A Scattering Model for Thin Dielectric Cylinders of Arbitrary Cross Section and Electrical Length

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***Abstract*—A scattering solution for long, thin, dielectric cylinders of arbitrary cross section and electrical length is presented. The infinite-cylinder scattering formulation is shown to be an asymptotic solution for the finite-cylinder case, regardless of cylinder electrical length or cross section. The generalized Rayleigh–Gans (GRG) approximation for circular cylinders is shown to be a specific case of this general formulation, and therefore, the assertions of GRG are explicitly proven. A moment-method (MM) solution for thin circular cylinders is likewise presented and is used to examine and quantify the asymptotic errors associated with this solution.**

I. Introduction

In activities such as radar remote sensing, accurate scattering models of elemental constituents are essential in constructing robust scattering models of random media such as vegetation. This challenge is often compounded by the arbitrary and complex nature of these constituent elements. For example, a type of element often encountered are long, thin dielectric cylinders of arbitrary cross section, including grasses and needle structures. In the microwave region, the radius of these cylinders are usually very small compared to the incident wavelength, whereas the electrical length may take any value. This generality in structure precludes the implementation of specific scattering solutions. The arbitrary value of electrical length $k\ell$ eliminates asymptotic solutions such as Rayleigh ($k\ell \ll 1$) or physical optics ($k\ell \gg 1$), and the generally noncanonical cross sections leave inapplicable solutions for circular and elliptical structures. Thus, a scattering solution is required which accurately comprehends these arbitrary particles.

One relevant analysis is that of Sarabandi and Senior [1], who explicitly derived the scattering solution of an electrically thin, but infinitely long, dielectric cylinder of arbitrary cross section. This work provides a general solution for the internal electric fields and demonstrates that the far-field scattering can be expressed in terms of a dipole moment per unit length. Using the high-frequency approximation, the scattering from a finite, but electrically long, ($k\ell \gg 1$) cylinder can be approximated by truncating the solution of the equivalent infinite length case. Although this solution is correct for arbitrary cross sections, its validity can apparently be justified only for cylinders of large electrical length $k\ell$.

A solution which is often employed to model circular cylinders of smaller $k\ell$ is the generalized Rayleigh–Gans approximation (GRG) introduced by Schiffer and Thielheim [2]. In this approximation, terms of the Borne (or Rayleigh–Gans) approximation are modified by the Rayleigh solution of a long, thin, spheroidal particle. The GRG approximation is said to be valid for electrically small, circular dielectric cylinders, provided that their normalized length ℓ/a is very large. No constraint is explicitly placed on electrical length $k\ell$. The GRG approximation was presented by first hypothesizing the solution and then successfully comparing the results to the asymptotic solutions known for both the long ($k\ell \ll 1$) and short ($k\ell \gg 1$) wavelength cases. On this basis, it was inferred that GRG validity is independent of electrical length. Whereas this presentation provides evidence as to the accuracy of the GRG approximation, it does not prove its general validity; the scattering from objects with dimensions on the order of a wavelength is often quite different from either the short or long wavelength cases. In addition, the analysis does not address the issue of cylinder cross section, only circular cylinders were considered.

In this paper, a scattering solution for the general case of an electrically-thin dielectric cylinder of arbitrary cross section and electrical length will be presented. The solution will be explicitly shown to be the unique asymptotic solution to the scattering problem as the electrical radius ka converges to zero. A moment-method (MM) solution will likewise be implemented to quantify the convergence of this asymptotic solution.

II. An Analysis of Thin Cylinder Scattering

Consider an infinite length dielectric cylinder lying along the z-axis. This cylinder is illuminated by a uniform plane wave $\boldsymbol{E}^i(\overline{r}) = \hat{e}\, e^{ik_0\hat{k}^i\cdot\overline{r}}$ where $\hat{e}\cdot\hat{k}^i = 0, \hat{e} = e^i_x\hat{x} + e^i_y\hat{y} + e^i_z\hat{z}$, and $\hat{k}^i$ is the propagation direction vector, $\hat{k}^i = \sin\beta\cos\phi\ \hat{x} + \sin\beta\sin\phi\ \hat{y} + \cos\beta\ \hat{z}$. As the electrical radius of the cylinder approaches zero ($ka \to 0$), the total electric field in the interior of the cylinder is given by Sarabandi and Senior [1] as

$$\boldsymbol{E}(\overline{r}) = (-e^i_x\ \nabla\Phi_1(\overline{\rho}) - e^i_y\ \nabla\Phi_2(\overline{\rho}) + e^i_z\ \hat{z})e^{ik_0\cos\beta z} \quad (1)$$

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where $\Phi_1(\overline{\rho})$ and $\Phi_2(\overline{\rho})$ are electrostatic potentials specified as the solution to the following integral equations [1]

$$\Phi_1(\overline{\rho}) + x + b_1 = \frac{(\epsilon_r - 1)}{2\pi} \int_A \nabla_t' \Phi_1(\overline{\rho}') \cdot \nabla_t' \ln |\overline{\rho} - \overline{\rho}'| \, d_A'$$
$$\Phi_2(\overline{\rho}) + y + b_2 = \frac{(\epsilon_r - 1)}{2\pi} \int_A \nabla_t' \Phi_2(\overline{\rho}') \cdot \nabla_t' \ln |\overline{\rho} - \overline{\rho}'| \, d_A'. \quad (2)$$

The constants b_1 and b_2 are arbitrary, and A defines the area of the cylinder cross section. Using the physical optics approximation, this solution can likewise be applied to finite cylinders, provided its electrical length $k\ell$ is sufficiently large. From this formulation, the scattering solution for a thin cylinder of circular cross section can be determined

$$\boldsymbol{E}(\overline{r}) = \left(\frac{2e_x^i}{\epsilon_r + 1}\hat{x} + \frac{2e_y^i}{\epsilon_r + 1}\hat{y} + e_z^i \hat{z} \right) e^{\imath k_0 \cos\beta z}. \quad (3)$$

This solution, derived from the infinite cylinder formulation, is identical to that provided by the GRG approximation. Thus, for a circular cylinder the two solutions resulting from each formulation are in agreement. However, their general validity regions are in conflict. The GRG approximation claims that the above expression is valid for all electrical lengths $k\ell$, whereas the infinite cylinder approximation has only a high-frequency justification of $k\ell \gg 1$. The requirement for the normalized cylinder length ($\ell/a \gg 1$) is implied in the infinite cylinder approximation (since $ka \ll 1$, then $k\ell/ka \gg 1$), and explicitly required by GRG.

If the assertions of the GRG approximation are correct, it suggests that the validity limits placed on the truncated infinite cylinder solution are too restrictive. That is, in addition to the high-frequency limit ($k\ell \gg 1$), the infinite-cylinder solution could likewise be applied to finite cylinders with electrical lengths in the resonance ($kl \approx 1$) and Rayleigh ($kl \ll 1$) regions. However, given the heuristic nature of the GRG approximation, this is strictly conjecture, particularly with regard to noncircular cross sections. Thus, we seek to determine under what conditions (1) and (2) define a valid scattering solution for thin, finite dielectric cylinders. Are they valid only for electrically-long cylinders, or does the validity extend to cylinders of other $k\ell$? If so, is this true only for circular cylinders, or is the solution generally valid for all cross sections?

If a formulation $\boldsymbol{E}(\overline{r})$ is a valid electromagnetic solution, then it will uniquely satisfy the integral equation which describes the scattering problem, $\boldsymbol{E}(\overline{r}) = \boldsymbol{E}^i(\overline{r}) + \boldsymbol{E}^s(\overline{r})$, where the scattered field $\boldsymbol{E}^s(\overline{r})$ is given as

$$\boldsymbol{E}^s(\overline{r}) = [k_0^2 + \nabla\nabla \cdot] \int_V (\epsilon_r - 1)\boldsymbol{E}(\overline{r}') g_0(|\overline{r} - \overline{r}'|) \, dv' \quad (4)$$

and $g_0(|\overline{r} - \overline{r}'|)$ is the free space Green's function. For a given type of scatterer (e.g., thin cylinders), a function $\boldsymbol{E}(\overline{r})$ may in general satisfy the integral equation, or perhaps satisfy only under specific conditions, such as a circular cross section or infinite electrical length. Therefore, to determine the validity of the truncated infinite-cylinder solution, (1) and (2) will be inserted into the integral equation for a thin finite cylinder and evaluated. The conditions under which the integral equation is satisfied will then be determined, thus defining the validity regions of this solution.

A. Transverse Components

Since (1) is a superposition of three terms, each proportional to a single component of the incident electric field vector (e_x^i, e_y^i, e_z^i), each term must individually satisfy the integral equation in order for the total solution to be valid. We first examine the transverse term proportional to e_x^i, given as

$$\boldsymbol{E}(\overline{r}) = -e_x^i \nabla \Phi_1(\overline{\rho}) e^{\imath k_0 \cos\beta z}. \quad (5)$$

Inserting (1) into (4), evaluating the integrals and making the substitution $\overline{kr} = k_0 \overline{r}$, the scattered field $\boldsymbol{E}^s(\overline{r})$ is given as

$$\begin{aligned}\boldsymbol{E}^s(\overline{kr}/k_0) = &-e_x^i \frac{(\epsilon_r - 1)}{4\pi} \int_{k^2 A} \nabla_{kt} \Phi_1(\overline{k\rho}'/k_0) \\ &\cdot \int_{k\ell} \frac{e^{\imath kz' \cos\beta} e^{\imath |\overline{kr} - \overline{kr}'|}}{|\overline{kr} - \overline{kr}'|} \, dkz' \, dk_A^{2'} \\ &+ e_x^i \frac{\imath(\epsilon_r - 1)}{4\pi} \int_{kC} \hat{n}' \cdot \nabla_{kt} \Phi_1(\overline{k\rho}'/k_0) \\ &\cdot \int_{k\ell} \frac{e^{\imath kz' \cos\beta} e^{\imath |\overline{kr} - \overline{kr}'|}}{|\overline{kr} - \overline{kr}'|^2} (\overline{kr} - \overline{kr}') \, dkz' \, dkc' \\ &- e_x^i \frac{(\epsilon_r - 1)}{4\pi} \int_{kC} \hat{n}' \cdot \nabla_{kt} \Phi_1(\overline{k\rho}'/k_0) \\ &\cdot \int_{k\ell} \frac{e^{\imath kz' \cos\beta} e^{\imath |\overline{kr} - \overline{kr}'|}}{|\overline{kr} - \overline{kr}'|^3} (\overline{kr} - \overline{kr}') \, dkz' \, dkc'. \end{aligned} \quad (6)$$

The electrical length $k\ell$ is finite but otherwise arbitrary, kc defines the outer contour of the arbitrary cylinder cross section, and $k^2 A$ similarly defines cross section area. However, since the inserted solution (1) is valid in the limit as electrical radius ka approaches zero, this constraint must likewise be placed on the above equation. Therefore, we seek to evaluate the above integral in the limit as ka approaches zero ($\lim_{ka \to 0} \boldsymbol{E}^s(\overline{kr}/k_0)$).

Each of the three terms in (6) contain an explicit integral of kz', but none can be directly evaluated. However, since the integration is over a finite region $k\ell$, the exponential term can be approximated in the region $-k\ell/2 < kz < k\ell/2$ as its Taylor series expansion

$$e^{\imath |\overline{kr} - \overline{kr}'| + \imath kz' \cos\beta} \approx \sum_{n=0}^{N} \frac{(\imath |\overline{kr} - \overline{kr}'| + \imath kz' \cos\beta)^n}{n!} \quad (7)$$

where N is arbitrarily large. Inserting this series into the kz' integrals, the order of integration and summation can be interchanged, since both N and $k\ell$ are finite. The integration of each term can now be directly evaluated, resulting in a series whose coefficients are in terms of $|\overline{k\rho} - \overline{k\rho}'|$. For example, the

kz' integral from the first term of (6) can be approximated as

$$\begin{aligned}
&\int_{k\ell} \frac{e^{\imath|\overline{kr}-\overline{kr}'|+\imath kz'\cos\beta}}{|\overline{kr}-\overline{kr}'|}\, dkz' \\
&= \sum_{n=0}^{N} \int_{k\ell} \frac{(\imath|\overline{kr}-\overline{kr}'| + \imath kz'\cos\beta)^n\, dkz'}{n!|\overline{kr}-\overline{kr}'|} \\
&\approx e^{\imath kz\cos\beta}(\mathrm{Ei}[\imath c_1(kl/2 - kz)] \\
&\quad + \mathrm{Ei}[\imath c_2(kl/2+kz)] - 2(\gamma + \imath\pi/2) \\
&\quad - 2\ln[\sin\beta|\overline{k\rho}-\overline{k\rho}'|/2]) + O(|\overline{k\rho}-\overline{k\rho}'|) \qquad (8)
\end{aligned}$$

where γ is Euler's constant, $c_1 = 1+\cos\beta, c_2 = 1-\cos\beta$ and $\mathrm{Ei}[x]$ is the exponential integral function defined as $\mathrm{Ei}[x] = -\int_{-x}^{\infty}(e^-t/t)\ dt$. Discarding the higher order elements of $O(|\overline{k\rho}-\overline{k\rho}'|)$, (8) provides an accurate approximation to the integral, providing $1 \gg ka > |\overline{k\rho}-\overline{k\rho}'|$. Fig. 1 graphically displays this, showing both the approximation and a numerical solution to a representative integral.

Similar approximations can be determined for the remaining two kz' integrals. Inserting these into (6), the scattered field expression can now be evaluated in the limit as electrical radius ka approaches zero. The first two terms of (6) vanish, but the third terms remains nonzero, and the scattered field reduces to

$$\begin{aligned}
&\lim_{ka\to 0} \boldsymbol{E}^s(\overline{kr}/k_0) \\
&= -e_x^i \frac{(\epsilon_r - 1)}{4\pi} e^{\imath kz\cos\beta} \int_{kc} \hat{n}' \\
&\cdot \nabla'_{kt}\Phi_1(\overline{k\rho}'/k_0) \frac{2(\overline{k\rho}-\overline{k\rho}')}{|\overline{k\rho}-\overline{k\rho}'|^2}\, dkc' \\
&= -e_x^i \frac{(\epsilon_r-1)}{2\pi} e^{\imath k_0 z\cos\beta}\nabla_t \int_A \nabla'_t\Phi_1(\overline{\rho}') \\
&\cdot \nabla'_t \ln|\overline{\rho}-\overline{\rho}'|\, d'_A. \qquad (9)
\end{aligned}$$

The task remaining is to therefore evaluate the integral over dA. Recall that the potential $\Phi_1(\overline{\rho})$ is not arbitrary, but is the unique solution to the integral (2). Notice the integral appearing in both (2) and (9) are identical, and from (2) is given as

$$\int_A \nabla'_t\Phi_1(\overline{\rho}') \cdot \nabla'_t \ln|\overline{\rho}-\overline{\rho}'|\, d'_A = \frac{2\pi}{(\epsilon_r - 1)}(\Phi_1(\overline{\rho}) + x + b_1). \qquad (10)$$

This condition is now enforced by replacing the integral in (9) with equation (10). The scattered field reduces to

$$\begin{aligned}
\boldsymbol{E}^s(\overline{r}) &= -e_x^i \nabla_t(\Phi_1(\overline{\rho}) + x + b_1)e^{\imath k_0 z\cos\beta} \\
&= -e_x^i(\nabla_t\Phi_1(\overline{\rho}) + \hat{x})e^{\imath k_0 z\cos\beta} \qquad (11)
\end{aligned}$$

and the original integral equation is therefore

$$\begin{aligned}
\boldsymbol{E}(\overline{r}) &= \boldsymbol{E}^i(\overline{r}) + \boldsymbol{E}^s(\overline{r}) \\
&= e_x^i e^{\imath k_0 z\cos\beta}\hat{x} - e_x^i(\nabla_t\Phi_1(\overline{\rho}) + \hat{x})e^{\imath k_0 z\cos\beta} \\
&= -e_x^i\nabla_t\Phi_1(\overline{\rho}) = \boldsymbol{E}(\overline{r}). \qquad (12)
\end{aligned}$$

Insertion of the second transverse term $-e_y^i\nabla\Phi_2(\overline{\rho})$ leads to an identical evaluation and result. Thus, the transverse terms of (1) and (2) asymptotically satisfy the integral equation as ka approaches zero.

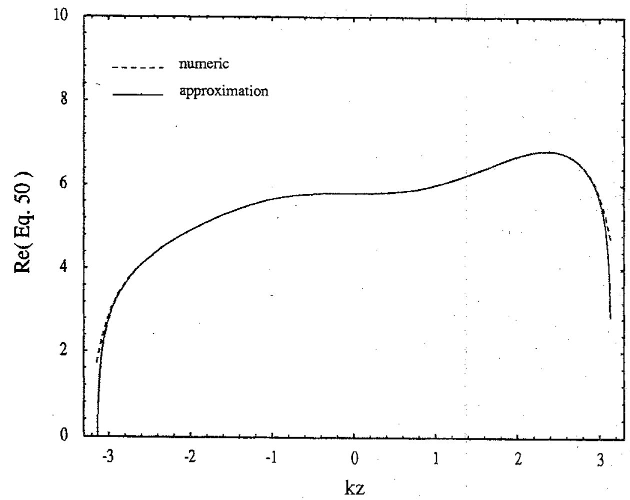

Fig. 1. The real part of (8), both the exact numerical evaluation and the analytic approximation ($ka = 0.1, k\ell = 2\pi, \epsilon_r = 10$).

B. Axial Solution

The remaining component of (1) is the axial, z-directed term given as

$$\boldsymbol{E}(\overline{r}) = \hat{z}e_z^i e^{\imath k_0\cos\beta z}. \qquad (13)$$

Again, inserting this equation into (4), the scattered electric field can be expressed as

$$\begin{aligned}
&\boldsymbol{E}^s(\overline{kr}/k_0) = \\
&e_z^i \frac{(\epsilon_r-1)}{4\pi}\int_{k^2A}\int_{k\ell} \frac{e^{\imath kz'\cos\beta}e^{\imath|\overline{kr}-\overline{kr}'|}}{|\overline{kr}-\overline{kr}'|}\, dkz'\, dk_A^{2'} \\
&- e_z^i \frac{\imath(\epsilon_r-1)}{4\pi}\int_{k^2A} \left.\frac{e^{\imath kz'\cos\beta}e^{\imath|\overline{kr}-\overline{kr}'|}}{|\overline{kr}-\overline{kr}'|^2}\right|_{-k\ell/2}^{k\ell/2} \\
&\cdot(\overline{kr}-\overline{kr}')\, dk_A^{2'} \\
&+ e_z^i \frac{(\epsilon_r-1)}{4\pi}\int_{k^2A} \left.\frac{e^{\imath kz'\cos\beta}e^{\imath|\overline{kr}-\overline{kr}'|}}{|\overline{kr}-\overline{kr}'|^3}\right|_{-k\ell/2}^{k\ell/2} \\
&\cdot(\overline{kr}-\overline{kr}')\, dk_A^{2'} \\
&- e_z^i \frac{\cos\beta(\epsilon_r-1)}{4\pi}\int_{k^2A}\int_{-k\ell/2}^{k\ell/2} \frac{e^{\imath kz'\cos\beta}e^{\imath|\overline{kr}-\overline{kr}'|}}{|\overline{kr}-\overline{kr}'|^2} \\
&\cdot(\overline{kr}-\overline{kr}')\, dkz'\, dk_A^{2'} \\
&- e_z^i \frac{\imath\cos\beta(\epsilon_r-1)}{4\pi}\int_{k^2A}\int_{-k\ell/2}^{k\ell/2} \frac{e^{\imath kz'\cos\beta}e^{\imath|\overline{kr}-\overline{kr}'|}}{|\overline{kr}-\overline{kr}'|^3} \\
&\cdot(\overline{kr}-\overline{kr}')\, dkz'\, dk_A^{2'}. \qquad (14)
\end{aligned}$$

The integrals involving kz' are the same as those encountered for the transverse case, therefore, their approximations can again be implemented in (14). As in the previous section, the limit for each term of (14) is determined as $ka \to 0$. However, we find that for the axial case, every term vanishes, not a single nonzero term remains. The scattered field is, therefore, approximately zero, and the integral equation reduces to $\boldsymbol{E}(\overline{r}) = \boldsymbol{E}^i(\overline{r})$. Since both $\boldsymbol{E}(\overline{r})$ and $\boldsymbol{E}^i(\overline{r})$ are equal to $e_z^i e^{\imath k_0 \cos\beta z}\hat{z}$, this equation is satisfied.

Thus, it has been explicitly demonstrated that the formulation of (1) and (2) obtained from the infinite cylinder solution, asymptotically satisfy the scattering-integral equation for a finite-dielectric cylinder as $ka \to 0$. It should be noted that nowhere in the preceding analysis was any specific condition or constraint required to satisfy the integral equation. No restriction or assumption was placed on cross section or electrical length. Thus, the infinite cylinder formulation is a valid scattering solution for electrically thin, finite-dielectric cylinders of all cross sections and electrical lengths. However, the solution is only asymptotically valid as the electrical radius approaches zero. Since kl is a fixed constant, as $ka \to 0$, the ratio $ka/kl = a/l$ likewise approaches zero. Therefore, ka must not only be numerically small ($ka \ll 1$), but small compared to the electrical length, as well $ka \ll k\ell$. To satisfy this last constraint, the normalized length ℓ/a is required to be large.

Finally, since this formulation is independent of $k\ell$, it is valid for Rayleigh cylinders where $k\ell \ll 1$. Thus, the electrostatic solutions $\Phi(\overline{\rho})$, derived for infinite cylinders, are likewise the asymptotic solutions for a Rayleigh cylinder as $\ell/a \to \infty$. The GRG approximation which considers circular cylinders is, therefore, a specific case of the more general approximation defined by (1) and (2). As such, the validity regions of the GRG approximation, being identical to the requirements stated above ($ka \ll 1, ka \ll k\ell$ for all $k\ell$), have been explicitly proven by the analysis of this section.

III. Asymptotic Error Evaluation

As this solution is asymptotic, it will exhibit a finite error which becomes diminishingly small as $ka \to 0$. To evaluate the asymptotic error of (1) and (2), a moment-method (MM) solution was constructed to evaluate the scattering from a thin, circular, dielectric cylinder. It was assumed that the electric field in the cylinder is dependent on axial position z only; that is, the fields are constant with respect to the transverse dimension ρ. The interior field is therefore described as

$$\boldsymbol{E}(\overline{r}) = e_x \frac{2\, f_x(z)}{(\epsilon_r + 1)}\hat{x} + e_y \frac{2\, f_y(z)}{(\epsilon_r + 1)}\hat{y} + f_z(z)\hat{z} \qquad (15)$$

where the expressions $f_x(z), f_y(z)$, and $f_z(z)$ are unknown complex scalar functions. Comparing the above equation with (3), the values of $f_w(z)(w \in \{x, y, z\})$ predicted by the asymptotic scattering solution are $f_w(z) = e^{\imath k_0 \cos\beta z}$, so that $|f_w(z)| = 1.0$ for all k_0, β, and z.

The ability of the MM solution to accurately reflect the exact scattering solution depends on the general validity of (15). To test this accuracy, the moment method code was applied to a circular Rayleigh cylinder ($k\ell \gg 1$) at a variety of dielectrics and normalized lengths ℓ/a. The results were used to determine the polarizability tensor elements for each cylinder, and were then compared to the known values for circular cylinders [5]. These results are presented in Fig. 2.

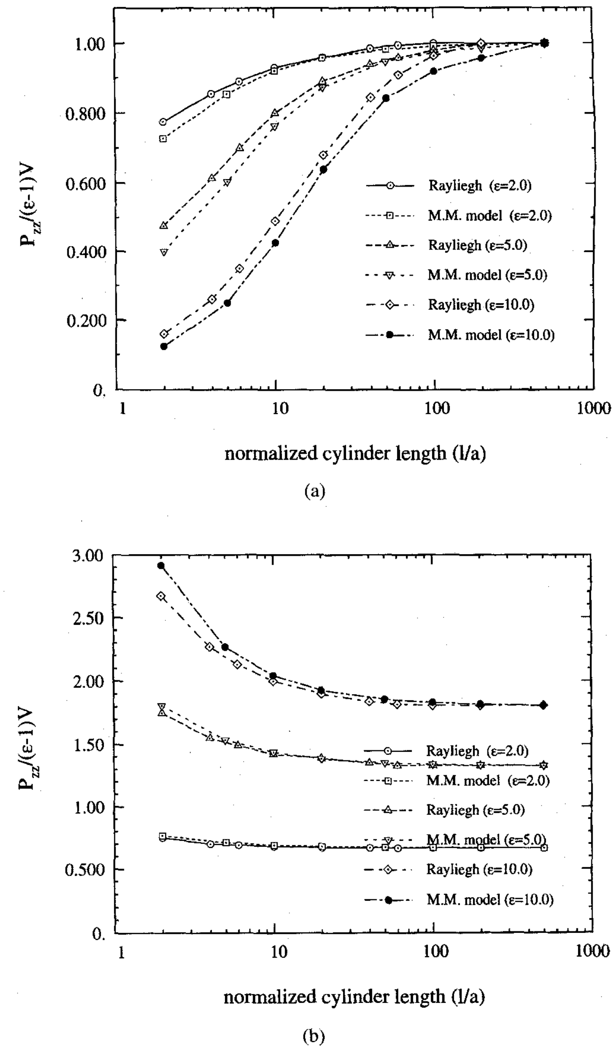

Fig. 2. (a) The axial and (b) transverse polarizability tensor elements of a circular cylinder, both the exact values and those determined using the MM model.

The MM solution matches the Rayleigh values well over all dielectrics and normalized lengths ℓ/a, thus providing evidence as to the accuracy of (15).

The MM solution was first used to evaluate the scattering from a thin cylinder with a large normalized length $\ell/a = 200$. The magnitude of $f(z)$ was determined at each point z along the cylinder for various electrical lengths. The results are given by Fig. 3, and show that the asymptotic solution ($|f_w(z)| = 1.0$) is valid at all points along the cylinder except for small regions near the cylinder ends. As expected, this is true regardless of electrical length $k\ell$. The error at the cylinder ends is likewise independent of $k\ell$, but is more pronounced for the axial component $f_z(z)$.

The analysis was then reversed, fixing the electrical length $k\ell = \pi/2$ and evaluating the MM solution at various normalized lengths ℓ/a. In contrast to $k\ell$, the scattering solution exhibits a strong dependence as a function of ℓ/a (Fig. 4). The result is a confirmation of the requirement that the

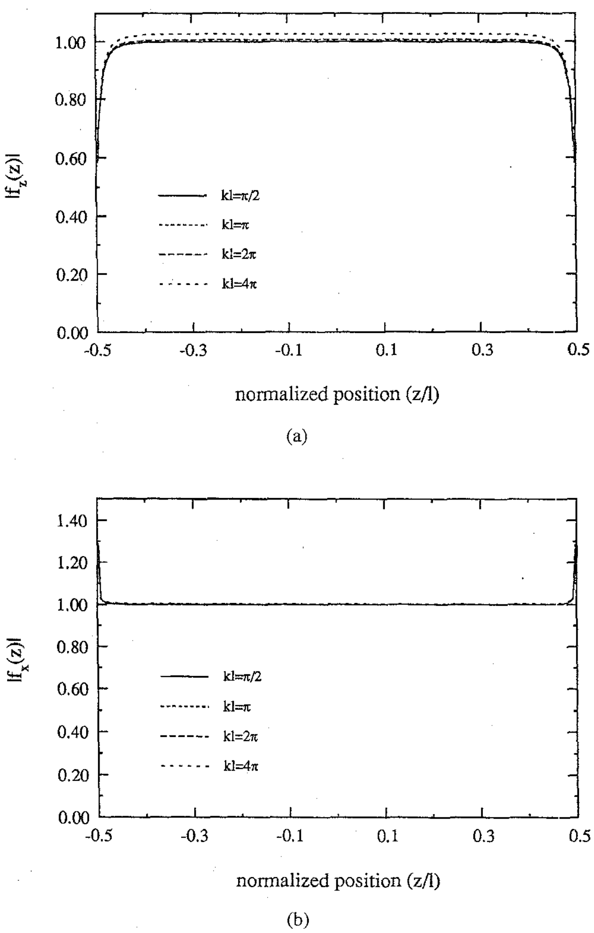

Fig. 3. (a) The axial and (b) transverse magnitude of the internal electric field in a thin circular cylinder for various electrical lengths ($\ell/a = 200, \epsilon_r = 5.0, \beta = \pi/2$).

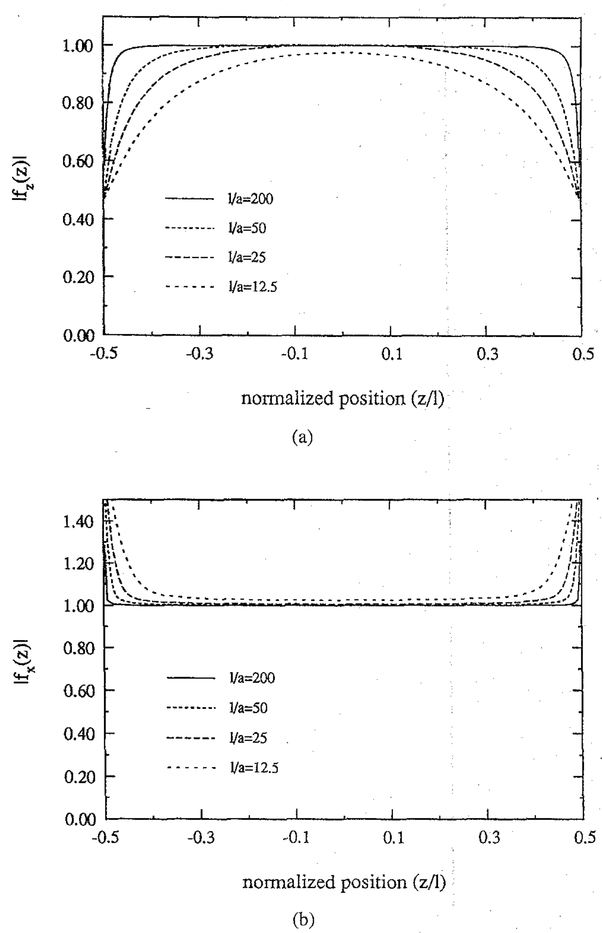

Fig. 4. (a) The axial and (b) transverse magnitude of the internal electric field in a thin circular cylinder for various normalized lengths ($k\ell = \pi/2, \epsilon_r = 5.0, \beta = \pi/2$).

normalized length ℓ/a be large (i.e., $ka \ll kl$) to ensure a valid approximation. As the normalized length becomes smaller, the MM solution greatly diverges from the asymptotic approximation of $|f(z)| = 1$. The error at the cylinder end expands as ℓ/a is reduced, occupying an increasingly greater portion of total cylinder length. Eventually, the formulation of (1) and (2) no longer provide an accurate approximation to the actual electric field $\boldsymbol{E}(\bar{r})$. Conversely, as ℓ/a increases, the error region will become diminishingly small.

To further examine its performance, the accuracy of the solution is examined as a function of both incidence angle and dielectric constant. Fig. 5 displays the MM solution calculated for an oblique incidence angle ($\beta = \pi/8$). Although the solution $f(z)$ is dependent on incidence angle β, almost no sensitivity to this parameter was detected in regard to approximation accuracy; the error regions at the ends of the cylinder remain constant regardless of incidence angle. Conversely, accuracy is greatly influenced by dielectric constant ϵ_r. Fig. 6 displays the MM solution for various dielectric constants. It is quite evident that as the value of ϵ_r is increased, so too does the region of significant error. This sensitivity to dielectric constant is observed almost entirely for the axial component $f_z(z)$; the transverse components display only minor sensitivity to ϵ_r.

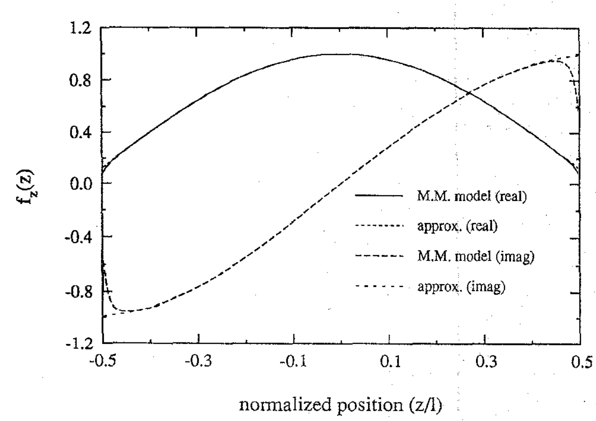

Fig. 5. The axial magnitude of the internal electric field in a thin circular cylinder for an oblique incidence angle ($\beta = 22.5$). Both the MM model and the infinite cylinder approximation are plotted ($k\ell = \pi, \ell/a = 200, \epsilon_r = 5.0$).

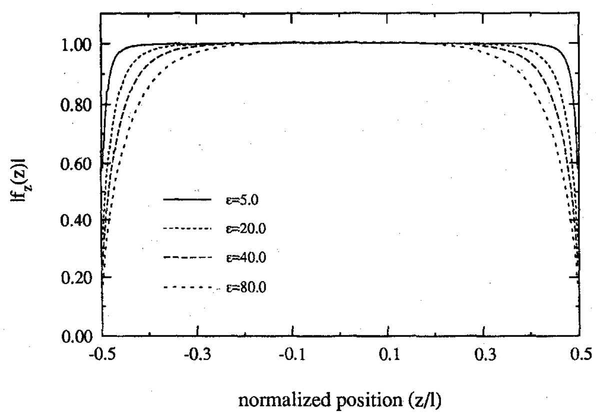

Fig. 6. The axial magnitude of the internal electric field in a thin circular cylinder for various dielectric values ($k\ell = \pi/2, \ell/a = 200, \beta = \pi/2$).

Using the MM solution as a standard, the validity limits of the asymptotic solution can be inferred. From the numerical results, we conclude that the accuracy of (1) and (2) are dependent mainly on normalized length ℓ/a and dielectric ϵ_r. The MM solution likewise demonstrates that for all conditions the axial component $\hat{z}$ exhibits significantly greater error than the transverse component. Therefore, the axial solution will be used to define limits on ℓ/a and ϵ_r.

The far-field scattering is a function of the internal field $\boldsymbol{E}(\overline{r})$ and the free-space Green's function integrated over the cylinder length. Therefore, the metric selected to define accuracy is proportional to the average magnitude of the internal electric field, defined by integrating $|f_z(z)|$ over cylinder length ℓ

$$m = \frac{1}{\ell}\int_{-\ell/2}^{\ell/2} |f_z(z)|\, dz \propto \frac{1}{\ell}\int_{-\ell/2}^{\ell/2} |E_z(z)|\, dz. \quad (16)$$

The asymptotic solution yields a value of $m = 1.0$ for all cases. As the solution breaks down, the actual value of m (as determined from the MM code) will decrease from this value. Placing an arbitrary error limit of 5% $(m > 0.95)$, we determine from the numeric solution the following criteria for the validity of (1) and (2).

$$\ell/a > 20\sqrt{|\epsilon_r|}. \quad (17)$$

In addition to the above requirement, the electrical radius ka must likewise be small. The upper bound on ka is determined by the error of the Rayleigh approximation, a topic which has been addressed previously and, therefore, will not be examined here [4, pp. 92–101].

IV. Far-Field Scattering from Thin Cylinders

The far-field scattering from a long, thin, dielectric cylinder can be determined by using the familiar far-field scattering equation [4, p. 55]

$$\boldsymbol{E}^s = -k_0^2 \hat{k}^s \times \hat{k}^s \times \Pi^e(\overline{r}) \quad (18)$$

where the Hertz potential $\Pi^e(\overline{r})$ in the far field is given as

$$\Pi^e(\overline{r}) = -\frac{e^{\imath k_0 r}}{r}\frac{(\epsilon_r - 1)}{4\pi}\int_V \boldsymbol{E}(\overline{r}')e^{-\imath k_0 \hat{k}^s\cdot\overline{r}'}\, dv'. \quad (19)$$

Since the cylinder is electrically thin, the phase kernel $\exp(\imath k_0 \hat{k}^s \cdot \overline{r}')$ is approximated as $\exp(\imath k_z^s z')$. Inserting (1) and integrating over the cross section A, the electric Hertz potential can be succinctly written as

$$\Pi(\overline{r}) = \frac{e^{\imath k_0 r}}{4\pi r}\int_\ell \boldsymbol{P}\cdot\hat{a}e^{-\imath k_0(\hat{k}^s\cdot\hat{z}-\cos\beta)z'}\, dz' \quad (20)$$

where $\boldsymbol{P}$ is defined as the polarizability tensor per unit length, whose elements can be determined from $\Phi(\overline{\rho})$ using the formulation of Sarabandi and Senior [1].

Finally, integrating over the cylinder length ℓ, the electric Hertz vector potential is given as

$$\Pi(\overline{r}) = \frac{e^{\imath k_0 r}}{4\pi r}\ell\boldsymbol{P}\cdot\hat{a}\frac{\sin U}{U} \quad \text{where}$$
$$U = \frac{k_0\ell}{2}(\hat{k}^s\cdot\hat{z} - \cos\beta). \quad (21)$$

Therefore, the far-field scattering for a long, thin, dielectric cylinder of arbitrary cross section and electrical length is expressed as

$$\boldsymbol{E}^s = -\frac{e^{\imath k_0 r}}{4\pi r}k_0^2\{\hat{k}^s \times \hat{k}^s \times [\ell\boldsymbol{P}\cdot\hat{a}]\}\frac{\sin U}{U}. \quad (22)$$

V. Conclusion

This paper has addressed the scattering from long, electrically-thin dielectric cylinders of arbitrary electrical length and cross section. As such, it provides a solution which eliminates the additional constraints required by methods which might otherwise be used on these thin cylinder structures. For example, the Rayleigh approximation is limited to small $k\ell$ [5], physical optics to large $k\ell$ [1], the Borne approximation to small ϵ_r [3], and generalized Rayleigh–Gans to circular cross sections [2].

Equations (1) and (2) are the scattering solution for an infinite cylinder $(kl = \infty)$ as $ka \to 0$. Yet, this paper has demonstrated that they also satisfy the integral equation defining the scattering from a finite-length cylinder with arbitrary $k\ell$, again as ka approaches zero. By definition, (1) and (2) are therefore the asymptotic solution to this scattering problem as $ka \to 0$. The MM solution was constructed merely to evaluate the asymptotic error associated with nonzero ka. Section II alone provides the general proof of this paper's hypothesis.

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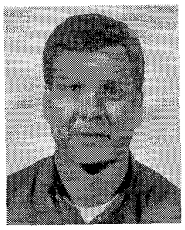

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