ANOTHER LOOK AT THE DEGREE CONSTRAINED SUBGRAPH PROBLEM

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1. Introduction

There are several versions of the degree constrained subgraph problem, and we refer to the following: Given an undirected graph $G = (V, E)$ with n vertices, and 2n integers $a_1, ..., a_n, b_1, ..., b_n$, find a subgraph $G' = (V, E')$ of G such that $a_i \leq d_{G'}(v_i) \leq b_i$ for $1 \leq i \leq n$ and $|E'|$ is maximized. Here $d_{G'}(v_i)$ denotes the degree of v_i restricted to G'.

This problem has been solved by Urquhart [5], and a polynomial solution to it can also be derived from Edmonds and Johnson's work [2]. Both papers use the linear programming approach. A more combinatorial approach is presented here.

In Section 2 we solve a restricted problem in which $a_i = 0$ for all i. This problem is reduced to the regular maximum matching problem via a simple construction[1]. The same construction also yields a reduction of the weighted version of this problem to the weighted maximum matching problem. (In the weighted problem a weight w(e) is assigned to each $e \in E$ and $\Sigma_{e \in E'} w(e)$ is maximized rather than $|E'|$.)

In Section 3 an alternating path technique is used to obtain a solution to the general problem from that of the restricted problem. The corresponding weighted problem is reduced to the weighted matching problem[2].

[1] Another reduction of the restricted unweighted problem to the matching problem is given in [1, ch. 8]. The author thanks the referee for bringing it to his attention.

[2] This nice reduction was suggested by the same referee, and the author is grateful for that too.

We conclude with an application to an edge-partitioning problem that is closely related to edge-coloring problems (see [3]).

2. The restricted problem: $a_i = 0$ for all i

Given are a graph $G = (V, E)$ with n vertices and non-negative integers $b_1, ..., b_n$. We wish to find a subgraph $G' = (V, E')$ of G such that $d_{G'}(v_i) \leq b_i$, for $i = 1, ..., n$ and $|E'|$ is maximized.

Let $H = (V_H, E_H)$ be an undirected graph obtained from G in the following way:

(1) For each vertex $v_i \in V$ we put b_i copies $v_i^1, ..., v_i^{b_i}$ in V_H.

(2) For each edge $e \in E$ we add two vertices u_e, w_e to V_H.

(3) An edge $e = (v_i, v_j) \in E$ is transformed to the subgraph H(e) of H shown in Fig. 1. An undirected graph G and its corresponding graph H are shown in Figs. 2a and 2b, respectively.

Theorem 1. Let $M \subseteq E_H$ be a maximum matching in H and let G′ be a solution to our problem. We then

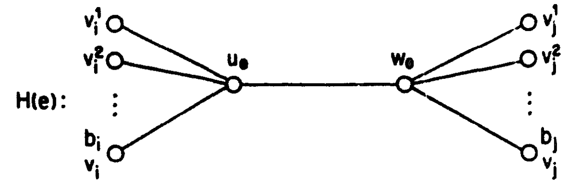

Fig. 1.

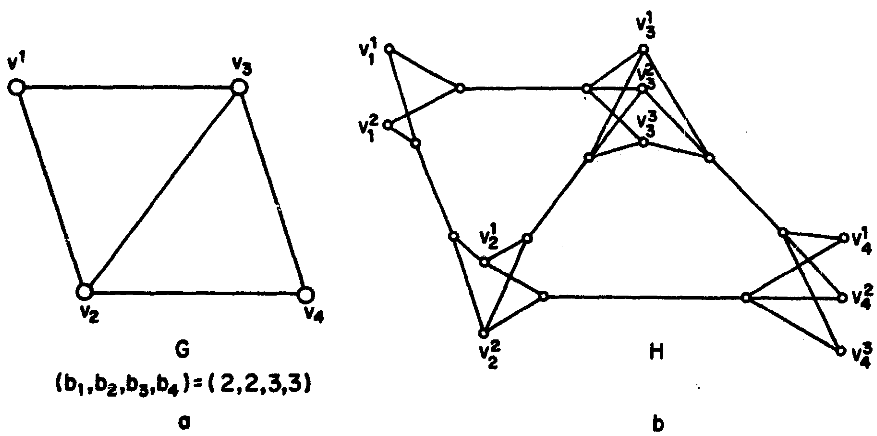

Fig. 2.

have

$$|M| = 2|E'| + |E \setminus E'| = |E| + |E'|.$$

Proof. We first show that $|M| \leqslant 2|E'| + |E \setminus E'|$. Obviously for all $e \in E$, $H(e)$ contains at least one and at most two edges of M. (It cannot contain more than two and it is not maximal if it contains none.)

Let $E'' = \{e \in E: |H(e) \cap M| = 2\}$. By the definition of E'' and the statement above:

$$|M| = 2|E''| + |E \setminus E''| = |E| + |E''|.$$

On the other hand it easily follows from the construction of H that

$$d_{G''}(v_i) \leqslant b_i \quad \text{for } i = 1, ..., n,$$

where $G'' = (V, E'')$. Thus, $|E''| \leqslant |E'|$ and this direction is proved.

Conversely, let $M' \subseteq E_H$ be a matching defined by:

(1) $(u_e, w_e) \in M'$ iff $e \notin E'$.

(2) If $e = (v_i, v_j) \in E'$, then $(v_i^k, u_e) \in M'$ and $(v_j^\ell, w_e) \in M'$ for some $1 \leqslant k \leqslant b_i$ and $1 \leqslant \ell \leqslant b_j$.

Since $d_{G'}(v_i) \leqslant b_i$ for all i, this matching can be accomplished without conflicts. Hence we have both:

$$|M'| = 2|E'| + |E \setminus E'| \quad \text{and} \quad |M| \geqslant |M'|$$

implying this direction.

It follows from Theorem 1 that if a maximum matching $M \subseteq E_H$ is found for H then E' is easily obtained by the rule:

$$e \in E' \quad \text{iff} \quad |H(e) \cap M| = 2.$$

Complexity. The maximum matching problem for H can be solved in $O(|M|^{1/2}|E_H| \log|E_H|)$ as shown in [K]. Since $|M| < 2|E|$ and $|E_H| = |E| + \Sigma_i b_i d(v_i) < 2|V||E|$, the running time for the restricted problem is $O(|V||E|^{1.5} \log|E|)$ in terms of the original graph G. This bound is very loose when G is sparse.

The weighted restricted problem. This problem is obtained from the restricted problem by assigning a weight $w(e)$ to each edge $e \in E$ and trying to maximize $w(E') = \Sigma_{e\in E'} w(e)$ rather than $|E'|$.

Let us assign weights to the edges of H by the rule:

$$w(e_H) = w(e) \quad \text{if} \quad e_H \in H(e).$$

Given a set $E' \subseteq E$, the matching M' associated with it satisfies:

$$w(M') = \sum_{e\in E'} 2w(e) + \sum_{e\in E\setminus E'} w(e) = w(E) + w(E').$$

Thus, maximizing $w(E')$ is equivalent to maximizing $w(M')$ and the restricted problem is reducible to the weighted matching problem.

3. The general problem

3.1. The unweighted general problem

Theorem 2. If $G_r = (V, E_r)$ is a solution to the restricted problem and $G_g = (V, E_g)$ is a solution to the general problem then $|E_r| = |E_g|$.

Proof. Obviously $|E_g| \leq |E_r|$. Assume that $|E_g| < |E_r|$. Let H be the graph obtained from G in the way described in the previous section. Let $M_r(M_g) \subseteq E_H$ be a matching of H obtained from $E_r(E_g)$ by the same two rules as in the proof of Theorem 1:

$$|M_g| = |E| + |E_g| < |E| + |E_r| = |M_r|.$$

Thus, M_g is not a maximum matching of H and can be augmented by the use of alternating paths. Let M' be a matching of H obtained from M_g by one alternating path between two exposed vertices of M_g. Let $E' = \{e \in E: |H(e) \cap M'| = 2\}$ and let $G' = (V, E')$. It is easy to see that $d_{G_g}(v) \leq d_{G'}(v)$ for all $v \in V$. Thus, G' is a feasible solution to the general problem contradicting the maximality of $|E_g|$ (since $|E'| = |E_g| + 1$).

Theorem 2 suggests the following approach to the general problem. First solve the restricted problem. If the solution does not satisfy the constraints of the general problem, try to modify it by alternating paths so that its cardinality is preserved and less constraints are violated. This natural idea is realized below. Let $G_r = (V, E_r)$ be a solution to the restricted problem, and let $V_A(V_B) = \{v_i \in V: d_{G_r}(v_i) < a_i (> a_i)\}$.

The deficiency of G_r ($def(G_r)$) is $\Sigma_{v_i \in V_A}(a_i - d_{G_r}(v_i))$. The following theorem guarantees that if there exists a solution to the general problem but G_r is not, we can use the alternating path technique to transform G_r into another solution G_r' of the restricted problem such that $def(G_r') = def(G_r) - 1$. A repeated application of this idea yields a solution to the general problem if one exists.

Theorem 3. Let $G_r = (V, E_r)$ be a solution to the restricted problem which is not a solution to the general problem, and let $v \in V_A$. If the general problem has a solution then there exists an alternating $(E_r/E \setminus E_r)$ path P (not necessarily simple) leaving v with an edge of $E \setminus E_r$ and terminating at a vertex of V_B with an edge of E_r.

Corollary. Let $G_r' = (V, E_r')$, where $E_r' = (E_r \setminus P) \cup (P \setminus E_r)$, then:

(a) G_r' is a solution to the restricted problem.
(b) $def(G_r') = def(G_r) - 1$.

Assuming that Theorem 3 is true, both assertions follow easily.

Proof of Theorem 3. Let $v_i \in V_A$, and let $G_g = (V, E_g)$ be a solution to the general problem. Since $d_{G_r}(v_i) < a_i \leq d_{G_g}(v_i)$, there exists an edge $e \in E_g \setminus E_r$ incident with v_i. Let P be an alternating (E_g/E_r) path (not necessarily simple) that starts at v_i and leaves it through e. Also assume that P is maximal, i.e. cannot be extended further.

Claim 1. P does not terminate with an edge of E_g.

Proof. It is easy to see that if Claim 1 does not hold then $|(E_r \setminus P) \cup (P \setminus E_r)| = |E_r| + 1$. This contradicts the maximality of E_r since $|(E_r \setminus P) \cup (P \setminus E_r)|$ is also a solution to the restricted problem.

Claim 2. If P terminates at v_j then $v_j \in V_B$.

Proof. By Claim 1 P terminates with an edge of E_r. Since P is maximal, $d_{G_r}(v_j) > d_{G_g}(v_j) \geq a_j$, which implies $v_j \in V_B$. (Note that P cannot terminate at v_i since $d_{G_r}(v_i) < d_{G_g}(v_i)$.)

The proof of Theorem 3 follows immediately from Claims 1 and 2 above.

Let $H = (V_H, E_H)$ be obtained from G as described in the beginning of Section 2 and let $M_r \subseteq E_H$ be a matching of H induced by E_r (see proof of Theorem 1). Let $v_i \in V_A$. There exists an alternating $(E_r/E \setminus E_r)$ path P between v_i and some vertex $v_j \in V_B$. Since $v_i \in V_A$, there exists k such that v_i^k ($\in V_H$) is exposed with respect to M_r. On the other hand since $v_j \in V_B$, there exists ℓ such that v_j^ℓ ($\in V_H$) is incident with an edge of M_r. It is easy to transform P into an alternating $(M_r/E_H \setminus M_r)$ path P_H in H, that starts at v_i^k (with an edge of $E_H \setminus M_r$) and ends at v_j^ℓ with an edge of M_r. The matching $M_r' = (M_r \setminus P_H) \cup (P_H \setminus M_r)$ induces a set $E_r' \subseteq E$ such that $G_r' = (V, E_r')$ is a solution to the restricted problem and $def(G_r') = def(G_r) - 1$. Thus, once G_r (and hence M_r) is given, improving it to G_r' amounts to finding an alternating $(M_r/E_H \setminus M_r)$ path in H that starts at a given exposed vertex v_i^k (such that $v_i \in V_A$) and ends at some vertex v_j^ℓ where $v_j \in V_B$. If the algorithm fails to find such a path, the problem has no feasible solution. Such a path can be found (if one exists) in $O(|E_H|)$ time (see [4]). A transformation of a solution G_r of the restricted problem to a solution of the general problem involves

$\text{def}(G_r)$ iterations of this process. Since $\text{def}(G_r) < |E|$, the time involved is $O(|E| \cdot |E_H|)$ or $O(|V||E|^2)$ in terms of the input parameters. The real time is much smaller though whenever G is sparse of $\text{def}(G_r) \ll |E|$.

3.2. The weighted general problem

This problem can be reduced to the weighted maximum matching problem. The following reduction, contributed by one of the referees, utilizes a 'middle' problem called the 'constrained' weighted matching problem.

The input to this problem, in addition to that of the common weighted matching problem, contains a distinguished set of vertices $V_1 \subseteq V$. The matchings in this problem must satisfy the additional requirement that each of the vertices of V_1 will have a mate. This additional requirement may, of course, cause that no solution will exist.

In order to reduce the weighted degree constrained problem to the constrained matching problem, we construct the same graph H as in Section 2 and let all the edges of H replacing an edge e of G have the same weight as e. The set V_1 is the defined by:

$$V_1 = \bigcup_{i=1}^{n} \{v_i^1, ..., v_i^{a_i}\},$$

and the correspondence between the solutions of the problems is defined as before. It is easy to see that the weighted degree constrained problem has a solution if and only if the corresponding constrained weighted matching problem has one. Moreover, it can be shown exactly as in the case of the restricted problem that both problems attain their maxima together.

Given a constrained matching problem, one can get rid of the constraints imposed by V_1 as follows: Add a sufficiently large weight W to the weight of each edge that is incident with one vertex of V_1 and 2W to each edge that is incident with two vertices of V_1 ($W > \Sigma_{e \in E'} w(e)$ is good enough). It is easy to see that the constrained problem has a solution if and only if the unconstrained problem associated with it has one with a total weight of at least $W \cdot |V_1|$. Moreover, the total weights of the two problems differ by exactly $W \cdot |V_1|$ and therefore attain their maxima together.

4. An application to an edge-partitioning problem

The following problem arises in the context of edge-coloring problems (see [3]).

Let $G = (V, E)$ be an undirected graph and let $\Delta(G) = \max_i d(v_i)$. Let α, β be positive integers such that $\alpha + \beta = \Delta$. The problem is to find (if one exists) a partition of E into two disjoint sets E_1 and E_2 such that $\Delta(G_1) = \alpha$ and $\Delta(G_2) = \beta$, where $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. This problem is easily reduced to our problem by setting $b_i = \beta$ and $a_i = d(v_i) - \alpha$ for all i. If $G' = (V, E')$ is a feasible solution to the degree constrained subgraph problem then $E_1 = E \setminus E'$, $E_2 = E'$ is the desired partition as one can easily verify.

Note. Degree constrained subgraph problems can as well be defined for directed graphs with upper and lower bounds on both the in-degree and out-degree of each vertex. In the directed case, however, they are easily reduced to max-flow problems which is not the case in the undirected problems.

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