Radio Science, Volume 27, Number 5, Pages 553–560, September–October 1992

Derivation of phase statistics from the Mueller matrix

K. Sarabandi

Radiation Laboratory, Department of Electrical Engineering and Computer Science,
University of Michigan, Ann Arbor

(Received April 18, 1991; revised December 5, 1991; accepted January 6, 1992.)

To answer the question of what radar polarimetry has to offer to the remote sensing of random media, statistics of the phase difference of the scattering matrix elements must be studied. Recent polarimetric measurements of rough surfaces have indicated that the statistical parameters of the phase difference (mean, standard deviation, etc.) are very sensitive to some of the physical parameters. In this paper the probability density function of the phase differences is derived from the Mueller matrix, assuming that the elements of the scattering matrix are jointly Gaussian. It is shown that the probability density functions of the copolarized and cross-polarized phase differences are similar in form, and each can be determined by two parameters (α and ζ) completely. The expressions for the probability density functions are verified by comparing the histograms, the mean, and the standard deviations of phase differences derived directly from polarimetric measurements of a variety of rough surfaces to the probability density function, its mean, and standard deviation derived from the Mueller matrices of the same data. The expressions for the probability density functions are of special interest for noncoherent polarimetric radars and noncoherent polarimetric models for random media such as vector radiative transfer.

1. INTRODUCTION

In the past decade, substantial effort within the microwave remote sensing community has been devoted to the development and improvement of polarimetry science. Polarimetric radars are capable of synthesizing the radar response of a target to any combination of the receive and transmit polarizations from coherent measurements of the target with two orthogonal channels. Polarimetric radars have demonstrated their abilities in improving point target detection and classification [Ioannidis and Hammers, 1979]. That is, for a point target in a clutter background the transmit and receive polarizations can be chosen such that the target to clutter response is maximum. Also, different point targets in the radar scene can be classified according to their optimum polarization. Although radar polarimeters have shown a great potential in point target detection and classification, their capabilities in remote sensing of distributed targets are not completely understood yet.

Paper number 92RS00195
0048-6604/92/92RS-00195$08.00

Considering the complexity involved in designing, manufacturing, and processing the data of an imaging polarimeter as opposed to a conventional imaging radar, it is necessary to examine the advantages that the imaging polarimeter provides with the targets of interest. For example, in retrieving the biophysical parameters from the polarimetric radar data, one should ask whether there exists a dependency between the parameters and the measured phase of the scattering matrix components. If the answer is negative, obviously gathering polarimetric data for inversion of that parameter is a waste of effort. One way of confirming this question is by collecting data over a range of the desired parameter while keeping other influential parameters constant. This procedure, if not impossible, is very difficult to conduct because of problems in repeatability of the experiment and difficulties in controlling the environmental conditions. Moreover, at high frequencies (millimeter wave frequencies and higher), coherent measurement of the scattering matrix is impossible because of instabilities of local oscillators and relative movements of the target and the radar platform [Meads and McIntosh, 1991]. At these frequencies, noncoherent radars are employed which provide the Mueller matrix of the target.

Another approach to examine the dependency of the radar response to the desired parameters of the targets is the application of theoretical models. One of the most successful polarimetric models for random media is the vector radiative transfer theory [Tsang et al., 1985]. This model is based on conservation of energy and the single scattering properties of the constituent particles. The solution of the radiative transfer equation relates the scattered wave Stokes vector to the incident wave Stokes vector via the Mueller matrix. The Mueller matrix, as computed by this method, is an ensemble- averaged quantity because of the inherent nature of the radiative transfer theory. Since the Mueller matrix is related to the scattering matrix through a nonlinear process and the components of the scattering matrix are statistically dependent, the information about the phase difference of the scattering matrix components cannot be obtained from the Mueller matrix directly. To achieve information about the phase statistics, one may resort to the Monte Carlo type models which are computationally inefficient and, in general, inaccurate.

Experimental observations of phase difference statistics from a polarimetric synthetic aperture radar at L-band [Ulaby et al., 1987; Zebker et al., 1987] over agricultural terrain and bare soil surfaces indicate that the statistics of the copolarized phase difference depends on the target type and its conditions. Recent measurements of bare soil surfaces by polarimetric scatterometers show that the variance of the copolarized phase difference is a function of the roughness parameters and incidence angle but is less sensitive to moisture content [Sarabandi et al., 1991].

In view of difficulties in measuring the scattering matrix at high frequencies and performing controlled experiments, it is necessary to establish a relationship between the Mueller matrix and the statistics of the phase differences of the scattering matrix elements. In the next section we derive the probability density function of the copolarized and cross-polarized phase difference in terms of the Mueller matrix elements assuming that the scattering matrix elements are jointly Gaussian. Then the assumptions and final results are compared with the experimental data acquired by polarimetric scatterometers in section 3.

2. THEORETICAL DERIVATION OF PHASE DIFFERENCE STATISTICS

The polarimetric response of a point or distributed target can be obtained by simultaneously measuring both the amplitude and phase of the scattered field using two orthogonal channels. If the incident and scattered field vectors are decomposed into their horizontal and vertical components, the polarimetric response can be represented by the scattering matrix $\mathbf{S}$, which for plane wave illumination we can write

$$\mathbf{E}^s = \frac{e^{ikr}}{r}\begin{bmatrix} S_{vv} & S_{vh} \\ S_{hv} & S_{hh} \end{bmatrix}\mathbf{E}^i \quad (1)$$

where r is the distance from the radar to the center of the distributed target. It should be noted that in the backscattering case, reciprocity implies that $S_{vh} = S_{hv}$. Each element of the scattering matrix, in general, is a complex quantity characterized by an amplitude and a phase. When the radar illuminates a volume of a random medium or an area of a random surface, many point scatterers contribute to the total scattered energy received by the radar, and therefore each element of the scattering matrix may be represented by

$$S_{pq} = |S_{pq}|e^{i\phi_{pq}} = \sum_{n=1}^{N} |s_{pq}^n| e^{i\phi_{pq}^n} \qquad p,q = v,h \quad (2)$$

Here N is the total number of scatterers each having scattering amplitude $|s_{pq}^n|$ and phase ϕ_{pq}^n. It should be mentioned that the phase of each scatterer, as given in (2), includes a phase delay according to the location of the scatterer with respect to the center of the distributed target. Without loss of generality all multiple scattering over the surface or in the medium can be included in (2). Since the location of the scatterers within the illuminated area (volume) is random, the process describing the phasor s_{pq} is a Wiener process (random walk) [Davenport, 1970]. If N is large enough, application of the central limit theorem shows that the real and imaginary parts of the scattering matrix element S_{pq} are independent identically distributed zero mean Gaussian random variables. Equivalently, it can also be shown that $|S_{pq}|$ and ϕ_{pq} are Rayleigh and uniform independent random variables, respectively. The three elements of the scattering matrix, in general, can be viewed as a six-element random vector, and it is again reasonable to assume that the six components are jointly Gaussian.

Observation of polarimetric data for a variety of distributed targets such as bare soil surfaces and different kinds of vegetation-covered terrain all indicates that the cross-polarized component of the scattering matrix (S_{hv}) is statistically independent of the copolarized terms (S_{vv} and S_{hh}). Therefore the statistical behavior of S_{hv} can be obtained from a single parameter, namely, the variance (σ_c^2) of the real or imaginary part of $S_{hv} = X_5 + iX_6$; that is,

$$f_{X_5,X_6}(x_5,x_6) = \frac{1}{2\pi\sigma_c^2}\exp[-\frac{x_5^2+x_6^2}{2\sigma_c^2}]$$

or equivalently the joint density function $|S_{vh}|$ and ϕ_{vh} is

$$f_{|S_{vh}|,\phi_{vh}}(|s_{vh}|,\phi_{vh}) = \frac{1}{2\pi\sigma_c^2}|s_{hv}|\exp[-\frac{|s_{hv}|^2}{2\sigma_c^2}] \quad , \quad (3)$$

which indicates that ϕ_{vh} is uniformly distributed between $(-\pi,+\pi)$.

Since measurement of the absolute phase of the scattering matrix elements is very difficult, it is customary to factor out the phase of one of the copolarized terms, for example S_{vv}, and therefore the phase difference statistics are of concern as opposed to the absolute phases. Since S_{hv} is assumed to be independent of S_{vv} (not a necessary assumption) and both ϕ_{hv} and ϕ_{vv} are uniformly distributed, it can be easily shown that the cross-polarized phase difference $\phi_c = \phi_{vh} - \phi_{vv}$ is also uniformly distributed between $(-\pi, +\pi)$.

The copolarized elements of the scattering matrix, however, are dependent random variables which can be denoted by a four-component jointly Gaussian random vector $\mathbf{X}$. Let us define

$$S_{vv} = X_1 + iX_2, \qquad S_{hh} = X_3 + iX_4$$

and since $X_1, \cdots, X_4$ are Gaussian, their joint probability density function (pdf) can be fully determined by a 4×4 symmetric positive definite matrix known as covariance matrix Λ whose entries are given by [Davenport, 1979]

$$\lambda_{ij} = \lambda_{ji} = < X_i X_j > \qquad i,j \in \{1, \cdots, 4\} \quad .$$

The joint probability density function in terms of the covariance matrix takes the following form:

$$f_{\mathbf{X}}(x_1, \cdots, x_4) = \frac{1}{4\pi^2 |\Lambda|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\tilde{\mathbf{X}}\Lambda^{-1}\mathbf{X}\right] \quad , \qquad (4)$$

where $\tilde{\mathbf{X}}$ is a transpose of the column vector $\mathbf{X}$. To characterize the covariance matrix, the following observations are in order. First, it was shown that the real and imaginary parts of the scattering matrix elements are mutually independent and identically distributed zero mean random variables; therefore

$$\lambda_{11} = \lambda_{22} = < X_1^2 > = < X_2^2 > \quad , \qquad (5)$$

$$\lambda_{12} = < X_1 X_2 > = 0 \quad , \qquad (6)$$

$$\lambda_{33} = \lambda_{44} = < X_3^2 > = < X_4^2 > \quad , \qquad (7)$$

$$\lambda_{34} = < X_3 X_4 > = 0 \quad . \qquad (8)$$

Second, it was shown that the absolute phase ϕ_{pp} is uniformly distributed and is independent of $|S_{pp}|$. Thus the random variable $\phi_{vv} + \phi_{hh}$ is also uniformly distributed and is independent of $|S_{vv}||S_{hh}|$ from which it can be concluded that

$$\begin{aligned} < |S_{vv}||S_{hh}| \cos(\phi_{vv} + \phi_{hh}) > &= 0 \quad , \\ < |S_{vv}||S_{hh}| \sin(\phi_{vv} + \phi_{hh}) > &= 0 \quad . \end{aligned} \qquad (9)$$

In fact, the complex random variable $S_{vv}S_{hh}$ is obtained from a similar Wiener process which led to the random variables S_{vv} and S_{hh}. On the other hand

$$\begin{aligned} X_1X_3 - X_2X_4 &= |S_{vv}||S_{hh}| \cos(\phi_{vv} + \phi_{hh}) \quad , \\ X_1X_4 + X_2X_3 &= |S_{vv}||S_{hh}| \sin(\phi_{vv} + \phi_{hh}) \quad . \end{aligned} \qquad (10)$$

In view of (9) and (10) it can easily be seen that

$$\lambda_{13} = \lambda_{24} \quad , \qquad (11)$$

$$\lambda_{14} = -\lambda_{23} \quad . \qquad (12)$$

The properties derived for the entries of the covariance matrix, as given by (5)-(8), (11), and (12), indicate that there are only four unknowns left in the covariance matrix, namely λ_{11}, λ_{13}, λ_{14}, and λ_{33}, which can be obtained directly from the Mueller matrix of the target as will be shown next. The Mueller matrix relates the scattered wave Stokes vector to the incident wave Stokes vector by [van Zyl and Ulaby, 1990]

$$\mathbf{F}^s = \frac{1}{r^2}\mathbf{M}\mathbf{F}^i \quad ,$$

where $\mathbf{F}^{i,s}$ are the modified incident and scattered wave Stokes vector defined by

$$\mathbf{F} = \begin{bmatrix} |E_v|^2 \\ |E_h|^2 \\ 2\Re[E_v E_h^*] \\ 2\Im[E_v E_h^*] \end{bmatrix} \quad .$$

The Mueller matrix can be expressed in terms of the elements of the scattering matrix as follows [Ulaby et al., 1987]

$$\mathbf{M} = \begin{bmatrix} |S_{vv}|^2 & |S_{vh}|^2 & \Re[S_{vh}^* S_{vv}] \\ |S_{hv}|^2 & |S_{hh}|^2 & \Re[S_{hh}^* S_{hv}] \\ 2\Re[S_{vv}S_{hv}^*] & 2\Re[S_{hv}S_{hh}^*] & \Re[S_{vv}S_{hh}^* + S_{hv}S_{vh}^*] \\ 2\Im[S_{vv}S_{hv}^*] & 2\Im[S_{hv}S_{hh}^*] & \Im[S_{vv}S_{hh}^* + S_{hv}S_{vh}^*] \end{bmatrix}$$

$$\cdots \begin{matrix} -\Im[S_{vh}^* S_{vv}] \\ -\Im[S_{hh}^* S_{hv}] \\ -\Im[S_{vv}S_{hh}^* - S_{hv}S_{vh}^*] \\ \Re[S_{vv}S_{hh}^* - S_{hv}S_{vh}^*] \end{matrix} \Bigg]$$

In the case of a random medium we are dealing with a partially polarized scattered wave, and the quantity of interest is the ensemble-averaged Mueller matrix. Using the assumption that the copolarized and cross-polarized terms of the scattering matrix are independent and employing the properties given by (5)-(8), (11), and (12), the Mueller matrix in terms of the entries of the covariance matrix is given by

$$\mathcal{M} = <\mathbf{M}> = \begin{bmatrix} 2\lambda_{11} & 2\sigma_c^2 & 0 & 0 \\ 2\sigma_c^2 & 2\lambda_{33} & 0 & 0 \\ 0 & 0 & 2\lambda_{13} + 2\sigma_c^2 & 2\lambda_{14} \\ 0 & 0 & -2\lambda_{14} & 2\lambda_{13} - 2\sigma_c^2 \end{bmatrix} . \qquad (13)$$

Equation (13) provides enough equations to determine the unknown elements of the covariance matrix and variance of the cross-polarized component, i.e.,

$$\begin{aligned} \sigma_c^2 &= \frac{\mathcal{M}_{12}}{2} \quad , \\ \lambda_{11} &= \frac{\mathcal{M}_{11}}{2} \quad , & \lambda_{33} &= \frac{\mathcal{M}_{22}}{2} \\ \lambda_{13} &= \frac{\mathcal{M}_{33} + \mathcal{M}_{44}}{4} \quad , & \lambda_{14} &= \frac{\mathcal{M}_{34} - \mathcal{M}_{43}}{4} \quad . \end{aligned}$$

With the covariance matrix the joint pdf of $X_1, \cdots, X_4$ can be obtained as given by (4). Using a rectangular to polar transformation, i.e.,

$$\begin{aligned} x_1 &= \rho_1 \cos\phi_{vv} \quad , & x_2 &= \rho_1 \sin\phi_{vv} \quad , \\ x_3 &= \rho_2 \cos\phi_{hh} \quad , & x_4 &= \rho_2 \sin\phi_{hh} \quad , \end{aligned}$$

the joint pdf of the amplitudes and phases takes the following form

$$f_{\rho_1,\rho_2,\Phi_{vv},\Phi_{hh}}(\rho_1,\rho_2,\phi_{vv},\phi_{hh}) = \frac{1}{4\pi^2\sqrt{\Delta}}\rho_1\rho_2 \cdot \exp\left\{-\frac{1}{2}[a_1\rho_1^2 + a_2\rho_2^2 - 2a_3\rho_1\rho_2]\right\}, \quad (14)$$

where

$$\begin{aligned} \Delta &= \quad |\Lambda| = (\lambda_{11}\lambda_{33} - \lambda_{13}^2 - \lambda_{14}^2)^2, \\ a_1 &= \lambda_{33}/\sqrt{\Delta}, \qquad a_2 = \lambda_{11}/\sqrt{\Delta}, \\ a_3 &= [\lambda_{13}\cos(\phi_{hh} - \phi_{vv}) + \lambda_{14}\sin(\phi_{hh} - \phi_{vv})]/\sqrt{\Delta}. \end{aligned}$$

To obtain the copolarized phase difference statistics, the joint density function of ϕ_{vv} and ϕ_{hh} is needed which can be obtained from

$$f_{\Phi_{vv},\Phi_{hh}}(\phi_{vv},\phi_{hh}) = \int_0^\infty\int_0^\infty f_{\rho_1,\rho_2,\Phi_{vv},\Phi_{hh}}(\rho_1,\rho_2,\phi_{vv},\phi_{hh}) \, d\rho_1 d\rho_2. \quad (15)$$

Noting that a_1 is a positive real number, the integration with respect to ρ_1 can be carried out which results in

$$f_{\Phi_{vv},\Phi_{hh}}(\phi_{vv},\phi_{hh}) = \frac{1}{4\pi^2\sqrt{\Delta}}\left\{\frac{1}{a_1}\int_0^\infty \rho_2 e^{-\frac{a_2}{2}\rho_2^2}d\rho_2 + \sqrt{\frac{\pi}{8a_1^3}}a_3\int_0^\infty \rho_2^2\left[1 \pm \operatorname{erf}\left(\frac{|a_3|}{\sqrt{8a_1}}\rho_2\right)\right] e^{-\frac{1}{2a_1}(a_1a_2 - a_3^2)\rho_2^2}d\rho_2\right\}, \quad (16)$$

where erf( ) is the error function and the plus or minus sign is used according to the sign of a_3. To evaluate the integrals in (16), we need to show that both a_2 and $a_1a_2 - a_3^2$ are positive numbers. By definition, a_2 is positive, and to show $a_1a_2 - a_3^2$ is positive, we note that Λ is a symmetric positive definite matrix, therefore its eigenvalues must be positive. It can be shown that Λ has two distinct eigenvalues, γ_1 and γ_2, each with multiplicity 2, and their product is given by

$$\gamma_1\gamma_2 = \lambda_{11}\lambda_{33} - \lambda_{13}^2 - \lambda_{14}^2 > 0.$$

Thus

$$a_1a_2 - a_3^2 = \gamma_1\gamma_2 + [\lambda_{13}\cos(\phi_{hh} - \phi_{vv}) - \lambda_{14}\sin(\phi_{hh} - \phi_{vv})]^2$$

is positive. After integrating the first integral and the first term of the second integral in (16) directly and using integration by parts on the second term of the second integral, (16) becomes

$$f_{\Phi_{vv},\Phi_{hh}}(\phi_{vv},\phi_{hh}) = \frac{1}{4\pi^2\sqrt{\Delta}}\left\{\frac{1}{a_1a_2} + \frac{a_3^2}{a_1a_2(a_1a_2 - a_3^2)} + \frac{\sqrt{\pi}|a_3|}{\sqrt{8a_1}(a_1a_2 - a_3^2)}\int_0^\infty \operatorname{erf}\left(\frac{|a_3|}{\sqrt{8a_1}}\rho_2\right) e^{-\frac{1}{2a_1}(a_1a_2 - a_3^2)\rho_2^2}d\rho_2\right\}.$$

By expanding the error function in terms of its Taylor series, interchanging the order of summation and integration, and then using the definition of the gamma function, it can be shown that

$$\int_0^\infty \operatorname{erf}\left(\frac{|a_3|}{\sqrt{8a_1}}\rho_2\right)e^{-\frac{1}{2a_1}(a_1a_2 - a_3^2)\rho_2^2}d\rho_2 = \sqrt{\frac{2a_1}{\pi(a_1a_2 - a_3^2)}} \tan^{-1}\left(\frac{|a_3|}{2\sqrt{a_1a_2 - a_3^2}}\right).$$

The joint density function of ϕ_{vv} and ϕ_{hh} is a periodic function of $\phi = \phi_{hh} - \phi_{vv}$, and therefore the random variable ϕ, after some algebraic manipulation, can be shown to have the following pdf over the interval $(-\pi, +\pi)$

$$f_\Phi(\phi) = \frac{\lambda_{11}\lambda_{33} - \lambda_{13}^2 - \lambda_{14}^2}{2\pi(\lambda_{11}\lambda_{33} - D^2)}\left\{1 + \frac{D}{\sqrt{\lambda_{11}\lambda_{33} - D^2}}\left[\frac{\pi}{2} + \tan^{-1}\frac{D}{\sqrt{\lambda_{11}\lambda_{33} - D^2}}\right]\right\}, \quad (17)$$

where we recall that

$$D = \lambda_{13}\cos\phi + \lambda_{14}\sin\phi$$

and the elements of the covariance matrix in terms of the Mueller matrix elements are given by

$$\lambda_{11} = \frac{\mathcal{M}_{11}}{2}, \qquad \lambda_{33} = \frac{\mathcal{M}_{22}}{2},$$
$$\lambda_{13} = \frac{\mathcal{M}_{33} + \mathcal{M}_{44}}{4}, \qquad \lambda_{14} = \frac{\mathcal{M}_{34} - \mathcal{M}_{43}}{4}.$$

Some limiting cases can be considered in order to check the validity of (17). For example, when S_{vv} and S_{hh} are uncorrelated, then both λ_{13} and λ_{14} are zero for which $f_\Phi(\phi) = 1/(2\pi)$, as expected. Also, for the case of completely polarized scattered wave where S_{vv} and S_{hh} are completely correlated, the determinant of Λ is zero, and so $f_\Phi(\phi)$ is a delta function.

It is interesting to note that the pdf of the phase difference is only a function of two parameters defined by

$$\alpha = \sqrt{\frac{\lambda_{13}^2 + \lambda_{14}^2}{\lambda_{11}\lambda_{33}}}, \qquad \zeta = \tan^{-1}\frac{\lambda_{14}}{\lambda_{13}}$$

where α and ζ can vary from 0 to 1 and $-\pi$ to π, respectively. In fact, if the wave were completely polarized, ζ

would have been the phase difference between the copolarized terms. The parameter ζ will henceforth be referred to as the polarized phase difference. In terms of these parameters, (17) can be written as

$$f_\Phi(\phi)=\frac{1-\alpha^2}{2\pi\left[1-\alpha^2\cos^2(\phi-\zeta)\right]}\left\{1+\frac{\alpha\cos(\phi-\zeta)}{\sqrt{1-\alpha^2\cos^2(\phi-\zeta)}}\left[\frac{\pi}{2}+\tan^{-1}\frac{\alpha\cos(\phi-\zeta)}{\sqrt{1-\alpha^2\cos^2(\phi-\zeta)}}\right]\right\}. \quad (18)$$

It can be shown that the maximum of the pdf occurs at $\phi = \zeta$ independent of α. However, the width of the pdf (e.g., the 3-dB angular width) is only a function of α which will be referred to as the degree of correlation. The probability distribution function given by (18) is the analog of Gaussian distribution for periodic random variables where ζ and α are the counterparts of the mean and variance for Gaussian random variables, respectively. Figure 1 shows the pdf for different values of ζ while keeping α constant, and Figure 2 shows the pdf for a fixed value of ζ while changing α as a parameter. The calculated mean and standard deviation of the phase difference as a function of both the polarized phase difference and the degree of correlation are depicted in Figures 3 and 4, respectively.

Last, it is necessary to point out that the formulation of the copolarized phase difference pdf, as given in (17), is not restricted to the backscattering case or to the copolarized and cross-polarized components being uncorrelated. In fact, we can derive the cross-polarized phase difference statistics in a similar manner, and the pdf in this case for the backscattering case can be obtained from (17) upon the following substitution for the elements of the cross-polarized covariance matrix

$$\lambda_{11} = \frac{\mathcal{M}_{11}}{2}, \quad \lambda_{33} = \frac{\mathcal{M}_{12}}{2},$$
$$\lambda_{13} = \frac{\mathcal{M}_{13}}{2}, \quad \lambda_{14} = \frac{\mathcal{M}_{14}}{2}.$$

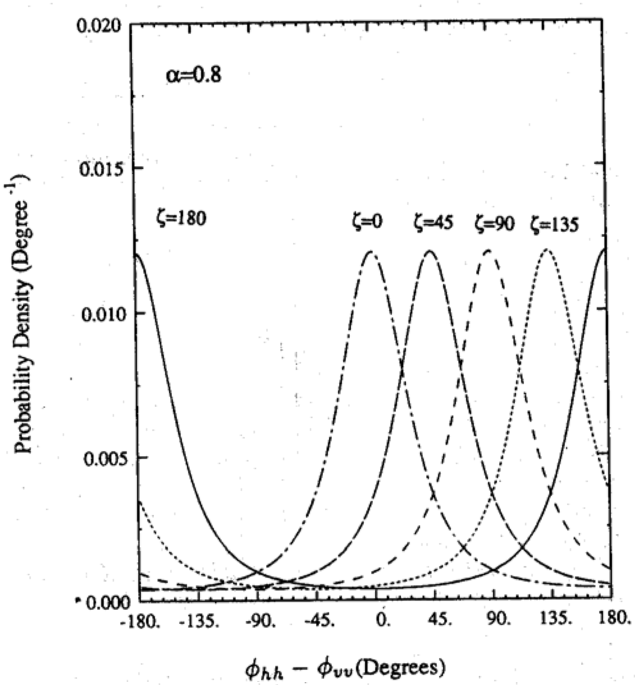

Fig. 1. The probability density function of the copolarized phase difference for a fixed value of α (degree of correlation) and five values of ζ (coherent phase difference).

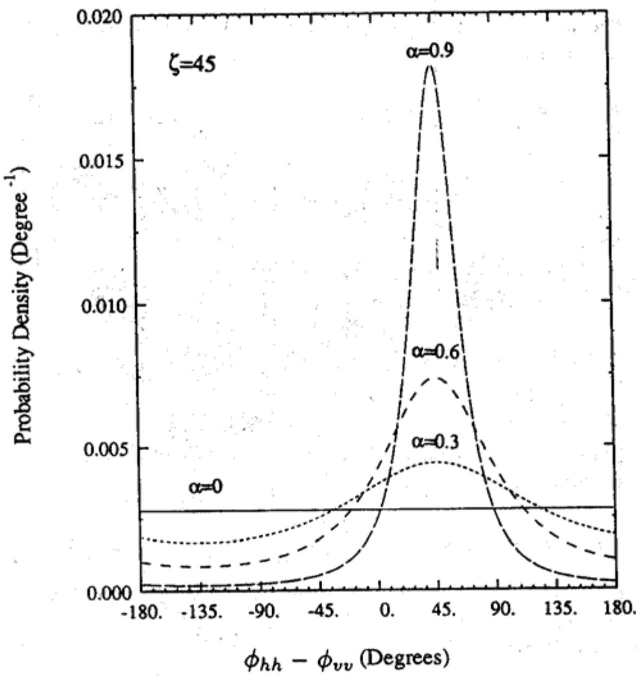

Fig. 2. The probability density function of the copolarized phase difference for a fixed value of ζ (coherent phase difference) and four values of α (degree of correlation).

3. COMPARISON WITH MEASUREMENTS

Using the polarimetric data gathered by scatterometers from a variety of natural targets, the assumptions leading to the pdf of phase differences as derived in the previous section are examined. Also, by generating the histograms, means, and standard deviations of the phase differences from the data and comparing them with the results based on the pdf derived from the measured Mueller matrices, validity of the model is also examined. The polarimetric radar measurements of bare soil surfaces were performed at L-, C-, and X-band frequencies for a total of eight different soil surface conditions (four roughness and two moisture conditions). For this experiment we tried to preserve the absolute phase of the measured scattering matrix by calibrating the surface data with a metallic sphere located

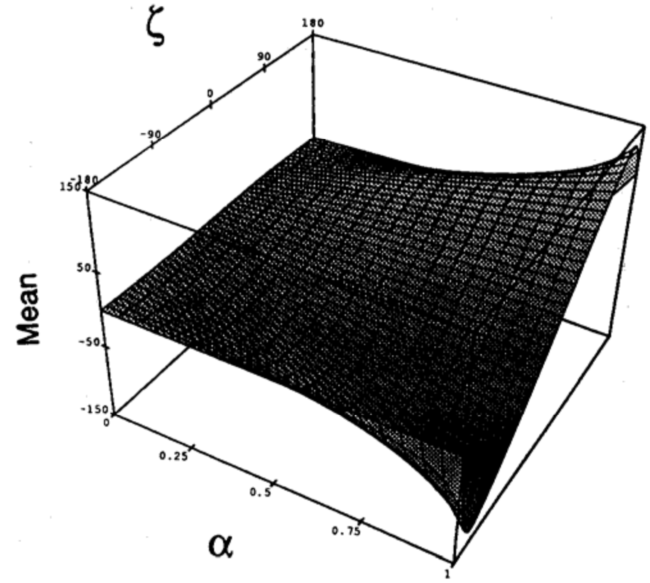

Fig. 3. The mean value of the copolarized phase difference as a function of α (degree of correlation) and ζ (coherent phase difference).

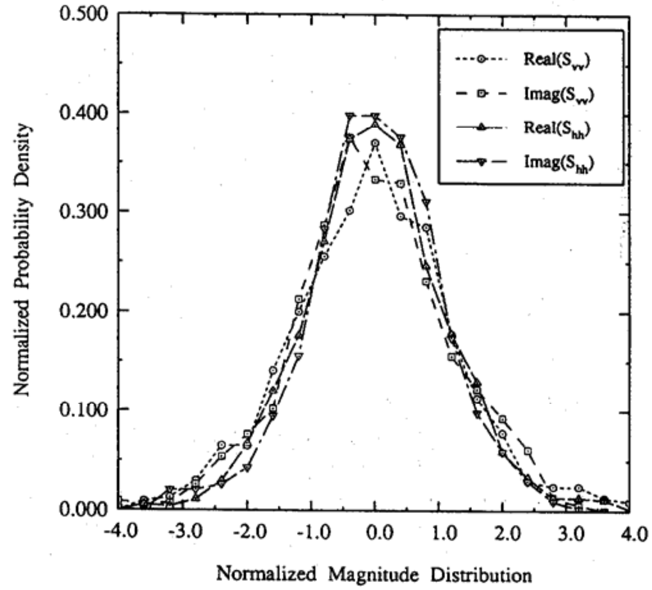

Fig. 5. The histogram of the real and imaginary parts of S_{vv} and S_{hh} for a rough surface with a rms height of 0.32 cm and a correlation length of 9.9 cm at C-band and at a 30° incidence angle.

at the same distance from the radar as the center of the surface target. For each frequency, surface condition, and incidence angle a minimum of 700 independent samples were collected. The detailed procedure of the data collection and calibration is given by Sarabandi et al. [1991].

By generating the histograms of the real and imaginary parts of the elements of the scattering matrix for all surfaces, it was found that they have a zero mean Gaussian distribution as we assumed. Figure 5 represents a typical case where the histogram of the real and imaginary

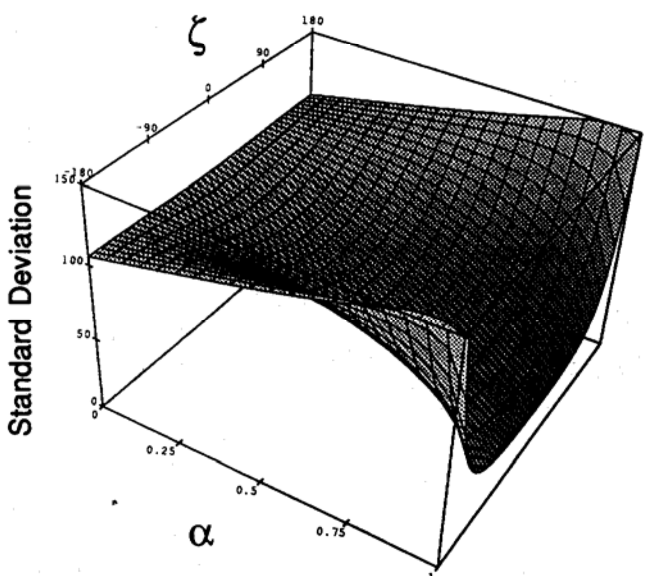

Fig. 4. The standard deviation of the copolarized phase difference as a function of α (degree of correlation) and ζ (coherent phase difference).

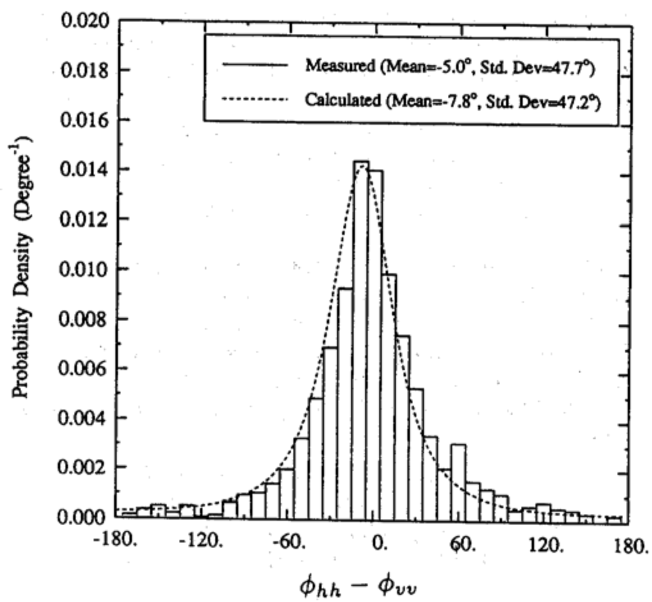

Fig. 6. The histogram and pdf of the copolarized phase difference for a rough surface with a rms height of 0.32 cm and a correlation length of 9.9 cm at C-band and at a 30° incidence angle.

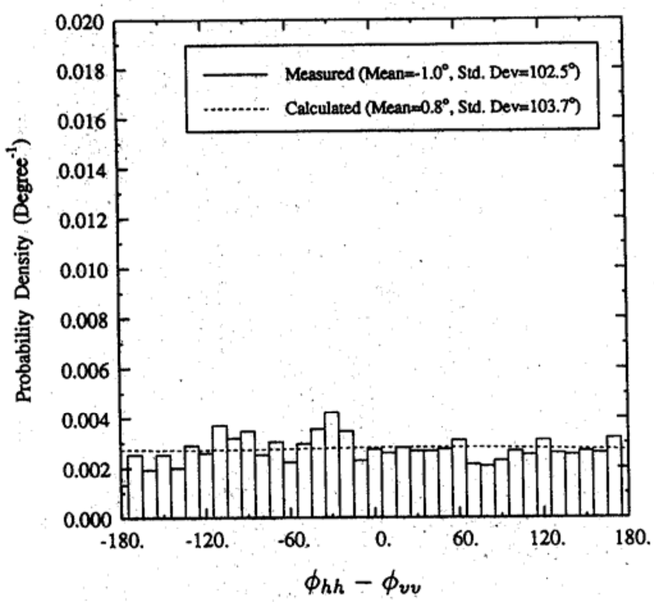

Fig. 7. The histogram and pdf of the cross-polarized phase difference for a rough surface with rms height of 0.32 cm and correlation length of 9.9 cm at C-band and at a 30° incidence angle.

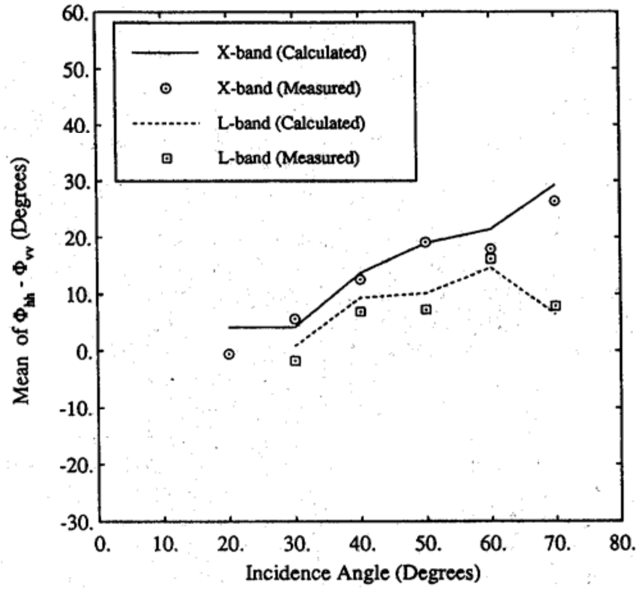

Fig. 8. Angular dependency of the mean of the copolarized phase difference for a dry rough surface with a rms height of 0.4 cm and a correlation length of 8.4 cm at L- and X-band.

parts of S_{vv} and S_{hh} of a dry surface with a rms height of 0.32 cm and correlation length of 9.9 cm at C-band has a bell-shaped distribution. The properties of the covariance matrix as given by (5)-(8) and (11)-(12), are verified by calculating the covariance matrices of the data for all cases. The normalized vocariance matrix of the surface with rms height .3 cm and correlation length 9 cm at C-band is given by

$$\Lambda = \begin{bmatrix} 1.00 & 0.03 & 0.75 & -0.12 \\ 0.03 & 0.90 & 0.08 & 0.68 \\ 0.75 & 0.08 & 0.77 & 0.05 \\ -0.12 & 0.68 & 0.05 & 0.69 \end{bmatrix}$$

where it possesses the mentioned properties approximately; that is, $\lambda_{11} \approx \lambda_{22}$, $\lambda_{12} \approx \lambda_{34} \approx 0$, $\lambda_{33} \approx \lambda_{44}$, $\lambda_{13} \approx \lambda_{24}$, and $\lambda_{14} \approx -\lambda_{23}$. The small discrepancies are due to the fact that the measurement of the scattering matrix with absolute phase has an uncertainty of $\pm 30°$.

The Mueller matrix of the typical surface at C-band is given by

$$\mathcal{M} = \begin{bmatrix} 1.000 & 0.030 & 0.000 & 0.000 \\ 0.028 & 0.767 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.770 & -0.11 \\ 0.000 & 0.000 & 0.110 & 0.711 \end{bmatrix}$$

from which the copolarized and cross-polarized phase difference pdfs are calculated using (17) and are compared with the measured phase histograms in Figures 6 and 7, respectively. Similar comparisons were also made for the rest of surfaces, frequencies, and incidence angles, and it was found that the expression (17) predicts the density functions very accurately. Some examples of these comparisons are shown in Figures 8 and 9. Figures 8 and 9 compare

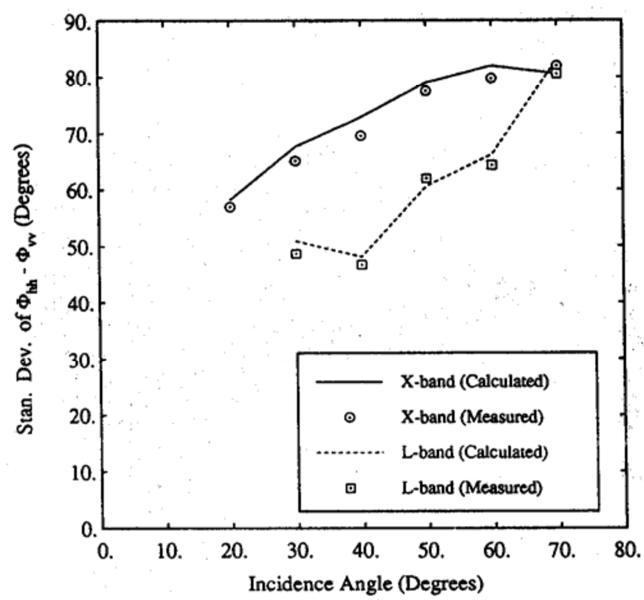

Fig. 9. Angular dependency of the standard deviation of the copolarized phase difference for a dry rough surface with a rms height of 0.4 cm and a correlation length of 8.4 cm at L- and X-band.

the mean and standard deviation of the copolarized phase difference versus incidence angle at L- and X-band for a surface with rms height of 0.4 cm and correlation length of 8.4 cm in dry conditions using the results based on the direct calculation and the results derived from (17).

4. CONCLUSIONS

Prompted by the experimental observations which show strong dependence of phase differences of the scattering matrix elements on the physical parameters of random media, the statistical behavior of the phase differences for distributed targets is studied. The pdfs of the phase differences are derived from the Mueller matrix of the target. In derivation of the density functions it is assumed that the real and imaginary parts of the copolarized and cross-polarized terms of the scattering matrix are jointly Gaussian and their covariance matrices are found in terms of the Mueller matrix elements. The functional forms of the copolarized and cross-polarized density functions are similar and are obtained independently. It is shown that the density function of the phase difference is completely determined in terms of only two parameters. The assumptions and final expressions are verified by using a set of polarimetric data acquired by scatterometers from rough surfaces.

Acknowledgment. This research was supported by NASA grant NAGW-2151 and ARO contract DAAL 03-91-G0202.

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K. Sarabandi, Radiation Laboratory, 3225 EECS Bldg., University of Michigan, Ann Arbor, MI 48109-2111.