matching x^0. Then the maximum value of $c \cdot x$ over matchings x of G satisfying (3.10.2) is equal to $1/2b(V)$. Choose $v \in V$ and define b' by

$$b'_i = \begin{cases} b_i & \text{for } i \in V - \{v\}, \\ b_i + 1 & \text{for } i = v. \end{cases}$$

Then the maximum of $c \cdot x$ over matchings x of G satisfying (3.10.2) is still $1/2b(V)$. Suppose that y^0 is an optimal dual solution relative to b' which satisfies (3.10.14). Then $\Sigma(b'_i y^0_i: i \in V) + \Sigma(q_S y^0_S: S \in Q^0) = 1/2b(V)$.
Hence y^0 is an optimum dual solution relative to b but y^0 satisfies (3.10.14), a contradiction. Hence no optimum

solution relative to b' can satisfy (3.10.14) and since G can have no perfect b'-matching, we assume

(3.10.15) b is chosen so that G has no perfect b-matching.

Let y^0 be an optimum solution relative to b satisfying (3.10.8), (3.10.9). Clearly we have $y_i^0 \in \{0, 1/2, 1\}$ for all $i \in V$ and $y_S^0 \in \{0, 1\}$ for all $S \in Q^0$. Let $W \equiv \{i \in V: y_i^0 = 1/2\}$. If $W = \phi$ then y^0 satisfies (3.10.14) and we are finished. If $W = V$ then $\Sigma\{b_i y_i^0: i \in V\} + \Sigma\{q_S y_S^0: S \in Q^0\} \geq 1/2b(V)$ implying G has a perfect matching, contradictory to (3.10.15). Thus we have

(3.10.16) $\phi \neq W \subset V$.

(3.10.17) For any $j \in \delta(W)$ we must have either $y_v^0 = 1$ where $\{v\} = \psi(j) - W$ or $j \in \gamma(S)$ for some $S \in Q^0$ such that $y_S^0 = 1$.

Otherwise we could have $y^0(\psi(j)) + y^0(Q^0(j)) = 1/2$ contradictory to (3.10.6).

By our minimality assumption for G and (3.10.16) there is an optimal solution y^1 satisfying (3.10.14) to the problem

$$\text{minimize } \Sigma(b_i y_i: i \in W) + \Sigma(q_S y_S: S \in Q_W^0)$$

subject to

$$y_i \geq 0 \text{ for all } i \in W$$

$$y_S \geq 0 \text{ for all } S \in Q_W^0$$

$$y(\psi(j)) + y(Q_W^0(j)) \geq 1 \text{ for all } j \in E(G[W])$$

where $Q_W^0 \equiv \{S \in Q^0: S \subseteq W\}$ and $Q_W^0(j) \equiv \{S \in Q_W^0: j \in \gamma(S)\}$

for all $j \in J$. If we define y^* by

$$y_i^* \equiv \begin{cases} y_i^0 & \text{for } i \in V - W, \\ y_i^1 & \text{for } i \in W; \end{cases}$$

$$y_S^* \equiv \begin{cases} y_S^0 & \text{for } S \in Q^0 - Q_W^0, \\ y_S^1 & \text{for } S \in Q_W^0 \end{cases}$$

then y^* satisfies (3.10.14) and by (3.10.17), y^* is a feasible solution to the problem of minimizing (3.10.3) subject to (3.10.4)-(3.10.6). Since

$$\Sigma(b_i y_i^1 : i \in W) + \Sigma(q_S y_S^1 : S \in Q_W^0)$$

$$\leq \Sigma(b_i y_i^0 : i \in W) + \Sigma(q_S y_S^0 : S \in Q_W^0)$$

and since y^0 was optimal it follows that y^* is optimal. This contradicts the choice of G and completes the proof.□

Combining (3.10.13) and (3.10.1), we obtain the following.

(3.10.18) Theorem: If $c_j \in \{0, 1\}$ then the maximum value of cx for any matching x of G satisfying (3.10.2) is equal to the minimum of (3.10.3) subject to (3.10.4)-(3.10.6) and a minimum y can be chosen so as to satisfy (3.10.14).

Theorem (3.10.18) can be specialized in the following manner. Let $G = (V, E, \psi)$ be a graph and let $b = (b_i : i \in V)$ be a vector of positive integers. For any $X \subseteq V$ we define

(3.10.19) $C(X) \equiv \{S \subseteq V - X : G[S]$ is a component of $G[V - X]\}$.

We partition $C(X)$ as follows.

(3.10.20) $C_0(X) \equiv \{S \in C(X): |S| = 1\}$,

(3.10.21) $C_1(X) \equiv \{S \in C(X): |S| > 1$ and $b(S)$ is odd$\}$

(3.10.22) $C_2(X) \equiv \{S \in C(X): |S| > 1$ and $b(S)$ is even$\}$.

(3.10.23) Theorem. Max$\{x(E)$: x is a matching of G satisfying (3.10.2)$\} = 1/2(b(V) + \min\{b(X) - |C_1(X)| - b(\cup(C_0(X))): X \subset V\}$). Moreover

(3.10.23a) there is a set $X^* \subseteq V$ which minimizes $b(X) - |C_1(X)| - b(\cup(C_0(X)))$ over $X \subseteq V$ and satisfies $C_2(X^*) = \phi$ and $C_1(X^*) \subseteq Q^0$.

Proof. Let x be any matching of G which satisfies (3.10.2). Let $X \subseteq V$. Then for any $S \in C_1(X)$ we have $x(\gamma(S)) \leq 1/2(b(S) - 1)$ (by (3.1.7)). Therefore

(3.10.24) $b(\cup(C_1(X))) - |C_1(X)| \geq 2\Sigma\{x(\gamma(S)): S \in C_1(X)\}$.

Let $J \equiv \delta(X) \cap (\delta(\cup(C_0(X) \cup C_1(X)))$. Then we have

(3.10.25) $b(X) + b(\cup(C_2(X))) \geq 2x(\gamma(X \cup \cup(C_2(X)))) + x(J)$.

We also have

(3.10.26) $b(X) \geq x(J)$.

Summing (3.10.24)-(3.10.26) we obtain

$$[b(X) + b(\cup(C_1(X))) + b(\cup(C_2(X)))] + b(X) - |C_1(X)| \geq 2x(E)$$

or

$$b(V) - b(\cup(C_0(X))) - |C_1(X)| + b(X) \geq 2x(E)$$

Therefore

(3.10.27) $\max\{x(E): x$ is a matching of G satisfying (3.10.2)$\} \le 1/2b(V) + 1/2\min\{b(X) - |C_1(X)| - b(\cup(C_0(X))): X \subseteq V\}$.

We now show that equality holds.

By (3.10.13) there is a y satisfying (3.10.14) which minimizes (3.10.3) subject to (3.10.4)-(3.10.6) taking $c_j \equiv 1$ for all $j \in E$. Let y^0 be such a solution for which the cardinality of $Z \equiv \{S \in Q^0: y^0_S = 1\}$ is as small as possible. Suppose $S, T \in Z$ are such that $S \cap T \neq \phi$. If $b(S \cap T) \ge 2$ then if we define y' by

$$y'_i \equiv \begin{cases} y^0_i & \text{if } i \in V - (S \cup T) \\ y^0_i + 1/2 & \text{if } i \in S \cup T \end{cases}$$

$$y'_R \equiv \begin{cases} y^0_R & \text{if } R \in Q^0 - \{S, T\} \\ 0 & \text{if } R \in \{S, T\} \end{cases}$$

it is easily seen that y' is a feasible solution to (3.10.4)-(3.10.6) for which (3.10.3) attains a smaller value than for y^0, a contradiction to our choice of y^0. If $b(S \cap T) = 1$, and hence $|S \cap T| = 1$, then $S \cup T \in Q^0$ and if we define y' by

$$y'_i \equiv y_i \quad \text{for all } i \in V$$

$$y'_R \equiv \begin{cases} y_R & \text{if } R \in Q^0 - \{S, T, S \cup T\} \\ 1 & \text{if } R = S \cup T \\ 0 & \text{if } R \in \{S, T\} \end{cases}$$

then y' is a feasible solution to (3.10.4)-(3.10.6) satisfying (3.10.14) for which the value of (3.10.3) is no greater than that obtained for y^0. But $|\{R \in Q^0: y'_R = 1\}| < |Z|$, contradictory to our choice of y^0. Hence

(3.10.28) the members of Z are pairwise disjoint.

Suppose $y^0_v = 1$ for some $v \in S \in Z$. Then if we define y' by

$$y'_i \equiv \begin{cases} y^0_i + 1/2 & \text{if } i \in V - \{v\} \\ y^0_i - 1 & \text{if } i = v \end{cases}$$

$$y'_R \equiv \begin{cases} y^0_R & \text{if } R \in Q^0 - \{S\} \\ 0 & \text{if } R = S \end{cases}$$

y is a feasible solution to (3.10.4)-(3.10.6) which causes (3.10.3) to assume a smaller value than for y^0, a contradiction. Hence

(3.10.29) $y^0_i = 0$ for all $i \in S \in Z$.

Let $X \equiv \{i \in V: y^0_i = 1\}$. Because of (3.10.29), in order for y^0 to be feasible we require

$$\delta(S) \subseteq \delta(X) \quad \text{for every } S \in Z,$$

$$\delta(i) \subseteq \delta(X) \quad \text{for every } i \in V - \cup(Z)$$

such that $y^0_i = 0$. Hence $C_0(X) = \{\{i\} \in V - V(Z): y^0_i = 0\}$,

(3.10.30) $C_1(X) = Z$

(3.10.31) $C_2(X) = \phi$.

Hence

$$(3.10.32)\quad \Sigma(b_i y_i^0 : i \in V) + \Sigma(q_S y_S^0 : S \in Q^0)$$

$$= b(X) + \Sigma(1/2(b(S) - 1) : S \in C_1(X)) + 1/2b(\cup(C_2(X)))$$

$$= 1/2b(X) + 1/2b(\cup(C_1(X))) + 1/2b(\cup(C_2(X))) + 1/2b(X) - 1/2|C_1(X)|$$

$$= 1/2b(V) + 1/2b(X) - |C_1(X)| - b(\cup(C_0(X)))).$$

Since by (3.10.18) and our choice of y^0,

$$\max\{x(E) : x \text{ is a matching of } G \text{ satisfying } (3.10.2)\}$$
$$= \Sigma(b_i y_i^0 : i \in V) + \Sigma(q_S y_S^0 : S \in Q)$$

it follows from (3.10.30) that equality holds in (3.10.27). Since $Z \subseteq Q^0$, (3.10.30) and (3.10.31) imply (3.10.23a) completing the proof.□

Theorem (3.10.23) (excluding (3.10.23a)) reduces to a theorem of Berge [B2] when it is further specialized to 1-matchings. G has a perfect matching if and only if $\max\{x(E) : x$ is a matching of G satisfying (3.10.2)$\} = 1/2b(V)$. Therefore, by (3.10.23), G has a perfect matching if and only if

$$\min\{b(X) - |C_1(X)| - b(\cup(C_0(X))) : X \subseteq V\} = 0 .$$

Thus we obtain the fundamental theorem of Tutte.

(3.10.33) Theorem (Tutte [T3]). <u>$G = (V, E, \psi)$ has a perfect matching if and only if for each $X \subseteq V$,</u>

$$(3.10.34)\quad \underline{b(X) \geq |C_1(X)| + b(\cup(C_0(X)))}.$$

In the case of 1-matchings this reduces to the well known theorem

(3.10.35) Theorem (Tutte [T1]). $G = (V, E, \psi)$ has a perfect 1-matching if and only if for any $X \subseteq V$ the number of components of $G[V - X]$ having an odd number of nodes is no greater than $|X|$.

The importance of (3.10.23a) to these theorems is discussed in Section 4.4 (see Theorems (4.4.21) and (4.4.22)).

Chapter 4

Facets and Vertices of Matching Polyhedra

Throughout this chapter we consider a graph $G = (V, E, \psi)$ and we take $b = (b_i : i \in V)$ to be a vector of positive integers. Since isolated nodes, that is nodes v for which $\delta(v) = \phi$, are of little interest in matching theory we assume G has no isolated nodes. In section 3.4 we defined the matching polyhedron $P(G, b)$ and proved the theorem of Edmonds that

$$P(G, b) = \{x \in \mathbb{R}^E:$$

(4.0.1) $x_j \geq 0$ for all $j \in E$,

(4.0.2) $x(\delta(i)) \leq b_i$ for all $i \in V$,

(4.0.3) $x(\gamma(S)) \leq q_S$ for all $S \in Q^0\}$,

where $Q^0 \equiv \{S \subseteq V: G[S]$ is shrinkable$\}$, and $q_S \equiv (1/2)(b(S)-1)$ for any set S such that $b(S)$ is odd. We now characterize the facets and vertices of $P(G, b)$ relating them to the structure of G and the value of b. In particular, for any G and b we prescribe a unique minimal subset of the inequalities (4.0.1)-(4.0.3) of which $P(G, b)$ is the solution set.

The material presented in this chapter does rely to an extent upon the material of Chapter 3. Sections 3.3 and 3.4 are used in characterizing the facets of $P(G, b)$, some of the material of Sections 3.6 and 3.7 is used in showing the equivalence of shrinkable graphs and b-critical graphs. The proof of the vertex characterization is related to the algorithm itself; in proving the theorem we also show that every matching obtained by the blossom algorithm is a vertex of $P(G, b)$. However we give an additional proof of this portion of the vertex characterization which is developed from basic properties of graph theory and polyhedra theory.

4.1. Dimension of $P(G, b)$ and Nonnegativity Facets

In order to characterize the facets of $P(G, b)$, we first determine its dimension.

(4.1.1) Proposition. $P(G, b)$ is of full dimension.

Proof. Since $P(G, b) \subseteq \mathbb{R}^E$ it follows that $\dim(P(G, b)) \le |E|$. We show that $\dim(P(G, b)) = |E|$ by exhibiting $|E| + 1$ affinely independent matchings belonging to $P(G, b)$. The result will then follow from (2.2.12).

For each $j \in E$ we define a matching x^j by

$$x_k^j = \begin{cases} 0 & \text{if } k \neq j, \\ 1 & \text{if } k = j. \end{cases}$$

Since $b_i \ge 1$ for all $i \in V$, we have $x^j(\delta(i)) \le b_i$ for all $i \in V$, for all $j \in E$. Let 0 be the zero vector in $\mathbb{R}^E$. Then $\{x^j: j \in E\} \cup \{0\} \subseteq P(G, b)$. The set of vectors $\{x^j: j \in E\} \cup \{0\}$ is easily seen to be affinely independent and the result follows.□

Let $a \in \mathbb{R}^E$, $\alpha \in \mathbb{R}$. We say that the linear inequality $ax \le \alpha$ gives a facet of $P(G, b)$ if $\{x \in P(G, b): ax = \alpha\}$ is a facet of $P(G, b)$. In characterizing which of the inequalities (4.0.1)-(4.0.3) give facets of $P(G, b)$ we use mainly the technique of showing that $ax \leq \alpha$ gives a facet of $P(G, b)$ by displaying $|E|$ affinely independent members x of $P(G, b)$ which satisfy $ax = \alpha$ and then appealing to (4.1.1) and (2.2.15).

(4.1.2) Theorem. <u>For every $j \in E$, $x_j \geq 0$ gives a facet of $P(G, b)$</u>.

Proof. For any $j \in E$ let P_j be the solution set of (4.0.2), (4.0.3) and

$$x_k \geq 0 \quad \text{for all} \quad k \in E - \{j\}.$$

We define x^j by

$$x_k^j \equiv \begin{matrix} 0 & \text{for} & k \neq j, \\ -1 & \text{for} & k = j. \end{matrix}$$

Then for each $j \in E$, $x^j \in P^j - P(G, b)$. Therefore by (2.3.30), $x_j \geq 0$ gives a facet of $P(G, b)$.□

The techniques used in this proof, showing that an inequality gives a facet by showing that if it is omitted we obtain a larger polyhedron, could possibly be used in proving the other facet characterizations of this chapter (theorems (4.2.1) and (4.3.49)). However we find it easier to show that $ax \leq \alpha$ gives a facet of $P(G, b)$ by exhibiting $|E|$ affinely independent members of $P(G, b)$. Theorem (4.1.2) is also easily proved by exhibiting $|E|$ affinely independent matchings of G, each such x satisfying $x_j = 0$. (Take the matchings 0, x^k: $k \in E - \{j\}$ defined in the proof of (4.1.1).)

We call $\{x \in P(G, b): x_j = 0\}$ a <u>nonnegativity facet</u> of P(G,b) for any $j \in E$.

4.2 Degree Constraint Facets.

In this section we characterize which of the inequalities $x(\delta(i)) \le b_i$ for $i \in V$ are facets of $P(G, b)$. For each $i \in V$ we let $N(i)$ be the set of nodes of G adjacent to i. Let v and w be nodes of G such that $N(v) = \{w\}$, $N(w) = \{v\}$ and $b_w = b_v$. Then $\{v, w\}$ is the node set of a component H of G containing at least one edge. We call H a <u>balanced edge</u>.

(4.2.1) <u>Theorem</u>. <u>For any $i \in V$, $x(\delta(i)) \le b_i$ gives a facet of $P(G, b)$ if and only if</u>

(4.2.2) <u>i is a node of a balanced edge</u>

or

(4.2.3) <u>$b(N(i)) > b_i$ and if $b(N(i)) = b_i + 1$ then $\gamma(N(i)) = \phi$.</u>

<u>Proof</u>. We first show the necessity of (4.2.2) and (4.2.3). Let i be a node violating (4.2.2) and (4.2.3). We will show that there are inequalities (4.0.1)-(4.0.3) which imply

$$(4.2.4) \qquad x(\delta(i)) \le b_i$$

and which are distinct from (4.2.4). Thus we can remove all copies of (4.2.4) from (4.0.1)-(4.0.3) without changing the solution set and the result follows from (2.3.30) and (4.1.1).

Suppose $b(N(i)) \leq b_i$. Summing the inequalities (4.0.2) for $v \in N(i)$ we obtain

$$x(\bigcup_{v \in N(i)} \delta(v)) \leq b(N(i))$$

and since $\delta(i) \subseteq \bigcup_{v \in N(i)} \delta(v)$, it follows that (4.0.1) implies

$$x(\delta(i)) \leq b(N(i)) \leq b_i.$$

Moreover if there were $v \in N(i)$ such that $x(\delta(v)) \leq b_v$ and (4.2.4) were the same inequality then $\delta(v) = \delta(i)$ and $b_v = b_i$ so since we do not allow isolated nodes we would have i and v being the nodes of a balanced edge, contradictory to i violating (4.2.2). Hence (4.2.4) is not a facet of $P(G, b)$.

Suppose $b(N(i)) = b_i + 1$ and there is some $j \in \gamma(N(i))$. Let $v \in \psi(j)$ and for each $u \in N(i)$ let $k(u)$ be an edge of G such that $\psi(k(u)) = \{i, u\}$. Let $J \equiv \{k(u): u \in N(i)\}$ and let the graph B be defined to be $(N(i) \cup \{i\}, J \cup \{j\}, \psi|J \cup \{j\})$. We show that B is a blossom. Clearly B is connected, has no even polygons and exactly one odd polygon. Moreover if we define a matching $\bar{x}$ of B by

$$\bar{x}_{k(u)} \equiv b_u \quad \text{for all } u \in N(i) - \{v\},$$

$$\bar{x}_{k(v)} \equiv b_v - 1$$

$$\bar{x}_j \equiv 0$$

we see that $\bar{x}$ is a matching of B satisfying (3.3.1)-(3.3.5) so that B is a blossom. Hence $G[N(i) \cup \{i\}]$ is shrinkable so $N(i) \cup \{i\} \in Q^0$. The inequality (4.0.3) for $N(i) \cup \{i\}$ is

$$(4.2.5) \quad x(\gamma(N(i) \cup \{i\})) \leq \frac{b(N(i)) + b(i) - 1}{2} = b_i$$

so since $\delta(i) \subseteq \gamma(N(i) \cup \{i\}) - \{j\}$, we see that (4.2.5) and (4.0.1) imply (4.2.4). Moreover (4.2.5) is different from (4.2.4). Hence (4.2.4) is not a facet of $P(G, b)$.

Now we prove the sufficiency of (4.2.2) or (4.2.3). Suppose that i is a node of a balanced edge H. For each $h \in \delta(i)$ we define a matching x^h by

$$x^h_k \equiv \begin{cases} b_i & \text{if } k = h, \\ 0 & \text{if } k \in E - \{h\}. \end{cases}$$

Let $j \in \delta(i)$. For each $h \in E - \delta(i)$ we define a matching x^h by

$$x^h_k \equiv \begin{cases} 1 & \text{if } k = h, \\ b_i & \text{if } k = j, \\ 0 & \text{if } k \in E - \{h, j\}. \end{cases}$$

Clearly the set $\{x^h: h \in E\}$ is linearly independent and $x^h(\delta(i)) = b_i$ for all $h \in E$. Since $\{x \in P(G, b): x(\delta(i)) = b_i\}$ is a proper face of $P(G, b)$ it follows from (2.2.15) that (4.2.4) gives a facet of $P(G, b)$.

Now suppose (4.2.3) is satisfied for $i \in V$. Let K be a minimal subset of $N(i)$ for which $b(K) > b_i$. For each $v \in N(i)$ let $j(v)$ be an edge joining i and v, let $E_K \equiv \{j(v): v \in K\}$. For every $j \in E_K$, let $\{v(j)\} \equiv \psi(j) \cap K$. Let $\bar{b} = (\bar{b}_j: j \in E_K)$ be defined by $\bar{b}_j \equiv b_{v(j)}$ for all $j \in E_K$. For each $k \in E_K$ we define $d^k = (d^k_j: j \in E_K)$ by

$$d_j^k \equiv \begin{cases} 0 & \text{if } j \in E_K - \{k\} \\ b(K) - b_i & \text{if } j = k. \end{cases}$$

Then $0 < d^k \leq \bar{b}$ for all $k \in E_K$ by our choice of K. For each $k \in E_K$, let $\bar{x}^k \equiv \bar{b} - d^k$. Since $\{d^k: k \in E_K\}$ is linearly independent, (2.2.4) implies $\{\bar{x}^k: k \in E_K\}$ is affinely independent. Each vector $\bar{x}^k$ can be extended to a matching x^k of G by letting $x_j^k \equiv 0$ for all $j \in E - E_K$. Then

(4.2.6) $\{x^k: k \in E_K\}$ is affinely independent.

Moreover,

(4.2.7) $x^k(\delta(v)) \leq b_v$ for all $v \in V - \{i\}$,

(4.2.8) $x^k(\delta(i)) = b_i$

so $x^k \in P(G, b)$ for each $k \in E_K$.

For each $j \in \delta(i) - E_K$ we define a matching x^j as follows. Let $\{v\} \equiv \psi(j) - \{i\}$. If $v \in K$ then let $k \in E_K$ be chosen such that $x_{j(v)}^k > 0$ and let x^j be defined by

$$x_\ell^j \equiv \begin{cases} x_\ell^k & \text{if } \ell \in E - \{j, j(v)\} \\ 0 & \text{if } \ell = j(v) \\ x_{j(v)}^k & \text{if } \ell = j. \end{cases}$$

If $v \notin K$ let k be any member of E_K and let $h \in \delta(i)$ be such that $x_h^k > 0$. Let x^j be defined by

$$x_\ell^j \equiv \begin{cases} x_\ell^k & \text{if } \ell \in E - \{j, h\}, \\ x_h^k - 1 & \text{if } \ell = h, \\ 1 & \text{if } \ell = j. \end{cases}$$

In either case, x^j is easily seen to satisfy (4.2.7) and (4.2.8) for every $j \in \delta(i) - K$. Since for each $j \in \delta(i) - E_K$ x^j is the unique matching x so far defined for which $x_j \neq 0$, (4.2.6) implies

(4.2.9) $\{x^j : j \in \delta(i)\}$ is affinely independent.

Finally, for each $j \in E - \delta(i)$ we define a matching x^j as follows.

$x_j^j \equiv 1$

$x_h^j \equiv 0$ for $h \in E - (\delta(i) \cup \{j\})$

x_h^j is defined for $h \in \delta(i)$ to be sufficiently large that (4.2.7), (4.2.8) are satisfied. This is possible for if $b(N(i)) = b_i + 1$ then by (4.2.3) at most one end of j is in $N(i)$. Therefore defining $x_j^j = 1$ restricts x_k^j to taking on a value one less than $b_{v(k)}$ for at most one edge $k \in \delta(i)$. Hence x^j can be defined as asserted. If $b(N(i)) \geq b_i + 2$ then it is easily seen that after defining $x_j^j = 1$ we can still assign values x_k^j for $k \in \delta(i)$ as required.

For any $j \in E - \delta(i)$, x^j is the only matching x defined for which $x_j \neq 0$. This together with (4.2.9) implies that $\{x^j : j \in E\}$ is affinely independent. Thus we have defined $|E|$ affinely independent members of $P(G, b)$ each of which satisfies (4.2.8). Moreover $F \equiv \{x \in P(G, b): x(\delta(i)) = b_i\}$ is a proper face of $P(G, b)$ since $0 \in P(G, b) - F$. Therefore by (2.2.15) it follows that (4.2.4) gives a facet of $P(G, b)$ completing the proof.□

We call $\{x \in P(G, b): x(\delta(i)) = b_i\}$ a <u>nonnegativity facet</u> for each $i \in V$ satisfying (4.2.2) or (4.2.3).

In the case of 1-matchings, (4.2.1) specializes to the following

(4.2.10) <u>Theorem</u>. <u>$x(\delta(i)) \le b_i$ gives a facet of $P(G, 1)$ if and only if</u>

(4.2.11) <u>i is a node of a balanced edge</u>

<u>or</u>

(4.2.12) <u>$|N(i)| > 1$ and if $|N(i)| = 2$ then $\gamma(N(i)) = \phi$</u>.

4.3. <u>Blossom Facets</u>.

In this section we give a first characterization of the inequalities $x(\gamma(S)) \le q_S$ for $S \in Q^0$ which are facets of $P(G, b)$. In fact for each $S \in Q^0$ we give the dimension of

$$F_S \equiv \{x \in P(G, b): x(\gamma(S)) = q_S\}.$$

These results are obtained by studying shrinkable graphs (as defined in Section 3.3).

In Section 4.4 we give two characterizations of shrinkable graphs and hence two more characterizations of the facets of this sort.

Recall that a np matching (near perfect) matching of G deficient at $v \in V$ is a matching x of G which satisfies

$$x(\delta(v)) = b_v - 1.$$

$$x(\delta(i)) = b_i \quad \text{for all} \quad i \in V - \{v\}.$$

The following lemma is useful when proving the independence of matchings.

(4.3.1) <u>Lemma</u>. <u>Let X be a set of np matchings of G and let $x^0 \in X$. If there exist $J(x^0) \subseteq E$ and $d(x^0) \in \mathbb{R}$ such that $x^0(J(x^0)) < d(x^0)$ but $x(J(x^0)) = d(x^0)$ for all $x \in X - \{x^0\}$ then x^0 is not a linear combination of $X - \{x^0\}$.</u>

<u>Proof</u>. Suppose that there are $\alpha_x \in \mathbb{R}$ for $x \in X' \equiv X - \{x^0\}$ such that

$$(4.3.2) \quad x^0 = \Sigma(\alpha_x x\colon x \in X').$$

By (3.3.24), $x(E) = 1/2(b(V) - 1)$ for all $x \in X$. Therefore by (4.3.2)

$$x^0(E) = \Sigma(\alpha_x x(E)\colon x \in X')$$

and hence

$$(4.3.3) \qquad \Sigma(\alpha_x\colon x \in X') = 1.$$

Therefore $\Sigma(\alpha_x x(J(x^0))\colon x \in X') = \Sigma(\alpha_x d(x^0)\colon x \in X') = d(x^0)$ by (4.3.3). Hence (4.3.2) implies that $x^0(J(x^0)) = d(x^0)$, a contradiction which proves the lemma.□

(4.3.4) We call $v \in V$ a <u>strong cut node</u> of $G = (V, E, \psi)$ relative to b if v is a cutnode of G (see (1.3.9)) and $b_v = 1$. A <u>weak block</u> of G relative to b is a maximal connected subgraph H of G such that $b_v > 1$ for any cutnode v of H. Thus a weak block consists of one or more blocks of G joined by cutnodes v for which $b_v > 1$. Notice that

(4.3.5) the edge sets of the weak blocks of G partition the edges of G.

(4.3.6) We let $\beta(G)$ denote the number of weak blocks of G.

In the case of 1-matchings, strong cutnodes and weak blocks are simply cutnodes and blocks respectively.

(4.3.7) <u>Proposition</u>. <u>G is shrinkable if and only if G is connected and every weak block of G is shrinkable</u>.

<u>Proof</u>. First suppose that G is connected and each weak block of G is shrinkable. We prove that G is shrinkable by induction on $\beta(G)$. If $\beta(G) = 1$ then the result is trivial. Suppose $\beta(G) > 1$ and assume the result is true for graphs having fewer than $\beta(G)$ weak blocks. Let D be a weak block of G, let R_D be a shrinking family for D. Each weak block of $G' \equiv G \times V(D)$ is isomorphic to a weak block of G and so is shrinkable. Moreover G' is connected. Since G is connected, $\beta(G') = \beta(G) - 1$

so by our induction hypothesis G' is shrinkable; let R' be a shrinking family of G'. For each $S \in R'$ we define a set $\zeta(S) \subseteq V$ as follows.

$$\zeta(S) \equiv \begin{cases} S & \text{if } V(D) \notin S, \\ S - \{V(D)\} \cup V(D) & \text{if } V(D) \in S \end{cases}$$

Let $R \equiv \{\zeta(S): S \in R'\} \cup R_D$. Then R is easily seen to be a shrinking family of G. The sufficiency now follows by induction.

Conversely, suppose that G is shrinkable. Let R be a shrinking family of G. Trivially G is connected. We prove that every weak block of G is shrinkable by induction on $|R|$. If $|R| = 0$, then G consists of a single node v and the result is trivial. Suppose that $|R| \geq 1$ and that the result is true for graphs having shrinking families of fewer than $|R|$ sets. Let S be a minimal member of R. By (3.3.16) $G[S]$ is spanned by a blossom B. By (3.3.9) only terminal nodes of B can be strong cutnodes so B is a subgraph of some weak block D of G. Let $G' \equiv G \times S$. For any $T \in R - \{S\}$ define $\zeta(T) \equiv T$ if $S \cap T = \phi$, define $\zeta(T) \equiv T - S \cup \{S\}$ if $S \subset T$ and let $R' = \{\zeta(T): T \in R - \{S\}\}$. R' is a shrinking family of G' and $|R'| = |R| - 1$ so by our induction hypothesis every weak block of G' is shrinkable. Hence every weak block of G different from D is shrinkable. Moreover every weak block of $D \times S$ is shrinkable so as we have already seen, $D \times S$ is shrinkable. Let R_D' be a shrinking family of $D \times S$ and for any $T \in R_0'$ let $\theta(T) \equiv T$ if $S \notin T$, let $\theta(T) \equiv T - \{S\} \cup S$ if $S \in T$.

Then $\{S\} \cup \{\theta(T)\colon T \in R_D'\}$ is a shrinking family of D and the proof now follows by induction.□

(4.3.8) Proposition. If Z is the set of weak blocks of a connected graph $G = (V, E, \psi)$ then

$$(4.3.9)\quad b(V) - 1 = \Sigma(b(V(D)) - 1\colon D \in Z).$$

Proof. We prove by induction on $|Z|$. If $|Z| = 1$ the result is trivial. Suppose $|Z| > 1$ and (4.3.9) holds for all graphs having fewer than $|Z|$ weak blocks. If every weak block of G contained two or more strong cutnodes then it is easily seen that G would contain a polygon having edges in more than one block, contrary to (1.3.10). Let B be a weak block of G containing exactly one strong cutnode v. Let $G' = G[V - (V(B) - \{v\})]$. Then G' is connected and $Z - \{B\}$ is the set of weak blocks of G'. Therefore by induction

$$b(V(G')) - 1 = \Sigma(b(V(D)) - 1\colon D \in Z - \{B\}).$$

Since $b(V) = b(V(G')) + b(V(B) - \{v\}) = b(V(G')) + b(V(B)) - 1$, (4.3.9) holds and the result follows by induction.□

(4.3.10) Proposition. Let $G = (V, E, \psi)$ be a shrinkable graph and suppose $x \in P(G, b)$ satisfies $x(E) = 1/2(b(V) - 1)$. Then for any weak block D of G, $x(E(D)) = 1/2(b(V(D)) - 1)$. (Note that x need not be integer valued.)

Proof. Let Z be the set of weak blocks of G. By (4.3.7) each $D \in Z$ is shrinkable so since $x \in P(G, b)$, x satisfies

$$(4.3.11) \quad x(E(D)) \le 1/2(b(V(D)) - 1) \quad \text{for all } D \in Z.$$

Therefore, summing for all $D \in Z$ we obtain

$$(4.3.12) \quad x(\bigcup_{D \in Z} E(D)) \le 1/2 \sum_{D \in Z} (b(V(D)) - 1).$$

By (4.3.5) $E = \bigcup_{D \in Z} E(D)$ so using (4.3.8) we obtain

$$(4.3.13) \quad x(E) \le 1/2(b(V) - 1).$$

But by hypothesis equality holds in (4.3.13) so equality must hold in (4.3.12) and (4.3.11) which proves the result.□

(4.3.14) Corollary. If x is a np matching of a shrinkable graph G then for any weak block D of G, $x|E(D)$ is a np matching of D.

Proof. The result follows from combining (4.3.10) and (3.3.24).□

Now we prove a main result used in characterizing the facets of $P(G, b)$ given by constraints (4.0.3).

(4.3.15) Theorem. If $G = (V, E, \psi)$ is shrinkable then G has $|E| - (\beta(G) - 1)$ linearly independent np matchings.

Proof. Let R be a shrinking family of G; we prove by induction on $|R|$. If $|R| = 0$ then G is degenerate, $|E| = 0$, $\beta(G) = 1$ and the result is trivial. Suppose $|R| \ge 1$ and the theorem holds for graphs having a shrinking family consisting of fewer than $|R|$ sets.

Let B be a blossom spanning $G \times R[V]$ which exists by (3.3.16). We partition $V(B)$ into $V_1 \cup V_2$ where $V_1 \equiv V(B) \cap V$ and $V_2 \equiv V(B) \cap R$. That is, V_1 is the set of real nodes of B and V_2 is the set of pseudonodes of B.

Let $C \equiv E(G \times R[V]) - E(B)$ and let G' be the graph obtained from G by deleting all the edges in C. Then R is a shrinking family of G' so by (3.3.21) for each $v \in V_1$ there is a np matching $\bar{x}^v$ of G' deficient at v and which satisfies

$$(4.3.16) \quad \bar{x}^v(\gamma(S)) = 1/2(b(S) - 1) \text{ for all } S \in R.$$

For each $v \in V_1$ we define a np matching x^v of G deficient at v by

$$(4.3.17) \quad x_j^v \equiv \begin{cases} \bar{x}_j^v & \text{for } j \in E' \\ 0 & \text{for } j \in C. \end{cases}$$

Let $X_1 = \{x^v : v \in V_1\}$. Since by (4.3.16) each $x \in X$ is a np matching of $G[S]$ for each $S \in V_2$, it follows from (4.3.14) that

(4.3.18) $x|E(D)$ is a np matching of D for every weak block D of $G[S]$ for every $S \in V_2$, for every $x \in X_1$.

For each $S \in V_2$ there are by induction $n(S) \equiv |\gamma(S)| - (\beta(G[S]) - 1)$ linearly independent np matchings $\{\bar{x}^{S,1}, \bar{x}^{S,2}, \ldots, \bar{x}^{S,n(S)}\}$ of $G[S]$ since $R[S] \cup \{S\}$

is a shrinking family of $G[S]$ and $|R[S] \cup \{S\}| \leq |R - \{V\}| < |R|$. By (4.3.14),

(4.3.19) $\bar{x}^{S,i}|E(D)$ is a np matching of D for every weak block D of $G[S]$ for every $i \in \{1, 2, \ldots, n(S)\}$.

We extend each to a np matching of G as follows. Let $\tilde{x}^S$ be the np matching of G deficient at S which exists by (3.3.12). For each $T \in V_2 - \{S\}$ let $j(T)$ be the edge of $\delta(T) \cap E(B)$ such that $\tilde{x}^S_{j(T)} = 1$, let $\{v(T)\} \equiv \psi(j(T)) \cap T$ and let $\bar{x}^{T,S}$ be a np matching of $G[T]$ deficient at $v(T)$. By (4.3.14),

(4.3.20) $\bar{x}^{T,S}|E(D)$ is a np matching of D for every weak block D of $G[T]$.

Now we define $x^{S,i}$ for all $i \in \{1, 2, \ldots, n(S)\}$ by

$$(4.3.21)\quad x^{S,i}_j \equiv \begin{cases} \bar{x}^{S,i}_j & \text{for } j \in \gamma(S), \\ \tilde{x}^S_j & \text{for } j \in E(B), \\ 0 & \text{for } j \in C, \\ \bar{x}^{T,S}_j & \text{for } j \in \gamma(T), \text{ for } T \in V_2 - \{S\}. \end{cases}$$

Let $X_2 = \{x^{S,i}: i \in \{1, 2, \ldots, n(S)\}, S \in V_2\}$. By (4.3.19) and (4.3.20),

(4.3.22) $x|E(D)$ is a np matching of D for every weak block D of $G[T]$ for every $T \in V_2$ for every $x \in X_2$.

Now we show

(4.3.23) $X_1 \cup X_2$ is linearly independent.

Suppose that $\alpha_v \in \mathbb{R} : v \in V_1$ and $\alpha_{S,i} \in \mathbb{R}$: $i \in \{1, 2,\ldots,n(S)\}$, $S \in V_2$ are such that

$$(4.3.24)\quad \Sigma(\alpha_v x^v : v \in V_1) + \Sigma(\alpha_{S,i} x^{S,i} : i \in \{1, 2,\ldots,n(S)\},\ S \in V_2) = 0.$$

If we let $\tilde{x}^v \equiv x^v|E(B)$ for each $v \in V_1$ we have

$$\Sigma(\alpha_v \tilde{x}^v : v \in V_1) + \Sigma(\bar{\alpha}_S \tilde{x}^S : S \in V_2) = 0$$

where

$$\bar{\alpha}_S \equiv \Sigma(\alpha_{S,i} : i \in \{1, 2,\ldots,n(S)\}) \text{ for } S \in V_2.$$

For each $v \in V(B)$, $\tilde{x}^v$ is a np matching of B deficient at v so if we let $J(\tilde{x}^v) \equiv \delta(v) \cap E(B)$ and $d(\tilde{x}^v) \equiv b_v$ for all $v \in V(B)$ then by (4.3.1), $\{\tilde{x}^v : v \in V(B)\}$ is linearly independent so

$$(4.3.25)\quad \alpha_v = 0 \text{ for all } v \in V_1,$$

$$(4.3.26)\quad \bar{\alpha}_S = 0 \text{ for all } S \in V_2.$$

Now let $S \in V_2$, let $V_2' \equiv V_2 - \{S\}$. By (4.3.21), (4.3.24) and (4.3.25) we have

$$\Sigma(\alpha_{S,i} \bar{x}^{S,i} : i \in \{1, 2,\ldots,n(S)\}) + \Sigma(\bar{\alpha}_T \bar{x}^{S,T} : T \in V_2') = 0$$

so by (4.3.26),

$$\Sigma(\alpha_{S,i}\bar{x}^{S,i} : i \in \{1, 2, \ldots, n(S)\}) = 0.$$

But the matchings $\{\bar{x}^{S,i} : i \in \{1, 2, \ldots, n(S)\}\}$ are by hypothesis linearly independent so

$$(4.3.27) \quad \alpha_{S,i} = 0 \quad \text{for all} \quad i \in \{1, 2, \ldots, n(S)\}.$$

This together with (4.3.25) proves (4.3.23).

Let $k \in C$. We define a np matching x^k as follows. Let v and w be the nodes of B met by k, let $\tilde{x}^v$ be the np matching of B deficient at v. There must be some edge $\ell \in E(B) \cap \delta(w)$ such that $\tilde{x}^v_\ell = 1$, we define a np matching $\hat{x}^k$ of $G \times R[V]$ by

$$(4.3.28) \quad \hat{x}^k_j \equiv \begin{cases} \tilde{x}^v_j & \text{for } j \in E(B) - \{\ell\}, \\ 0 & \text{for } j \in (C - \{k\}) \cup \{\ell\}, \\ 1 & \text{if } j = k. \end{cases}$$

Let $T \in V_2$. If $\hat{x}^k(\delta(T)) = 0$ we let $\bar{x}^T$ be any np matching of $G[T]$. If there is $\ell \in \delta(T)$ such that $\hat{x}^k_\ell = 1$ then let $\{v\} \equiv \psi(\ell) \cap T$ and let $\bar{x}^T$ be a np matching of $G[T]$ deficient at v. Now define x^k by

$$(4.3.29) \quad x^k_j \equiv \begin{cases} \hat{x}^k_j & \text{for } j \in E(G \times R[V]), \\ \bar{x}^T_j & \text{for } j \in \gamma(T) \quad \text{for } T \in V_2. \end{cases}$$

Let $X_3 \equiv \{x^k : k \in C\}$. Every $x \in X_3$ is a np matching of G and for any $S \in V_2$, $x|\gamma(S)$ is a np matching of $G[S]$. Therefore by (4.3.14), (4.3.30) $x|E(D)$ is a np matching of every weak block D of $G[S]$ for every $S \in V_2$ for every $x \in X_3$.

Moreover, by (4.3.17), (4.3.21), (4.3.28) and (4.3.29) for each $k \in C$, x^k is the unique member of $X_1 \cup X_2 \cup X_3$ such that $x_k \neq 0$, so by (4.3.23),

(4.3.31) $X_1 \cup X_2 \cup X_3$ is linearly independent.

Now let D be a weak block of $G[S]$ such that D is not a weak block of G for some $S \in V_2$. First observe that since $b_S = 1$ by (3.3.9) S must be a terminal node of B and consequently $|\delta_B(S)| \leq 2$. As before we let $G' \equiv (V, E - C, \psi|E - C)$. We distinguish two main cases.

<u>Case 1</u>. <u>D is not a weak block of G'</u>.

<u>Case 1a</u>. <u>An edge h of $\delta_B(S)$ is incident with a node $w \in V(D)$ for which $b_w \geq 2$.</u> (See Figure 4.1).

Since $b_w \geq 2$, w is not a strong cutnode of $G[S]$ and so every edge of $G[S]$ incident with w is an edge of D. Let x^S be a np matching of S deficient at w. Since $b_w \geq 2$ there is some $\ell \in E(D) \cap \delta(w)$ such that $x^S_\ell = 1$. Let $\{t\} \equiv \psi(h) - \{S\}$ and let u be the node of $V(B) - \{S\}$ met by h. If $u \in V_1$, then $u = t$, if $u \in V_2$ then $t \in u$. Let $\tilde{x}$ be the np matching of B deficient at u.

(4.3.32) For each $T \in V_2 - \{u\}$ let $j(T)$ be the unique edge j of $\delta_B(T)$ such that $\tilde{x}_j = 1$ and let x^T be a np matching of $G[T]$ deficient at $v(T)$, where $\{v(T)\} \equiv \psi(j(T)) \cap T$.

FIGURE 4.1

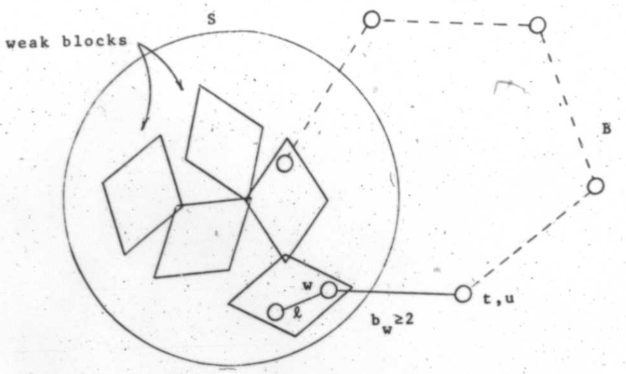
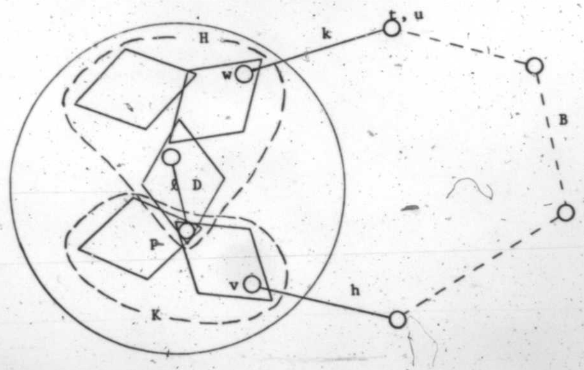


FIGURE 4.2



If $u \in V_2$ then

(4.3.33) let x^u be a np matching of $G[u]$ deficient at t.

We now define a np matching x^D of G by

$$x_j^D \equiv \begin{cases} x_j^S & \text{for } j \in \gamma(S) - \{\ell\}, \\ x_\ell^S - 1 & \text{if } j = \ell, \\ 0 & \text{for } j \in C \\ \tilde{x}_j & \text{for } j \in E(B) - \{h\}, \\ \tilde{x}_h + 1 & \text{if } j = h \\ x_j^T & \text{for } j \in \gamma(T) \text{ for } T \in V_2 - \{S\}. \end{cases}$$

It can be seen that

(4.3.34) $x^D|E(A)$ is a np matching of each weak block A of $G[T]$ for $T \in V_2$ unless $A = D$ and

(4.3.35) $x^D(E(D)) = \dfrac{b(V(D)) - 3}{2}$.

Case 1b. $b_i = 1$ for every node $i \in V(D)$ met by an edge of B. (see Figure 4.2) Then by our case 1 hypothesis there must be distinct $v, w \in S$ incident with edges h, $k \in \delta_B(S)$ respectively and every path in $G[S]$ from v to w must contain an edge of D. Since D is a weak block of $G[S]$ there is a unique node $p \in V(D)$ which is the first node of D in any such path. If $p \neq v$ then p is a strong cutnode of $G[S]$ and hence is not a cutnode of D.

If $p = v$ then $b_p = 1$ by our Case 1b hypothesis so p cannot be a cutnode of D. Thus there is a component $\bar{H}$ of $G[S - \{p\}]$ such that $V(D) - \{p\} \subseteq V(\bar{H})$. Let $H \equiv G[V(\bar{H}) \cup \{p\}]$, let $K \equiv G[S - V(\bar{H})]$. ($K$ may consist of just the single node p.) Then $V(H) \cup V(K) = S$ and $V(H) \cap V(K) = \{p\}$. Clearly the weak blocks of $G[S]$ are the weak blocks of H and K so by (4.3.7), H and K are shrinkable. Moreover, $v \in V(K)$, $w \in V(H)$ and $p \neq w$. Let x^H be a np matching of H deficient at w. Since $p \neq w$, there is some $\ell \in E(D) \cap \delta(p)$ such that $x^H_\ell = 1$. Let x^K be a np matching of K deficient at v. Let $\{t\} \equiv \psi(k) - S$, let u be the node of $V(B) - \{S\}$ met by k. Let $\tilde{x}$ be the np matching of B deficient at u. Since $|\delta_B(S)| = 2$ and since S is a terminal node of B, S must belong to the odd polygon of B. Therefore h is the first edge in a path of length two from a node in the polygon to u. Therefore by (3.3.12) and (3.3.5) $\tilde{x}_h = 1$ and $\tilde{x}_k = 0$. For each $T \in V_2 - \{u\}$ define x^T as in (4.3.32) and if $u \in V_2$ then define x^u as in (4.3.33). Now define x^D by

$$x^D_j \equiv \begin{cases} x^H_j & \text{for } j \in E(H) - \{\ell\} \\ 0 & \text{for } j \in \{\ell\} \cup C, \\ x^K_j & \text{for } j \in E(K) \\ \tilde{x}_j & \text{for } j \in E(B) - \{k\} \\ 1 & \text{for } j = k \\ x^T_j & \text{for } j \in \gamma(T) \text{ for } T \in V_2 - \{S\}. \end{cases}$$

It can now be seen that x^D is a np matching of G satisfying (4.3.34) and (4.3.35).

Case 2. D is a weak block of G'. (See Figure 4.3). Let W be the set of nodes of S incident with edges of B. There must be a node $p \in V(D)$ which is the first node of D in any path in $G[S]$ from a node in W to a node in D, otherwise D would not be a weak block of G'. p is a strong cutnode of $G[S]$ unless $W = \{p\}$. Since D is not a weak block of G, there is some edge $e \in C \cap \delta(S)$ such that where $\{q\} \equiv \psi(e) \cap S$, there is a path in $G[S]$ from q to a node of B which does not contain p. Let $\bar{H}$ be the component of $G[S - \{p\}]$ which contains q, let $H \equiv G[V(\bar{H}) \cup \{p\}]$, let $K \equiv G[S - V(\bar{H})]$. (If $W = \{p\}$ then K may simply consist of p.) Let u be the node of $V(B) - \{S\}$ met by e and let $\tilde{x}$ be the np matching of B deficient at u. Let x^H be a np matching of H deficient at q. There must be $\ell \in E(D) \cap \delta(p)$ such that $x^H_\ell = 1$. Let x^K be a np matching of K deficient at the node $w \in W$ met by an edge $h \in E(B)$ for which $\tilde{x}_h = 1$. For each $T \in V_2 - \{u\}$ define x^T as in (4.3.32). If $u \in V_2$ then let $\{t\} \equiv \psi(e) - S$ and define x^u as in (4.3.33). Now define x^D as follows.

$$x^D_j \equiv \begin{cases} x^H_j & \text{for } j \in E(H) - \{\ell\}, \\ 0 & \text{for } j \in \{\ell\} \cup C - \{e\}, \\ x^K_j & \text{for } j \in E(K), \\ \tilde{x}_j & \text{for } j \in E(B), \\ 1 & \text{for } j = e, \\ x^T_j & \text{for } j \in \gamma(T) \text{ for } T \in V_2 - \{S\}. \end{cases}$$