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A SEMANTICAL STUDY OF CONSTRUCTIBLE FALSITY

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In [6], DAVID NELSON develops an analogue of KLEENE's notion of recursive realizability in which false number-theoretic statements, as well as true ones, are regarded as established simultaneously by constructive means. The syntactical formulation of number theory presented in NELSON's paper is similar to systems studied by FITCH (e.g., in [1]), who in papers such as [2] has also dealt with mathematical theories in which true and false statements are defined simultaneously by a constructive process.

In this paper we will consider only the first-order predicate logic, say **CF**, corresponding to this notion of constructible falsity. We will give a syntactical treatment of **CF** resembling the ones given in [7] and [6], and then go on to develop a semantical analysis of this logic. This latter treatment resembles the modelings given by KRIPKE for modal and intuitionistic logics,[2]) and unlike the accounts of NELSON and FITCH, is general. That is, our semantics holds for applications of **CF** to any subject matter, rather than just number theory or set theory. The principal result of this paper has to do with semantical completeness; in particular, in section 4, below we will show that any formula of **CF** valid with respect to our interpretation is provable.

1. Syntax

A *morphology* $\boldsymbol{M}$ for **CF** consists of an infinite set $V_{\boldsymbol{M}}$ of individual variables, a set $C_{\boldsymbol{M}}$ of individual constants, and, for each i, $i \geqq 0$, a set $P^i_{\boldsymbol{M}}$ of predicate parameters. In this paper, we will assume that all of these sets are at most denumerable. The formulas of a morphology $\boldsymbol{M}$ are defined in the usual way, being constructed from atomic formulas using the connectives $\vee$, $\supset$, and $\sim$, and the universal quantifier. Conjunction and existential quantification are defined, as follows:

$$A \wedge B =_{\mathrm{df}} \sim(\sim A \vee \sim B)$$
$$(\exists x) A =_{\mathrm{df}} \sim(x) \sim A .$$

The set $T_{\boldsymbol{M}}$ of *terms* of $\boldsymbol{M}$ is $V_{\boldsymbol{M}} \cup C_{\boldsymbol{M}}$; where s and t are terms of $\boldsymbol{M}$ and A a formula of $\boldsymbol{M}$, $A^s/_t$ is the result of replacing all free occurrences of t in A by occurrences of s, relettering bound occurrences of s in A, if necessary, to avoid rendering any of the new occurrences of s bound.

A deductive formulation of **CF** will give rise to a relation $\vdash$ of *deducibility*; where Γ and Δ are sets of formulas, $\Gamma \vdash \Delta$ holds in case for some subset $\{A_1, \ldots, A_n\}$ of Δ there is a deduction of $A_1 \vee \cdots \vee A_n$ from Γ. Rather than present such a

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[2]) See KRIPKE [3] and [4], and THOMASON [10] and [11].

formulation axiomatically, we will use certain rules for the relation $\vdash$ in our proof of semantical completeness. These rules fall into two groups: structural rules for $\vdash$, and rules for specific logical signs. For the sake of brevity we will write, e.g., "$\Gamma; A$" for "$\Gamma \cup \{A\}$" and "$\Gamma; \Delta$" for "$\Gamma \cup \Delta$" in stating these rules.

Group I.

R: If Γ and Δ are not disjoint, then $\Gamma \vdash \Delta$.

T: $\dfrac{\Gamma \vdash \Delta}{\Gamma; \Theta \vdash \Xi; \Delta}$.

C: If $\Gamma \vdash \Delta$ then for some finite subsets Γ' and Δ' of Γ and Δ, respectively, $\Gamma' \vdash \Delta'$.

Group II.

$\vee$**I**: $\dfrac{\Gamma \vdash A; B; \Delta}{\Gamma \vdash A \vee B; \Delta}$ $\qquad$ $\vee$**E**: $\dfrac{\Gamma; A \vdash \Delta \quad \Gamma; B \vdash \Delta \quad \Gamma \vdash A \vee B; \Delta}{\Gamma \vdash \Delta}$

$\supset$**I**: $\dfrac{\Gamma; A \vdash B}{\Gamma \vdash A \supset B}$ $\qquad$ $\supset$**E**: $\dfrac{\Gamma \vdash A \supset B; \Delta \quad \Gamma \vdash A; \Delta}{\Gamma \vdash B; \Delta}$

$\sim$**E**: $\dfrac{\Gamma \vdash A; \Delta \quad \Gamma \vdash \sim A; \Delta}{\Gamma \vdash \Delta}$

$\forall$**I**: $\dfrac{\Gamma \vdash A; \Delta}{\Gamma \vdash (x)\, A^{x}/_{t}; \Delta}$ $\qquad$ $\forall$**E**: $\dfrac{\Gamma \vdash (x)\, A; \Delta}{\Gamma \vdash A^{t}/_{x}; \Delta}$

where in $\forall$**I**, t has no free occurrences in A or in any member of Γ or of Δ;

$\sim\vee$**I**: $\dfrac{\Gamma \vdash \sim A; \Delta \quad \Gamma \vdash \sim B; \Delta}{\Gamma \vdash \sim(A \vee B); \Delta}$ $\qquad$ $\sim\vee$**E**: $\dfrac{\Gamma \vdash \sim(A \vee B); \Delta}{\Gamma \vdash \sim A; \Delta}$ $\quad$ $\dfrac{\Gamma \vdash \sim(A \vee B); \Delta}{\Gamma \vdash \sim B; \Delta}$

$\sim\supset$**I**: $\dfrac{\Gamma \vdash A; \Delta \quad \Gamma \vdash \sim B; \Delta}{\Gamma \vdash \sim(A \supset B); \Delta}$ $\qquad$ $\sim\supset$**E**: $\dfrac{\Gamma \vdash \sim(A \supset B); \Delta}{\Gamma \vdash A; \Delta}$ $\quad$ $\dfrac{\Gamma \vdash \sim(A \supset B); \Delta}{\Gamma \vdash \sim B; \Delta}$

$\sim\sim$**I**: $\dfrac{\Gamma \vdash A; \Delta}{\Gamma \vdash \sim\sim A; \Delta}$ $\qquad$ $\sim\sim$**E**: $\dfrac{\Gamma \vdash \sim\sim A; \Delta}{\Gamma \vdash A; \Delta}$

$\sim\forall$**I**: $\dfrac{\Gamma \vdash \sim A^{t}/_{x}; \Delta}{\Gamma \vdash \sim(x)\, A; \Delta}$ $\qquad$ $\sim\forall$**E**: $\dfrac{\Gamma \vdash \sim(x)\, A; \Delta \quad \Gamma; \sim A^{t}/_{x} \vdash \Delta}{\Gamma \vdash \Delta}$

where in $\sim\forall$**E**, t has no free occurrences in $\sim(x)\, A$, or in any member of Γ or of Δ.

2. Saturated sets

A set Γ of formulas of a morphology $\boldsymbol{M}$ is *consistent* with respect to the above rules if for some A it is not the case that $\Gamma \vdash A$, and is *$\boldsymbol{M}$-ω-complete* if for all formulas A and individual variables x of $\boldsymbol{M}$, $\Gamma \vdash (x)\, A$ if $\Gamma \vdash A^{t}/_{x}$ for all $t \in T_{\boldsymbol{M}}$. Such a set Γ is *$\boldsymbol{M}$-saturated* if it meets the following five conditions:

1) Γ is consistent;
2) For all formulas A of $\boldsymbol{M}$, if $\Gamma \vdash A$ then $A \in \Gamma$;
3) For all formulas A and B of $\boldsymbol{M}$, if $\Gamma \vdash A \vee B$ then $\Gamma \vdash A$ or $\Gamma \vdash B$;
4) For all formulas A and individual variables x of $\boldsymbol{M}$, if $\sim(x)\, A \in \Gamma$ then for some term t of $\boldsymbol{M}$, $\sim A^{t}/_{x} \in \Gamma$;
5) Γ is $\boldsymbol{M}$-ω-complete.

We will use underlined capital Greek letters to range over saturated sets. The following two lemmas concerning $\boldsymbol{M}$-saturated sets will be needed in our proof of semantical completeness.

Lemma 1. *Let Γ be a set of formulas of a morphology $\boldsymbol{M}$, and E a formula of $\boldsymbol{M}$ such that not $\Gamma \vdash E$. Then there is an extension $\boldsymbol{M}'$ of $\boldsymbol{M}$ such that some $\boldsymbol{M}'$-saturated set $\underline{\Gamma}$ is a superset of Γ, and it is not the case that $\underline{\Gamma} \vdash E$.*

Proof. Let Γ be a set of formulas of $\boldsymbol{M}$ such that not $\Gamma \vdash E$, and let $\boldsymbol{M}'$ be like $\boldsymbol{M}$ except that $C_{\boldsymbol{M}'} = C_{\boldsymbol{M}} \cup \{c_1, c_2, \ldots\}$, where $c_1, c_2, \ldots$ are symbols foreign to $\boldsymbol{M}$. Let $\Gamma_0 = \Gamma$ and $\Delta_0 = \{E\}$. In defining Γ_i and Δ_i, distinguish four cases, according as $i = 4k$, $i = 4k+1$, $i = 4k+2$, or $i = 4k+3$ for some k.

Case 1. Let A be the alphabetically first formula of $\boldsymbol{M}'$ such that not $\Gamma_i; A \vdash \Delta_i$ and $A \notin \Gamma_i$. Then $\Gamma_{i+1} = \Gamma_i \cup \{A\}$, and $\Delta_{i+1} = \Delta_i$.

Case 2. Let $A \vee B$ be the alphabetically first formula of $\boldsymbol{M}'$ of the kind $C \vee D$, such that $\Gamma_i \vdash A \vee B; \Delta_i$ and $A \notin \Gamma_i$, $B \in \Gamma_i$. Then $\Gamma_{i+1} = \Gamma_i \cup \{A\}$ in case not $\Gamma_i; A \vdash \Delta_i$, and $\Gamma_{i+1} = \Gamma_i \cup \{B\}$ otherwise. And in either case, $\Delta_{i+1} = \Delta_i$.

Case 3. Let $(x)\, A$ be the alphabetically first formula of $\boldsymbol{M}'$ of the kind $(y)\, B$, such that $\Gamma_i; (x)A \vdash \Delta_i$, and for all k, $A^{c_k}/_{x} \notin \Delta_i$. Let c_k be the first member of $\{c_1, c_2, \ldots\}$ not to occur in $(x)A$ or in any member of Γ_i or of Δ_i. Then $\Gamma_{i+1} = \Gamma_i$, and $\Delta_{i+1} = \Delta_i \cup \{A^{c_k}/_{x}\}$.

Case 4. Let $\sim(x)A$ be the alphabetically first formula of $\boldsymbol{M}'$ of the kind $\sim(y)\, B$ such that $\Gamma_i \vdash \sim(x)\, A$, and for all k, $\sim A^{c_k}/_{x} \notin \Gamma_i$. Let c_k be the first member of $\{c_1, c_2, \ldots\}$ not to occur in $\sim(x)\, A$ or in any member of Γ_i or of Δ_i. Then $\Gamma_{i+1} = \Gamma_i \cup \{\sim A^{c_k}/_{x}\}$ and $\Delta_{i+1} = \Delta_i$.

By induction, we show that for all i the sets Γ_i and Δ_i are defined, and it is not the case that $\Gamma_i \vdash \Delta_i$. In case 1, the inductive hypothesis guarantees that not $\Gamma_i \vdash \Delta_i$; and since there are infinitely many formulas of $\boldsymbol{M}'$ which are not formulas of $\boldsymbol{M}$, there will be some formula C of $\boldsymbol{M}'$ such that $\Gamma_i \vdash C$ and $C \notin \Gamma_i$; C will then be such that not $\Gamma_i; C \vdash \Delta_i$ and $C \notin \Gamma_i$. Thus, Γ_{i+1} and Δ_{i+1} are defined in this case, and clearly it is not the case that $\Gamma_{i+1} \vdash \Delta_{i+1}$. In the remaining cases, Γ_{i+1} and Δ_{i+1} are defined for similar reasons. It remains to be shown that in cases 2–4, not $\Gamma_{i+1} \vdash \Delta_{i+1}$.

In case 2, suppose for *reductio* that $\Gamma_{i+1} \vdash \Delta_{i+1}$. Then Γ_{i+1} must be $\Gamma_i \cup \{B\}$, and $\Delta_{i+1} = \Delta_i$, so that both $\Gamma_i; A \vdash \Delta_i$ and $\Gamma_i; B \vdash \Delta_i$. But then, by $\vee$ E, $\Gamma_i \vdash \Delta_i$, which is impossible. Therefore, not $\Gamma_{i+1} \vdash \Delta_{i+1}$.

Similarly, suppose in case 3 that $\Gamma_{i+1} \vdash \Delta_{i+1}$; i.e., that $\Gamma_i \vdash A^{c_k}/_x; \Delta_i$. But then, by ∀I, $\Gamma_i \vdash (x)\,A; \Delta_i$. And since $\Gamma_i; (x)\,A \vdash \Delta_i$, the rules of Group II can be used to show that $\Gamma_i \vdash \Delta_i$. In case 4, suppose again that $\Gamma_{i+1} \vdash \Delta_{i+1}$; then $\Gamma_i; \sim A^{c_k}/_x \vdash \Delta_i$. By assumption, $\Gamma_i \vdash \sim(x)\,A; \Delta_i$, so that, in view of ∼∀E, $\Gamma_i \vdash \Delta_i$. In these cases also, then, it is false that $\Gamma_{i+1} \vdash \Delta_{i+1}$, and the induction is complete.

Now, let $\Gamma_\omega = \bigcup_{i<\omega} \Gamma_i$, and $\Delta_\omega = \bigcup_{i<\omega} \Delta_i$; we will show that Γ_ω is **M**′-saturated. First, in view of rule **C**, it is not the case that $\Gamma_\omega \vdash \Delta_\omega$; hence Γ_ω is consistent. Because of case 2 of the construction of Γ_ω, for all formulas A and B of **M**′, $A \in \Gamma$ or $B \in \Gamma_\omega$ if $\Gamma_\omega \vdash A \vee B$ and therefore $A \in \Gamma_\omega$ or $B \in \Gamma_\omega$ if $A \vee B \in \Gamma_\omega$. Also, if $\Gamma_\omega \vdash A$ then $\Gamma_\omega \vdash A \vee A$, so that $A \in \Gamma_\omega$. Similarly, case 4 of the construction ensures that if $\sim(x)\,A \in \Gamma_\omega$ for any formula A and individual variable x of **M**′, then $\sim A^t/_x \in \Gamma_\omega$ for some term t of **M**′.

Case 1 of the construction of Γ_ω guarantees that $\Gamma_\omega = F_{M'} - \Delta_\omega$, where $F_{M'}$ is the set of formulas of **M**′. For, assume that $A \notin \Delta_\omega$. Then for all i, it is not the case that $\Gamma_i; A \vdash \Delta_i$. (Otherwise we would have $\Gamma_i; (x)\,A \vdash \Delta_i$, where x is an individual variable not occurring in A, and hence case 3 of the construction would place A in Δ_ω.) But then, in view of case 1 of the construction, $A \in \Gamma_\omega$.

Finally, then, it follows that Γ_ω is **M**-ω-complete. For, suppose that for some formula A and individual variable x, $(x)\,A \notin \Gamma_\omega$. Then $(x)\,A \in \Delta_\omega$, so that $\Gamma_\omega; (x)\,A \vdash \Delta_\omega$. But then case 3 of the construction ensures that for some term t of **M**′, $A^t/_x \in \Delta_\omega$, so that it is not the case that $\Gamma_\omega \vdash A^t/_x$.

Combining these results, we see that Γ_ω is indeed **M**′-saturated. Also, since $E \in \Delta_\omega$, it is not the case that $\Gamma_\omega \vdash E$. Thus, the proof of Lemma 1 is complete.

Lemma 2. *Let $\underline{\Gamma}$ be an **M**-saturated set of formulas, and A and B be formulas of **M** such that $A \supset B \notin \underline{\Gamma}$. Then there is an **M**-saturated set $\underline{\Theta}$ such that $\underline{\Gamma} \cup \{A\} \subseteq \underline{\Theta}$, and $B \notin \underline{\Theta}$.*

Proof. Suppose that $\underline{\Gamma}$ is **M**-saturated, and that $A \supset B \notin \underline{\Gamma}$; then, because of ⊃E, it is not the case that $\underline{\Gamma}; A \vdash B$. Let $\Gamma_0 = \underline{\Gamma} \cup \{A\}$, and $\Delta_0 = \{B\}$. As in the proof of Lemma 2, define sets Γ_i and Δ_i inductively according to four cases. Cases 1 and 2 of the new definition are just like the corresponding cases of the old construction, and cases 3 and 4 (in which $i = 4k + 2$ for some k and $i = 4k + 3$ for some k, respectively) are as follows.

Case 3. In case there exist formulas $(x)\,C$ of **M** such that $\Gamma_i; (x)\,C \vdash \Delta_i$ and for all terms t of **M**, $C^t/_x \notin \Delta_i$, let $(x)\,C$ be the alphabetically first formula of **M** of the kind $(y)\,D$ such that $\Gamma_i; (x)\,C \vdash \Delta_i$ and for all terms t of **M**, $C^t/_x \notin \Delta_i$. Let t be the alphabetically first term of **M** such that it is not the case that $\Gamma_i \vdash C^t/_x; \Delta_i$. Then $\Gamma_{i+1} = \Gamma_i$, and $\Delta_{i+1} = \Delta_i \cup \{C^t/_x\}$. Otherwise $\Gamma_{i+1} = \Gamma_i$ and $\Delta_{i+1} = \Delta_i$.

Case 4. In case there exist formulas $\sim(x)\,C$ of **M** such that $\Gamma_i \vdash \sim(x)\,C; \Delta_i$ and for all terms t of **M**, $\sim C^t/_x \notin \Gamma_i$, let $\sim(x)\,C$ be the alphabetically first formula of **M** of the kind $\sim(y)\,D$ such that $\Gamma_i \vdash \sim(x)\,C; \Delta_i$ and for all terms t of **M**, $\sim C^t/_x \notin \Gamma_i$. Let t be the alphabetically first term of **M** such that it is not the case that $\Gamma_i; \sim C^t/_x \vdash \Delta_i$. Then $\Gamma_{i+1} = \Gamma_i \cup \{\sim C^t/_x\}$, and $\Delta_{i+1} = \Delta_i$. Otherwise $\Gamma_{i+1} = \Gamma_i$ and $\Delta_{i+1} = \Delta_i$.

By assumption, Γ_0 is **M**-ω-complete. Assume that Γ_i is defined and is **M**-ω-complete; if Γ_{i+1} is defined, then $\Gamma_{i+1} = \Gamma_i$—and hence is **M**-ω-complete—or for some formula C, $\Gamma_{i+1} = \Gamma_i \cup \{C\}$. In the latter case, suppose that for all terms t of **M**, $\Gamma_i; C \vdash D^t/_x$. Then for all terms t of **M**, $\Gamma_i \vdash C \supset D^t/_x$, so since Γ_i is **M**-ω-complete, $\Gamma_i \vdash (y)\,(C \supset D^y/_x)$, where y is an individual variable of **M** not occurring in C or in any member of Γ_i. But for such a choice of y, $(y)\,(C \supset D^y/_x); C \vdash (x)\,D$, and therefore $\Gamma_{i+1} \vdash (x)\,D$. Thus, Γ_{i+1} is **M**-ω-complete.

The above argument shows that if Γ_i is defined, then it is **M**-ω-complete; we will use this fact below. Assume now as hypothesis of induction for all i, Γ_i and Δ_i are defined, and not $\Gamma_i \vdash \Delta_i$; in cases 1 and 2 of the construction one shows that Γ_{i+1} and Δ_{i+1} are defined and that not $\Gamma_{i+1} \vdash \Delta_{i+1}$, in the same way as in the proof of Lemma 1. In case 3, if there is no formula $(x)\,C$ of **M** such that $\Gamma_i; (x)\,C \vdash \Delta_i$ and for all terms t of **M**, $C^t/_x \notin \Delta_i$, then Γ_{i+1} and Δ_{i+1} are Γ_i and Δ_i, respectively, and so are defined. If there is such a formula, let $(x)\,C$ be the alphabetically first, and suppose for *reductio* that $\Gamma_i \vdash C^t/_x; \Delta_i$ for all terms t of **M**. Since Γ_i is **M**-ω-complete, it follows that $\Gamma_i \vdash (x)\,C; \Delta_i$, so that $\Gamma_i \vdash \Delta_i$, contrary to our hypothesis. Therefore, there is a term t of **M** such that not $\Gamma_i \vdash C^t/_x; \Delta_i$. Thus, Γ_{i+1} and Δ_{i+1} are defined; and clearly, it is not case that $\Gamma_{i+1} \vdash \Delta_{i+1}$.

In case 4, one uses the **M**-ω-completeness of Γ_i in the same way to show that Γ_{i+1} and Δ_{i+1} are defined. In fact, suppose for *reductio* that for all terms t of **M**, $\Gamma_i; \sim C^t/_x \vdash \Delta_i$, where $\Gamma_i \vdash \sim(x)\,C; \Delta_i$. Then $\Gamma_i \vdash (y)\,(\sim C^y/_x \supset D)$, where $\Delta_i = \{D_1, \ldots, D_n\}$ and D is $D_1 \vee \cdots \vee D_n$, and y is an individual variable not occurring in $\sim(x)\,C$ or in D. But for such a choice of y, it can be shown, using ∼∀E, that $(y)\,(\sim C^y/_x \supset D); \sim(x)\,C \vdash D$, and therefore $\Gamma_i; \sim(x)\,C \vdash D$, so that $\Gamma_i \vdash D$ and, eventually, $\Gamma_i \vdash \Delta_i$, contrary to hypothesis. It follows in this case also, then, that Γ_{i+1} and Δ_{i+1} are defined.

To show that $\Gamma_\omega = \bigcup_{i<\omega} \Gamma_i$ is **M**-saturated, one argues in the same way as in the proof of Lemma 1. This completes the proof of Lemma 2.

3. Semantics

The intuitive idea underlying our interpretation of **CF** is that of a process of construction in which questions can be settled positively or negatively. This may be thought of in terms of what is *known* at a given stage of the construction; at any such stage, various relations will be known to obtain or not to obtain. From our standpoint in interpreting **CF**, this knowledge that a relation fails to obtain is *irreducible*. The falsity, for instance, of a formula such as $P(a)$ at a stage of construction is not defined in terms of the failure of the individual named by a to have the property corresponding to P; rather, this falsity is conceived of as a feature which is discovered directly, as the truth of $P(a)$ is discovered.

As we go from earlier to later stages of construction, various relations may be discovered to obtain or to fail to obtain. We assume, however, that information is never lost, so that what is true (or false) at a given stage of construction is true (false) at any later stage. It is also important that our notion of a process of construction be general enough to permit that from the standpoint of a given stage of construction, the possibility is allowed of settling certain questions in either way. Thus, for instance, if at a stage α a formula, say $P(a)$, is neither true nor false, there may be a later stage β at which $P(a)$ is true, and another stage γ at which $P(a)$ is false. (This does not mean, of course, that $P(a)$ can be *simultaneously* true and false, but that from the standpoint of α both possibilities are open.) We will not require, then, that the relation of being a later stage of construction be linear.[1]

We make these ideas rigorous in the following way. A CF *model structure* (CFms) is a triple $\langle \mathscr{K}, \mathscr{R}, D\rangle$, where $\mathscr{K}$ is a nonempty set, $\mathscr{R}$ a binary reflexive and transitive relation on $\mathscr{K}$, and D a nonempty domain. $\mathscr{K}$ is to be thought of as the set of stages of a process of construction, $\mathscr{R}$ is the relation ordering these stages of construction, and D the domain over which the quantifiers are to range.

An *interpretation* I of a morphology $\boldsymbol{M}$ on a CFms $\langle \mathscr{K}, \mathscr{R}, D\rangle$ is a function such that:

1) For each $t \in T_{\boldsymbol{M}}$, $I(t)$ is a member of D;

2) For each $P \in P^i_{\boldsymbol{M}}$ and $\alpha \in \mathscr{K}$, $I_\alpha(P)$ is a partial function from the cartesian product D^i into $\{T, F\}$ (where $i = 0$, $I_\alpha(P)$ is either T, F, or undefined).

We require of every interpretation I of $\boldsymbol{M}$ on $\langle \mathscr{K}, \mathscr{R}, D\rangle$ that for all $\alpha, \beta \in D$, if $\alpha \mathscr{R} \beta$ then for all i and all $P \in P^i_{\boldsymbol{M}}$, $I_\beta(P)$ is an extension of $I_\alpha(P)$. I.e., if $I_\alpha(P)$ is defined for any given argument, then $I_\beta(P)$ must also be defined for that argument, and take the same value.

Where I is an interpretation of $\boldsymbol{M}$ on $\langle \mathscr{K}, \mathscr{R}, D\rangle$, x an individual variable of $\boldsymbol{M}$ and d a member of D, $I^d/_x$ is the interpretation of $\boldsymbol{M}$ on $\langle \mathscr{K}, \mathscr{R}, D\rangle$ differing from I only in that $I^d/_x(x) = d$.

Where I is an interpretation of $\boldsymbol{M}$ on $\langle \mathscr{K}, \mathscr{R}, D\rangle$ and $\alpha \in \mathscr{K}$, the truth-value $I_\alpha(A)$ (if any) given by I to a formula A of $\boldsymbol{M}$ at the stage of construction α is defined as follows, by induction on the complexity of A.

1) $I_\alpha(P(t_1, \ldots, t_n)) = T$ if and only if $I_\alpha(P)(\langle I(t_1), \ldots, I(t_n)\rangle) = T$,
$I_\alpha(P(t_1, \ldots, t_n)) = F$ if and only if $I_\alpha(P)(\langle I(t_1), \ldots, I(t_n)\rangle) = F$;

2) $I_\alpha(B \vee C) = T$ if and only if $I_\alpha(B) = T$ or $I_\alpha(C) = T$,
$I_\alpha(B \vee C) = F$ if and only if $I_\alpha(B) = F$ and $I_\alpha(C) = F$;

3) $I_\alpha({\sim}B) = T$ if and only if $I_\alpha(B) = F$,
$I_\alpha({\sim}B) = F$ if and only if $I_\alpha(B) = T$;

4) $I_\alpha(B \supset C) = T$ if and only if for all β such that $\alpha \mathscr{R} \beta$, $I_\beta(C) = T$ if $I_\beta(B) = T$,
$I_\alpha(B \supset C) = F$ if and only if $I_\alpha(B) = T$ and $I_\alpha(C) = F$;

5) $I_\alpha((x)\,B) = T$ if and only if for all $d \in D$, $I^d/_{x_\alpha}(B) = T$,
$I_\alpha((x)\,B) = F$ if and only if for some $d \in D$, $I^d/_{x_\alpha}(B) = F$.

[1] This requirement would render formulas of the sort $(A \supset B) \vee (B \supset A)$ valid.

Note that, in general, I_α will only be a *partial* function on formulas of $\boldsymbol{M}$; there may well be formulas A which are assigned neither truth-value in α by I, so that $I_\alpha(A)$ is undefined.

The following lemma concerning substitution is easily proved by induction on the complexity of A; it will be used below in proving semantical completeness.

Lemma 3. *Let I be an interpretation of $\boldsymbol{M}$ on $\langle \mathscr{K}, \mathscr{R}, D\rangle$, and $\alpha \in \mathscr{K}$. Let A be any formula and t any term of $\boldsymbol{M}$. Then* $I_\alpha(A^t/_x) = I^{I(t)}/_{x_\alpha}(A)$.

Notions of *implication*, *simultaneous satisfiability*, and *validity* are defined in the usual way in terms of satisfaction. A set Γ of formulas of $\boldsymbol{M}$ *implies* a formula A of $\boldsymbol{M}$, $\Gamma \Vdash A$, if for every CFms $\langle \mathscr{K}, \mathscr{R}, D\rangle$ and every $\alpha \in \mathscr{K}$, $I_\alpha(A) = T$ if $I_\alpha(B) = T$ for all $B \in \Gamma$, for all interpretations I of $\boldsymbol{M}$ on $\langle \mathscr{K}, \mathscr{R}, D\rangle$. Γ is *simultaneously satisfiable* if there is a formula A such that not $\Gamma \Vdash A$, and A is *valid* if $\emptyset \Vdash A$. For brevity, we may express the fact that A is valid by writing simply „$\Vdash A$".

4. Semantical completeness

Our main result will follow from the following lemma, together with Lemma 1.

Lemma 4. *Let $\underline{\Gamma}$ be $\boldsymbol{M}$-saturated. Then there exists a CFms $\langle \mathscr{K}, \mathscr{R}, D\rangle$, an $\alpha \in \mathscr{K}$, and an interpretation I of $\langle \mathscr{K}, \mathscr{R}, D\rangle$ such that $\underline{\Gamma} = \{A / I_\alpha(A) = T\}$.*

Proof. Let $\mathscr{K} = \{\underline{\Delta} / \underline{\Gamma} \subseteq \underline{\Delta}$ and $\underline{\Delta}$ is $\boldsymbol{M}$-saturated$\}$, and $\underline{\Delta}\, \mathscr{R}\, \underline{\Theta}$ if and only if $\underline{\Delta} \subseteq \underline{\Theta}$. Let $D = T_{\boldsymbol{M}}$. $\langle \mathscr{K}, \mathscr{R}, D\rangle$ is a CFms; we define an interpretation I of $\boldsymbol{M}$ on $\langle \mathscr{K}, \mathscr{R}, D\rangle$ as follows. Where $t \in T_{\boldsymbol{M}}$, let $I(t) = t$. Where $P \in P^0_{\boldsymbol{M}}$, let $I_{\underline{\Delta}}(P) = T$ if and only if $P \in \underline{\Delta}$, and $I_{\underline{\Delta}}(P) = F$ if and only if ${\sim}P \in \underline{\Delta}$. Where $P \in P^n_{\boldsymbol{M}}$, $n > 0$, let $I_{\underline{\Delta}}(P)(\langle t_1, \ldots, t_n\rangle) = T$ if and only if $P(t_1, \ldots, t_n) \in \underline{\Delta}$, and $I_{\underline{\Delta}}(P)(\langle t_1, \ldots, t_n\rangle) = F$ if and only if ${\sim}P(t_1, \ldots, t_n) \in \underline{\Delta}$. I, thus defined, is clearly an interpretation of $\boldsymbol{M}$ on $\langle \mathscr{K}, \mathscr{R}, D\rangle$.

By induction on the complexity of A, we will now show that for all $\underline{\Delta} \in \mathscr{K}$, $I_{\underline{\Delta}}(A) = T$ if and only if $A \in \underline{\Delta}$. The basis step of the induction follows immediately from the definition of satisfaction; in the inductive step we distinguish cases according as A has the form $B \vee C$, ${\sim}(B \vee C)$, $B \supset C$, ${\sim}{\sim}B$, $(x)\,B$, or ${\sim}(x)\,B$.

Case 1. $I_{\underline{\Delta}}(B \vee C) = T$ if and only if $I_{\underline{\Delta}}(B) = T$ or $I_{\underline{\Delta}}(C) = T$, and by the hypothesis of induction this if and only if $B \in \underline{\Delta}$ or $C \in \underline{\Delta}$. But from the definition of $\boldsymbol{M}$-saturation and $\vee$I, it clearly follows that $B \in \underline{\Delta}$ or $C \in \underline{\Delta}$ if and only if $B \vee C \in \underline{\Delta}$.

Case 2. $I_{\underline{\Delta}}({\sim}(B \vee C)) = T$ if and only if $I_{\underline{\Delta}}({\sim}B) = T$ and $I_{\underline{\Delta}}({\sim}C) = T$, and by the hypothesis of induction this if and only if ${\sim}B \in \underline{\Delta}$ and ${\sim}C \in \underline{\Delta}$. But in view of ${\sim}\vee$I and ${\sim}\vee$E, ${\sim}B \in \underline{\Delta}$ and ${\sim}C \in \underline{\Delta}$ if and only if ${\sim}(B \vee C) \in \underline{\Delta}$.

Case 3. If $B \supset C \in \underline{\Delta}$ then by $\supset$E, for all $\underline{\Theta} \in \mathscr{K}$ such that $\underline{\Delta}\, \mathscr{R}\, \underline{\Theta}$, $C \in \underline{\Theta}$ if $B \in \underline{\Theta}$. By the hypothesis of induction, then, for all such $\underline{\Theta}$, $I_{\underline{\Theta}}(C) = T$ if $I_{\underline{\Theta}}(B) = T$;

and therefore $I_{\underline{\Delta}}(B \supset C) = T$. Conversely, suppose that $B \supset C \in \underline{\Delta}$; then by Lemma 2, there is a $\underline{\Theta} \in \mathscr{K}$ such that $\underline{\Delta}\ \mathscr{R}\ \underline{\Theta}$ and $B \in \underline{\Theta}$ but $C \notin \underline{\Theta}$. By the hypothesis of induction, $I_{\underline{\Theta}}(B) = T$ and $I_{\underline{\Theta}}(C) = \overline{T}$; but then, $I_{\underline{\Theta}}(B \supset C) \neq T$.

Case 4. $I_{\underline{\Delta}}(\sim(B \supset C)) = T$ if and only if $I_{\underline{\Delta}}(B) = T$ and $I_{\underline{\Delta}}(\sim C) = T$, and by the hypothesis of induction this if and only if $B \in \underline{\Delta}$ and $\sim C \in \underline{\Delta}$. But in view of $\sim\supset$**I** and $\sim\supset$**E**, $B \in \underline{\Delta}$ and $\sim C \in \underline{\Delta}$ if and only if $\sim(B \supset C) \in \underline{\Delta}$.

Case 5. $I_{\underline{\Delta}}(\sim\sim B) = T$ if and only if $I_{\underline{\Delta}}(B) = T$, and by the hypothesis of induction this if and only if $B \in \underline{\Delta}$. But in view of $\sim\sim$**I** and $\sim\sim$**E**, $B \in \underline{\Delta}$ if and only if $\sim\sim B \in \underline{\Delta}$.

Case 6. $I_{\underline{\Delta}}((x)\, B) = T$ if and only if for all $t \in T_{\boldsymbol{M}}$, $I^t/_{x\underline{\Delta}}(B) = T$, and by Lemma 3, this if and only if for all $t \in T_{\boldsymbol{M}}$, $I_{\underline{\Delta}}(B^t/_x) = T$. The hypothesis of induction ensures that for all $t \in T_{\boldsymbol{M}}$, $I_{\underline{\Delta}}(B^t/_x) = T$ if and only if for all $t \in T_{\boldsymbol{M}}$ $B^t/_x \in \underline{\Delta}$. But in view of $\forall$E and the definition of $\boldsymbol{M}$-saturation, this if and only if $(x)\, B \in \underline{\Delta}$.

Case 7. $I_{\underline{\Delta}}(\sim(x)\, B) = T$ if and only if for some $t \in T_{\boldsymbol{M}}$, $I^t/_{x\underline{\Delta}}(\sim B) = T$. By Lemma 3, this if and only if for some $t \in T_{\boldsymbol{M}}$, $I_{\underline{\Delta}}(\sim B^t/_x) = T$, and the hypothesis of induction ensures that for some $t \in T_{\boldsymbol{M}}$, $\sim B^t/_x \in \underline{\Delta}$. But in view of $\exists$I and the definition of $\boldsymbol{M}$-saturation, this if and only if $\sim(x)\, B \in \underline{\Delta}$.

This completes the induction, which shows in particular that $\underline{\Gamma} = \{A / I_{\underline{\Gamma}}(A) = T\}$. The proof of Lemma 4 is therefore complete.

Theorem 1. *If $\Gamma \Vdash A$, then $\Gamma \vdash A$.*

Proof. Suppose that not $\Gamma \vdash A$; by Lemma 1, there is for some morphology $\boldsymbol{M}$ an $\boldsymbol{M}$-saturated set $\underline{\Gamma}$ such that $\Gamma \subseteq \underline{\Gamma}$ and $A \notin \underline{\Gamma}$. By Lemma 4, then, there is a **CF**ms $\langle \mathscr{K}, \mathscr{R}, D \rangle$, an $\alpha \in \mathscr{K}$, and an interpretation I of $\boldsymbol{M}$ on D such that for all $B \in \Gamma$, $I_\alpha(B) = T$, but $I_\alpha(A) \neq T$. Therefore, not $\Gamma \Vdash A$.

By inspection of the rules of Groups I and II, above, it is readily verified that if $\Gamma \vdash A$, then $\Gamma \Vdash A$; thus, we have the following theorem.

Theorem 2. *$\Gamma \Vdash A$ if and only if $\Gamma \vdash A$.*

The following special case of Theorem 2 is of particular interest.

Theorem 3. *$\Vdash A$ if and only if $\vdash A$.*

Theorem 4. *Let **S** be any syntactical system in terms of which a deducibility-relation $\vdash_{\boldsymbol{S}}$ may be defined for formulas of **CF**. Then if $\Gamma \Vdash A$ whenever $\Gamma \vdash_{\boldsymbol{S}} A$ and $\vdash_{\boldsymbol{S}}$ satisfies the rules of Groups* I *and* II, *above, $\Gamma \Vdash A$ if and only if $\Gamma \vdash_{\boldsymbol{S}} A$; and, in particular, $\Vdash A$ if and only if $\vdash_{\boldsymbol{S}} A$.*

Theorem 3 and Theorem 4 are immediate corollaries of Theorem 2. Theorem 4 permits one to establish the semantical completeness of a variety of formulations of **CF**. One such example is the natural deduction system of Fitch [1], when this system is supplemented by introduction and elimination rules for negated implication, corresponding to $\sim\supset$**I** and $\sim\supset$**E**.[1]) And it is not difficult to devise an axiomatic system, with *modus ponens* its sole rule of inference, which meets the conditions of Theorem 4.

5. Applications

In this section, we will use the above lemmas and theorems to obtain some further results concerning **CF**. First, Theorem 2 may be used to establish the finitary character of implication for **CF**; and the proofs of Lemma 1 and Lemma 4 yield a Löwenheim-Skolem theorem.

Theorem 5. *If $\Gamma \Vdash A$, then for some finite subset Γ' of Γ, $\Gamma' \Vdash A$.*

Theorem 6. *If not $\Gamma \Vdash A$, then there is an interpretation I on a **CF**ms $\langle \mathscr{K}, \mathscr{R}, D \rangle$, where D is at most denumerable, such that for some $\alpha \in \mathscr{K}$, $I_\alpha(B) = T$ for all $B \in \Gamma$, but $I_\alpha(A) = F$.*

A closer examination of interpretations of **CF** yields the result that the set of valid formulas of **CF** is saturated; thus, there is a minimal member α of the **CF**ms of the proof of Lemma 4, such that $\{A / I_\alpha(A) = T\} = \{A / A$ is a formula of $\boldsymbol{M}$ and $\Vdash A\}$, where I is the interpretation defined in that proof. We can then use this fact to show, e.g., that if $\vdash A \vee B$, then $\vdash A$ or $\vdash B$. This is done in the following sequence of lemmas and theorems.

Lemma 5. *Let I be an interpretation of $\boldsymbol{M}$ on a **CF**ms $\langle \mathscr{K}, \mathscr{R}, D \rangle$, and let $\alpha\ \mathscr{R}\ \beta$, where $\alpha, \beta \in \mathscr{K}$, and $\Gamma = \{A / I_\alpha(A) = T\}$, $\Delta = \{A / I_\beta(A) = T\}$. Then $\Gamma \subseteq \Delta$.*

Proof. This is established readily by induction on the complexity of formulas of $\boldsymbol{M}$.

Lemma 6. *Let $\boldsymbol{M}$ be a morphology for **CF**. There exists an $\boldsymbol{M}$-saturated set $\underline{\Gamma}$ such that for all $\boldsymbol{M}$-saturated sets $\underline{\Delta}$, $\underline{\Gamma} \subseteq \underline{\Delta}$.*

Proof. Let $\mathscr{K} = \{\underline{\Delta} / \underline{\Delta}$ is $\boldsymbol{M}$-saturated$\}$, and let $\mathscr{R}$, D, and I be defined as in the proof of Lemma 4, so that for all $\underline{\Delta} \in \mathscr{K}$, $\underline{\Delta} = \{A / I_{\underline{\Delta}}(A) = T\}$. Let α be an object not in $\mathscr{K}$, and let $\mathscr{K}' = \mathscr{K} \cup \{\alpha\}$, and extend $\mathscr{R}$ to $\mathscr{R}'$ on $\mathscr{K}'$ by letting $\alpha\ \mathscr{R}\ \beta$ for all $\beta \in \mathscr{K}'$. Extend I to an interpretation I' on the **CF**ms $\langle \mathscr{K}', \mathscr{R}', D \rangle$ in such a way that no atomic formulas of $\boldsymbol{M}$ are true or false in α. I.e., for all $P \in P^0_{\boldsymbol{M}}$, $I_\alpha(P)$ is undefined, and for all $P \in P^i_{\boldsymbol{M}}$, $i > 0$, $I_\alpha(P)$ is undefined for all arguments in D^i. Let $\Gamma = \{A / I_\alpha(A) = T\}$; it is easy to see that Γ meets the conditions of $\boldsymbol{M}$-saturation, and by Lemma 5, $\Gamma \subseteq \underline{\Delta}$ for all $\underline{\Delta} \in \mathscr{K}$.

Theorem 5. *Let $\Delta = \{A / A$ is a formula of $\boldsymbol{M}$ and $\Vdash A\}$, where $\boldsymbol{M}$ is a morphology for **CF**. Then Δ is $\boldsymbol{M}$-saturated.*

[1]) Professor Fitch has told me that he thought of these rules in formulating this system, but omitted them in order not to complicate his consistency proof in chapter 4 of [1]. The rules are added in Prawitz' formulation of **CF**, in [7]. It is possible to devise a semantical theory characterizing the system without rules for negated implication, as indicated in [9]. This, however, does not seem to possess sufficient interest to merit a detailed presentation.

Proof. By Lemma 6, there exists an $\boldsymbol{M}$-saturated set Γ such that $\underline{\Gamma} \subseteq \underline{\Theta}$ for all $\boldsymbol{M}$-saturated sets $\underline{\Theta}$. But clearly $A \in \Delta$ if and only if for every $\boldsymbol{M}$-saturated $\underline{\Theta}$, $A \in \underline{\Theta}$. Thus, $\Delta = \underline{\Gamma}$.

Theorem 6. *If* $\vdash A \vee B$ *then* $\vdash A$ *or* $\vdash B$.

Theorem 7. *If* $\vdash (\exists x) A$ *then for some term* t, $\vdash A^t/_x$.

Theorem 6 and Theorem 7 follow immediately from Theorem 2 and Theorem 6, together with the definition of $\boldsymbol{M}$-saturation.

Our final result concerns a translation of **CF** into a modal logic; we will present a mapping τ of formulas of **CF** into modal formulas, such that $\Vdash A$ if and only if $\tau(A)$ is a valid modal formula. This mapping resembles known translations of intuitionistic formulas into modal ones,[1]) and thus is useful for comparing **CF** with intuitionistic as well as modal logic.

The modal logic into which we will translate **CF** is a version of **S4** with quantifiers, called **Q1 S4** in [11] and [12]. An axiomatic formulation of **Q1 S4** is obtained by adding classical axioms for quantifiers to axioms for **S4**, together with axioms of the sort $(x) \Box A \supset \Box (x) A$. **Q1 S4** is interpreted semantically by considering model structures of the same kind used in connection with **CF**. A **Q1 S4** interpretation I on $\langle \mathscr{K}, \mathscr{R}, D \rangle$ behaves for each $\alpha \in \mathscr{K}$ just like an interpretation of the classical predicate calculus; and $I_\alpha(\Box A) = T$ if and only if for all β such that $\alpha \mathscr{R} \beta$, $I_\beta(A) = T$. This semantics gives rise to notions of implication and validity: thus, where A is a formula and Γ a set of formulas, we will say that $\Gamma \Vdash_{\mathbf{Q1\,S4}} A$ (or, for short, that $\Gamma \Vdash_{\mathbf{Q1}} A$), if Γ implies A with respect to this interpretation.

We define the translation τ by induction on the complexity of formulas of **CF**, as follows.

1) If A is atomic, $\tau(A) = \Box A$ and $\tau(\sim A) = \Box \sim A$;

2) $\tau(A \vee B) = \tau(A) \vee \tau(B)$ and $\tau(\sim(A \vee B)) = \tau(\sim A) \wedge \tau(\sim B)$;

3) $\tau(A \supset B) = \Box(\tau(A) \supset \tau(B))$ and $\tau(\sim(A \supset B)) = \tau(A) \wedge \tau(\sim B)$;

4) $\tau(\sim\sim A) = \tau(A)$;

5) $\tau((x) A) = (x) \tau(A)$ and $\tau(\sim(x) A) = (\exists x) \tau(\sim A)$.

Making use of the results we have established concerning **CF**, we will show that for all formulas A of **CF**, $\Vdash A$ if and only if $\Vdash_{\mathbf{Q1}} \tau(A)$.

Theorem 8. *If* $\Vdash_{\mathbf{Q1}} \tau(A)$ *then* $\Vdash A$.

Proof. Where Γ and Δ are sets of formulas of **CF**, say that $\Gamma \vdash_\tau \Delta$ in case $\tau(\Gamma) \Vdash_{\mathbf{Q1}} \tau(\Delta)$. It is readily verified that the translations of the rules of Groups I and II are correct in **Q1 S4**. (For instance, one shows by induction that for all formulas A of **CF**, $\Vdash_{\mathbf{Q1}} \tau(A) \equiv \Box \tau(A)$. Using this fact, it is easy to show that

[1]) See McKinsey and Tarski [5]. The translation given there are easily extended to include quantifiers, as in Rasiowa and Sikorski [8].

$\supset$I translates into a correct rule concerning $\Vdash_{\mathbf{Q1}}$.) Hence, if $\Gamma \vdash_\tau \Delta$ then $\Gamma \vdash \Delta$; therefore, in view of Theorem 2, $\Gamma \Vdash \Delta$. As a special case of this, if $\Vdash_{\mathbf{Q1}} \tau(A)$ then $\Vdash A$.

Theorem 9. *If* $\Vdash A$ *then* $\Vdash_{\mathbf{Q1}} \tau(A)$.

Proof. Let A be a formula of a morphology $\boldsymbol{M}$ for **CF**, and let $\boldsymbol{M}'$ be the corresponding morphology for **Q1 S4**. Suppose that not $\Vdash_{\mathbf{Q1}} (A)$; then for some $\langle \mathscr{K}, \mathscr{R}, D \rangle$ there is a **Q1 S4** interpretation J of $\boldsymbol{M}'$ on $\langle \mathscr{K}, \mathscr{R}, D \rangle$ such that $J_\alpha(A) = F$ for some $\alpha \in \mathscr{K}$. Define a **CF** interpretation I of $\boldsymbol{M}$ by setting $I_\alpha(t) = J_\alpha(t)$ for all terms t of $\boldsymbol{M}$ and $\alpha \in \mathscr{K}$, and $I_\alpha(P) = T$ if and only if $J_\alpha(\Box P) = T$ and $I_\alpha(\sim P) = F$ if and only if $J_\alpha(\Box \sim P) = T$ for all $P \in P^0_{\boldsymbol{M}}$ and $\alpha \in \mathscr{K}$. Finally, let $I_\alpha(P)(\langle d_1, \ldots, d_n \rangle) = T$ if and only if $J_\beta(P)(\langle d_1, \ldots, d_n \rangle) = T$ for all β such that $\alpha \mathscr{R} \beta$ and $I_\alpha(P)(\langle d_1, \ldots, d_n \rangle) = F$ if and only if $J_\beta(P)(\langle d_1, \ldots, d_n \rangle) = F$ for all β such that $\alpha \mathscr{R} \beta$, for all $P \in P^n_{\boldsymbol{M}}$ and $\alpha \in \mathscr{K}$. By induction on the complexity of formulas B of $\boldsymbol{M}$, one shows that $I_\beta(B) = T$ if and only if $J_\beta(\tau(B)) = T$, for all $\beta \in \mathscr{K}$. Thus, in particular, $I_\alpha(A) = F$, and thus not $\Vdash A$. This completes the proof of Theorem 9.

Bibliography

[1] Fitch, F., Symbolic logic. New York 1952.

[2] Fitch, F., The system CΔ of combinatory logic. Journal Symb. Log. **28** (1963), 87–97.

[3] Kripke, S., A completeness theorem in modal logic. Journal Symb. Log. **24** (1959), 1–14.

[4] Kripke, S., Semantical analysis of intuitionistic logic I. Formal systems and recursive functions, ed. J. Crossley and M. Dummett, Amsterdam 1965, pp. 92–130.

[5] McKinsey, J., and A. Tarski, Some theorems about the sentential calculi of Lewis and Heyting. Journal Symb. Log. **13** (1948), 1–15.

[6] Nelson, D., Constructible falsity. Journal Symb. Log. **14** (1949), 16–26.

[7] Prawitz, D., Natural deduction. Stockholm 1965.

[8] Rasiowa, H., and R. Sikorski, The mathematics of metamathematics. Warsaw 1963.

[9] Thomason, R., The semantical completeness of a system of Fitch (abstract). Journal Symb. Log. **31** (1966), 695–696.

[10] Thomason, R., On the strong semantical completeness of the intuitionistic predicate calculus. Journal Symb. Log. **33** (1968), 1–7.

[11] Thomason, R., Modal logic and metaphysics. The logical way of doing things, ed. K. Lambert, New Haven 1969.

[12] Thomason, R., Some completeness results for modal predicate calculi, forthcoming.

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