# A fictitious play approach to large-scale optimization*

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**Abstract**

Fictitious play is a widely used model to describe learning. In this process, players in a "game" assume that their opponents have a fixed, but unknown, mixed strategy. At each stage, each player counters with a best reply under the belief that other players are employing the mixed strategy represented by the empirical distribution of their past plays. In this paper we discuss how the game-theoretic tool of fictitious play under identical interests can be used as an optimization heuristic. The fictitious play process essentially decomposes the original n-dimensional optimization problem into a series of 1-dimensional optimization problems. With the success of algorithms based on"natural" optimization methods such as simulated annealing and genetic algorithms, this may be viewed as exploiting the "behavioral" paradigm of competition. We show that fictitious play yields a joint strategy that gets arbitrarily close to a local optimum as the number of iterations increases.

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# 1 Introduction

Optimization of real problems can pose a formidable task. There are several common challenges one may face when approaching these problems. Many times the objective function does not posses a closed form expression, but instead comes from, in effect, a "black box," e.g., a simulation model or even program code. Moreover, even with a closed form expression for the objective available, the problem may remain too complex for direct analysis. Often times in such problems, even finding some form of locally-optimal solution may be very difficult.

In the face of such challenges, many use heuristic methods to solve these ill structured problems. Others simplify the problems, often making the assumptions necessary to apply well developed and theoretically sound techniques, thus arriving at solutions of the simplified problems. The second approach can be also viewed as an application of a particular heuristic, although the question of how well the simplified problem approximates the real problem is rarely addressed. With this in mind, we have chosen the heuristic path.

A vast array of optimization heuristics already exists. Some of these are based on naturally self optimizing systems. For example, Romeijin and Smith [6] use simulated annealing, which as its name implies, comes from an analogy to the annealing process of metals, to solve constrained optimization problems. Genetic algorithms, which can be thought of as evolutionary processes, are used by, for example, Boettcher and Percus [1] also as an optimization tool. With the success of these algorithms based on "natural" optimization methods, a next logical step would be to examine a "behavioral" optimization method. For this we turn to the game theoretic concept of fictitious play. With fictitious play we animate the components of a system to play a non-cooperative game under identical interests, representing the performance of the system, to reach a Nash equilibrium, i.e., a coordinate-wise optimum. The Nash equilibrium will be our surrogate for the optimal solution.

The practical potential for a fictitious play approach to optimization has already been demonstrated in Garcia, Reaume and Smith [3]. Their use of fictitious play to find system optimal routings in a large scale dynamic traffic network yielded substantial reductions in computational requirements at no deterioration in quality of solution obtained. However, in the implementation simplifications were made in the fictitious play algorithm in order to ensure tractability. Our objective is to establish a rigorous foundation for a version of fictitious play which can be directly implemented for optimizing large scale real systems.

This paper is organized as follows: in Section 2 we define the concepts of a Nash equilibrium and an approximate equilibrium, as well as provide proofs of several preliminary results. We end the section with the definition of $\varepsilon_t$-fictitious play which plays a crucial role in the implementation of a fictitious play approach to optimization. Sec-

tion 3 discusses the use of fictitious play as an optimization tool, introducing *sampled fictitious play*, and contains the main theoretical results of this paper. In particular, Theorem 1 gives conditions under which every $\varepsilon_t$-fictitious play process converges in beliefs to equilibrium in finite games with common utility or payoff function, thus guaranteing convergence of our basic fictitious play algorithm for optimization. This result is then used in Theorem 2 to show that every *sampled fictitious play process*, with sample sizes growing fast enough, converges in beliefs to equilibrium with probability 1 in finite games with common utility. This allows the development of a more general algorithm which makes use of statistical sampling, addressing possible computational or analytical difficulties which may arise in the decomposed single player optimization problems. Theorem 4 provides conditions under which an $\varepsilon_t$-fictitious play process has a unique limit in beliefs.

# 2 Fictitious Play

Following Monderer and Shapley [4] let $\Gamma$ be a finite game in strategic form with the set of players $N = \{1, 2, \ldots, n\}$. We denote the finite set of strategies of player $i \in N$ by $\mathcal{Y}^i$, and let $\mathcal{Y} = \mathcal{Y}^1 \times \mathcal{Y}^2 \times \cdots \times \mathcal{Y}^n$. The *payoff function* of player $i \in N$ is $u^i : \mathcal{Y} \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers.

For $i \in N$, let $\Delta^i$ be the set of mixed strategies of player $i$. That is,

$$\Delta^i = \left\{ f^i : \mathcal{Y}^i \to [0,1] : \sum_{y^i \in \mathcal{Y}^i} f^i(y^i) = 1 \right\}.$$

Each $f^i \in \Delta^i$ can be viewed as an assignment of probabilities, or *beliefs*, to the elements of $\mathcal{Y}^i$; in particular, with a slight abuse of notation we identify the pure strategy $y^i \in \mathcal{Y}^i$ with the extreme point of $\Delta^i$ which assigns a probability 1 to $y^i$. Set $\Delta = \Delta^1 \times \Delta^2 \times \cdots \Delta^n$.

For $i \in N$, we extend $u^i$ to be the payoff function of player $i$ in the mixed extension of $\Gamma$. That is, for any $f \in \Delta$,

$$u^i(f) = u^i(f^1, f^2, \ldots, f^n) = \sum_{y \in \mathcal{Y}} u^i(y^1, y^2, \ldots, y^n) f^1(y^1) f^2(y^2) \cdots f^n(y^n). \qquad (1)$$

Note that we have assumed players choose their strategies independently.

Let $g \in \Delta$, and let $\varepsilon \geq 0$. We say that $g$ is an *$\varepsilon$-equilibrium* if for each $i \in N$

$$u^i(g) \geq u^i(f^i, g^{-i}) - \varepsilon \text{ for all } f^i \in \Delta^i,$$

where $(f^i, g^{-i}) = (g^1, \ldots, g^{i-1}, f^i, g^{i+1}, \ldots, g^n)$. A *Nash equilibrium* is a 0-equilibrium, and will be simply referred to as an *equilibrium*.

Denote by $K_\varepsilon$ the set of all $\varepsilon$-equilibria, and by $K$ the set of all equilibria of $\Gamma$, and denote by $\|\cdot\|$ the Euclidean norm on the Euclidean space that may be viewed as containing $\Delta$. For $\delta > 0$ set

$$B_\delta(K) = \{g \in \Delta : \min_{f \in K} \|g - f\| < \delta\}.$$

A *belief path* is a sequence $(f(t))_{t=1}^\infty$ in $\Delta$. We say that the belief path $(f(t))_{t=1}^\infty$ *converges to equilibrium* if each accumulation point of $(f(t))_{t=1}^\infty$ is an equilibrium point; that is, if for every $\delta > 0$ there exists an integer $T$ such that $f(t) \in B_\delta(K)$ for all $t \geq T$. In other words, a belief path that converges to equilibrium is eventually arbitrarily close to some equilibrium of $\Gamma$.

In the following propositions we provide proofs of two claims from Monderer and Shapley [4].

**Proposition 1** *Suppose the belief path $(f(t))_{t=1}^\infty$ is such that for every $\varepsilon > 0$ there exists an integer $T$ such that $f(t)$ is an $\varepsilon$-equilibrium for every $t \geq T$. Then the belief path $(f(t))_{t=1}^\infty$ converges to equilibrium.*

**Proof:** See Appendix A for the proof of Proposition 1. ∎

We say that the belief path $f = (f(t))_{t=1}^\infty$ *converges to equilibrium in the Cesaro mean* if

$$\lim_{T\to\infty} \frac{\#\{1 \leq t \leq T : f(t) \notin B_\delta(K)\}}{T} = 0 \text{ for every } \delta > 0. \tag{2}$$

Clearly, convergence to equilibrium implies convergence to equilibrium in the Cesaro mean. The following proposition gives a sufficient condition for a belief path converging to equilibrium in the Cesaro mean.

**Proposition 2** *Suppose the belief path $(f(t))_{t=1}^\infty$ is such that for every $\varepsilon > 0$,*

$$\lim_{T\to\infty} \frac{\#\{1 \leq t \leq T : f(t) \notin K_\varepsilon\}}{T} = 0. \tag{3}$$

*Then the belief path converges to equilibrium in the Cesaro mean.*

**Proof:** See Appendix A for the proof of proposition 2. ∎

A *path* in $\mathcal{Y}$ is a sequence $(y(t))_{t=1}^\infty$ of elements of $\mathcal{Y}$. To each path $(y(t))_{t=1}^\infty$ we naturally associate a belief path $(f_y(t))_{t=1}^\infty$ by letting

$$f_y(t) = \frac{1}{t}\sum_{s=1}^{t} y(s) \text{ for every } \ t \geq 1.$$

In the previous equation, the $y(s)$ should be viewed as elements of $\Delta$. Note that for any integer $k \geq 0$,

$$f_y(t+k) = f_y(t) + \frac{1}{t+k} \sum_{s=t+1}^{t+k} (y(s) - f_y(t)) \,. \tag{4}$$

We say that the path $(y(t))_{t=1}^{\infty}$ *converges in beliefs to equilibrium (in the Cesaro mean)* if the associated belief path converges to equilibrium (in the Cesaro mean). The following lemma establishes the equivalence of these two types of convergence.

**Lemma 1 (Monderer and Shapley[4])** *For every game in strategic form, a path $(y(t))_{t=1}^{\infty}$ converges in beliefs to equilibrium if and only if it converges in beliefs to equilibrium in the Cesaro mean.*

**Proof:** See Appendix A for the proof of Lemma 1. ∎

We now formally define a fictitious play process, an idea first introduced by Brown[2]. For $i \in N$ and for $f \in \Delta$, let

$$v^i(f) = \max\{u^i(g^i, f^{-i}) : g^i \in \Delta^i\}.$$

That is, $v^i(f)$ is the value of player $i$'s best response to the other players strategies $f^{-i}$. Notice from the definition of $u^i(f)$, that $v^i(f)$ can always be attained by an extreme point of $\Delta$, i.e., $\max\{u^i(g^i, f^{-i}) : g^i \in \Delta^i\} = \max\{u^i(y^i, f^{-i}) : y^i \in \mathcal{Y}^i\}$. A path $(y(t))_{t=1}^{\infty}$ is a *fictitious play process* if for every $i \in N$,

$$u^i(y^i(t+1), f_y^{-i}(t)) = v^i(f_y(t)) \text{ for every } \ t \geq 1. \tag{5}$$

Notice that, as defined by (5), $y^i(t+1)$ is a best response of player $i$ to the mixed strategies of the other players, as represented by the beliefs $f_y^{-i}(t)$.

For a sequence of vectors $(\varepsilon_t)_{t=1}^{\infty}$, where $\varepsilon_t \in \mathbb{R}^n$ and $\varepsilon_t \geq 0$ for all $t$, we say that a path $(y(t))_{t=1}^{\infty}$ is an *$\varepsilon_t$-fictitious play process* if for every $i \in N$

$$u^i(y^i(t+1), f_y^{-i}(t)) \geq v^i(f_y(t)) - \varepsilon_t^i \text{ for every } \ t \geq 1. \tag{6}$$

The $\varepsilon_t^i$ can be viewed as player errors in their respective optimizations. A standard fictitious play process can be characterized as an $\varepsilon_t$-fictitious play process with $\varepsilon_t = 0$ for every $t \geq 1$, i.e., the players make no errors.

As above, the ($\varepsilon_t$-)fictitious play process $(y(t))_{t=1}^{\infty}$ *converges in beliefs to equilibrium (in the Cesaro mean)* if the associated belief path converges to equilibrium (in the Cesaro mean).

# 3 Fictitious Play and Optimization

Now that $\varepsilon_t$-fictitious play has been properly defined, let us discuss how it will be used as an optimization heuristic. Consider the following optimization problem:

$$\max\{u(y) : y = (y^1, \ldots, y^n) \in \mathcal{Y}^1 \times \cdots \times \mathcal{Y}^n\}, \tag{7}$$

where the $\mathcal{Y}^i$, $i \in N = \{1, \ldots, n\}$ are finite sets. We can view the optimization problem (7) as a game $\Gamma$ in which $n$ players,each player $i \in N$ having the finite set of strategies $\mathcal{Y}^i$, share a common utility, i.e., $u^1(y) = \cdots = u^n(y) = u(y)$. That is, we have a game of identical interests. We will use fictitious play as an algorithmic way to obtain a Nash equilibrium of the mixed extension of $\Gamma$. The obtained equilibrium will serve as a surrogate for the optimal solution. In fact, an equilibrium is, by definition, a coordinate-wise maximum of the mixed extension of the payoff function. Thus by finding a Nash equilibrium we find a local optimum of the mixed extension of (7), in the above sense.

The Nash equilibrium found may be a mixed strategy. While the pure strategy response structure of fictitious play favors pure strategy limiting behavior, simple examples which converge to mixed strategy equilibria can be easily constructed. As optimizers we would like to derive a pure strategy solution, this being the native form of solution for the original optimization problem, which performs as well or better than the mixed strategy solution derived by fictitious play. The following proposition guarantees the existence of such a solution and, as the proof is constructive provides a method for finding one.

**Proposition 3** *Let $\Gamma$ be a finite n-player game in strategic form with common utility $u$. Given a mixed strategy equilibrium $f \in \Delta$ there exists $y \in \mathcal{Y}$ such that $u(y) \geq u(f)$.*

**Proof:** Remember $u(f^1, f^2, \ldots, f^n) = \sum_{y \epsilon \mathcal{Y}} f^1(y^1) f^2(y^2) \cdots f^n(y^n) u(y^1, y^2, \ldots, y^n)$. Let

$$y^1_* = \text{argmax}_{(y^1 \in \mathcal{Y}^1)} \left( \sum_{y \in \mathcal{Y}} f^2(y^2) \cdots f^n(y^n) u(y^1, y^2, \ldots, y^n) \right)$$

Since

$$\sum_{y^1 \in \mathcal{Y}^1} f^1(y^1) = 1$$

and

$$f^2(y^2) \cdots f^n(y^n) u(y^1_*, y^2, \ldots, y^n) \geq f^2(y^2) \cdots f^n(y^n) u(y^1, y^2, \ldots, y^n) \, \forall y^1 \in \mathcal{Y}^1$$

we have

$$\sum_{y\in\mathcal{Y}} f^2(y^2)\cdots f^n(y^n)u(y^1_*, y^2, \ldots, y^n)$$
$$\geq \sum_{y\in Y} f^1(y^1)f^2(y^2)\cdots f^n(y^n)u(y^1, y^2, \ldots, y^n)$$

So $u(y^1_*, f^2, \ldots, f^n) \geq u(f^1, f^2, \ldots, f^n)$.
For each player $i$ follow the same procedure to get $y^i_*$, taking $f^{i-1}, \ldots, f^1$ to be 1 for the pure strategies $y^{i-1}_*, \ldots, y^1_*$ and 0 otherwise. Applying the above argument inductively $u(y^1_*, y^2_*, \ldots, y^n_*) \geq u(f^1, f^2, \ldots, f^n)$. ∎

A straightforward algorithmic implementation of the fictitious play paradigm for finding a local optimum to the optimization problem (7) would be:

**Fictitious Play Algorithm**

**Initialization:** Set $t = 1$ and select $y(1) \in \mathcal{Y} = \mathcal{Y}^1 \times \mathcal{Y}^2 \times \ldots \times \mathcal{Y}^n$ arbitrarily; set $f_y(1) = y(1)$.

**Iteration $t \geq 1$:** Given $f_y(t)$, find

$$y^i(t+1) \in \operatorname*{argmax}_{y^i\in\mathcal{Y}^i} u^i(y^i, f_y^{-i}(t)),\ i = 1, \ldots, n. \tag{8}$$

Set $f_y(t+1) = f_y(t) + \frac{1}{t+1}(y(t+1) - f_y(t))$ and increment $t$ by 1.

However, straightforward implementation of the above algorithm may be impractical for many problems. Performing the $n$ single variable optimizations (8) in the iterative step of the algorithm can be prohibitively expensive — in the extreme, if the vector $f_y(t)$ is strictly positive, computing $u(y^i, f_y(t)^{-i})$ can require evaluating the payoff function $u(\cdot)$ for *all* combinations of pure strategies, an effort equivalent to solving our original optimization problem by pure enumeration of all feasible policies! We propose to sidestep these difficulties by replacing exact computations in (8) with an approximation.

## 3.1 Fictitious Play and Sampled Fictitious Play Algorithms

We begin by showing that if $\varepsilon_t$ converges to zero fast enough, an $\varepsilon_t$-fictitious play process does in fact converge to a Nash Equilibrium in finite games with identical interests. The following theorem is a generalization of a theorem first proved by Monderer and Shapley[4]. The original result applied to fictitious play processes (i.e., $\varepsilon_t$-fictitious play processes with $\varepsilon_t = 0$) in finite games with identical interests, and thus is a special case of Theorem 1. The proof of Theorem 1 borrows from the proof of Monderer and Shapley.

**Theorem 1** *Let $\Gamma$ be a finite game in strategic form with identical payoff functions $u^1(\cdot) = \cdots = u^n(\cdot) = u(\cdot)$. Then any $\varepsilon_t$-fictitious play process with $\max_i\{\varepsilon_t^i\} = O\left(\frac{1}{t^\alpha}\right)$, for some $\alpha > 0$, converges in beliefs to the set of Nash equilibria.*

**Proof:** See appendix A for the proof of Theorem 1. ∎

The previous theorem guarantees convergence to the set of Nash Equilibria as long as $\max_i\{\varepsilon_t^i\} \to 0$ at an asymptotic order of $1/t^\alpha$, for some $\alpha > 0$. The vector $\varepsilon_t$ can be interpreted as the errors in the players responses at iteration $t$ as reflected by the utility function, which means the players need not start out as good optimizers as long as they continue to get better over time. The mechanism by which we propose to achieve the necessary error reduction over iterations is based on statistical sampling.

Recall that the components of the belief vector $f(t)$ can be viewed as components of a probability distribution over the pure strategies of the players. From this perspective, the one-dimensional optimization (8) can be viewed as

$$\max_{y^i \in \mathcal{Y}^i}\{E_{Y^{-i}}(u(y^i, Y^{-i}))\},$$

where $Y^{-i}$ is a random vector with probability distribution described by $f^{-i}(t)$.

We define the function $\bar{U}_k^i(\cdot, f_y^{-i}(t)) : \mathcal{Y}^i \to \mathbb{R}$ by

$$\bar{U}_k^i(y^i, f_y^{-i}(t)) = \sum_{j=1}^{k} \frac{u^i(y^i, Y_j^{-i}(t))}{k} \tag{9}$$

where the $Y_j^{-i}(t)$ are iid random vectors drawn from the distribution given by $f_y^{-i}(t)$. Then $\bar{U}_k^i(y^i, f_y^{-i}(t))$ is a sample mean (with sample size $k$) of player $i$'s utility when playing $y^i$. Let $\bar{u}_{k_t}^i(y^i, f_y^{-i}(t))$ denote the realization of $\bar{U}_{k_t}^i(y^i, f_y^{-i}(t))$. If the "best response" of each player is chosen based on sample means instead of the actual means, i.e., $y^i(t+1)$ is chosen so that $y^i(t+1) \in \operatorname{argmax}\{\bar{u}_{k_t}^i(y^i, f_y^{-i}(t)) : y^i \in \mathcal{Y}^i\}$ for some $k_t \in \{1, 2, \ldots\}$, we will call the stochastic process $(y(t))_{t=1}^\infty$ a *sampled fictitious play process*.

The sampled fictitious play algorithm is summarized below:

**Sampled Fictitious Play Algorithm**

**Initialization:** Set $t = 1$ and select $y(1) \in \mathcal{Y} = \mathcal{Y}^1 \times \mathcal{Y}^2 \times \ldots \times \mathcal{Y}^n$ arbitrarily; set $f_y(1) = y(1)$.

**Iteration $t \geq 1$:** Given $f_y(t)$, find

$$y^i(t+1) \in \operatorname*{argmax}_{y^i \in \mathcal{Y}^i}\{\bar{u}_{k_t}^i(y^i, f_y^{-i}(t))\},\ i = 1, \ldots, n, \tag{10}$$

where $\bar{u}_{k_t}^i(y^i, f_y^{-i}(t))$ is the realization of $\bar{U}_{k_t}^i(y^i, f_y^{-i}(t))$ as defined by (9). Set $f_y(t+1) = f_y(t) + \frac{1}{t+1}(y(t+1) - f_y(t))$, increment $t$ by 1 .

Prior to the use of sampling $(y(t))_{t=1}^{\infty}$ was a deterministic sequence, assuming a deterministic tie breaking rule for the single player optimizations, as was the associated belief path $(f_y(t))_{t=1}^{\infty}$. With sampling $(y(t))_{t=1}^{\infty}$ is now a stochastic process and we will denote the associated belief process as $(F_y(t))_{t=1}^{\infty}$. The reader should note that in equation (10) the sample mean was generated conditional on $F_y^{-i}(t) = f_y^{-i}(t)$.

The next theorem shows that it suffices to choose our sample sizes $k_t = \lceil t^{\beta} \rceil$ for $\beta > \frac{1}{2}$, where $\lceil t^{\beta} \rceil$ is the smallest integer greater than or equal to $t^{\beta}$, to ensure convergence of a sampled fictitious play process in a finite game with common utility.

**Theorem 2** *Let $\Gamma$ be a finite game in strategic form with identical payoff functions. Then every sampled fictitious play process $y(t)$ with sample sizes $k_t = \lceil t^{\beta} \rceil$ for $\beta > \frac{1}{2}$, will converge in beliefs to equilibrium with probability 1.*

**Proof:** Let $(y(t))_{t=1}^{\infty}$ be a sampled fictitious play process with sample sizes $k_t = \lceil t^{\beta} \rceil$, and $(F_y(t))_{t=1}^{\infty}$ be the associated belief process. We begin by establishing a bound on the random variable

$$\bar{U}_{k_t}^i(y^i, F_y^{-i}(t)) - u(y^i, F_y^{-i}(t))$$

for an arbitrary $y^i \in \mathcal{Y}^i$. Fix $t$, let $y^i \in \mathcal{Y}^i$, and define

$$X_j(t) = u(y^i, Y_j^{-i}(t)) - u(y^i, F_y^{-i}(t)),\ j = 1, \ldots, k_t,$$

where the $Y_j^{-i}(t)$ are random vectors with distribution $F_y^{-i}(t)$. The $X_j(t)$ are not independent random variables as they are describe by functions containing the same random variable $F_y^{-i}(t)$, also the $Y_j^{-i}(t)$ are dependent on $F_y^{-i}(t)$ and each other. Conditional on $F_y^{-i}(t) = f_y^{-i}(t)$ the $Y_j^{-i}(t)$ are iid random vectors with distribution $f_y^{-i}(t)$ and since $u(y^i, f_y^{-i}(t))$ is a constant, $X_1(t), \ldots, X_{k_t}(t)$ are iid random variables with, by the law of the unconscious statistician, mean 0. Let $H = max_{f \in \Delta} |u(f)|$ and let $A$ denote the event that $F_y^{-i}(t) = f_y^{-i}(t)$ . Then, $E[X_j(t)^4|A] \leq (2H)^4$ and $E[X_j(t)^2|A] \leq (2H)^2$ for all $j = 1, \ldots, k_t$.

Let $S(t) = \sum_{j=1}^{k_t} X_j(t)$. Then

$$\begin{aligned} E[S(t)^4|A] &= k_t E[X_1(t)^4|A] + \binom{4}{2}\binom{k_t}{2} E[X_1(t)^2 X_2(t)^2|A] \\ &\leq k_t (2H)^4 + (3{k_t}^2 - 3k_t) E[X_1^2|A] E[X_2^2|A] \leq 3(2H)^4 {k_t}^2. \end{aligned}$$

The previous bound did not depend on $f_y^{-i}(t)$, therefore unconditionally,

$$E\left[\frac{S(t)^4}{{k_t}^4}\right] \leq \frac{3(2H)^4 {k_t}^2}{{k_t}^4} = \frac{3(2H)^4}{\lceil t^{\beta} \rceil^2} \leq \frac{3(2H)^4}{t^{2\beta}}. \tag{11}$$

By the Markov inequality, for any $\delta > 0$,

$$P\left\{\frac{S(t)^4}{{k_t}^4} > \delta t^{0.5-\beta}\right\} \leq \frac{E\left[\frac{S(t)^4}{{k_t}^4}\right]}{\delta t^{0.5-\beta}}.$$

Combining this with (11),

$$\sum_{t=1}^{\infty} P\left\{\frac{S(t)^4}{{k_t}^4} > \delta t^{0.5-\beta}\right\} \leq \sum_{t=1}^{\infty} \frac{E\left[\frac{S(t)^4}{{k_t}^4}\right]}{\delta t^{0.5-\beta}} \leq \sum_{t=1}^{\infty} \frac{3(2H)^4}{\delta t^{\beta+0.5}} < \infty.$$

By the Borel-Cantelli lemma, which does not require independence of events (see Ross [7]), we have, with probability 1, $(S(t)^4/{k_t}^4) \to 0$ at an asymptotic order of $1/t^{\beta-0.5}$. The previous argument implies that, with probability 1, $(S(t)/k_t) \to 0$ at an asymptotic order of $1/t^{\alpha}$, where $\alpha = (\beta - 0.5)^{0.25}$. Since

$$\frac{S(t)}{k_t} = \bar{U}^i_{k_t}(y^i, F_y^{-i}(t)) - u(y^i, F_y^{-i}(t)),$$

we conclude that $\bar{U}^i_{k_t}(y^i, F_t^{-i}) - u(y^i, F_y^{-i}(t))$ converges to 0 at an asymptotic rate of $1/t^{\alpha}$, where $\alpha = (\beta - 0.5)^{0.25}$, for any $y^i \in \mathcal{Y}^i$, with probability 1.

For all $i \in N$ and $t \geq 1$, let

$$\varepsilon^i_t = v^i(F_y(t)) - u(y^i(t+1), F_y^{-i}(t)) \geq 0. \tag{12}$$

(Note that the vector $\varepsilon_t$ can be interpreted as the errors in the players responses at iteration $t$ as reflected by the utility function.) Then $(y(t))_{t=1}^{\infty}$ is for any realization, by definition, an $\varepsilon_t$-fictitious play process with $\varepsilon_t$ derived from (12). We will show that, with probability 1, for any player $i \in N$, $\varepsilon^i_t \to 0$ as $t \to \infty$ at an asymptotic order of $t^{-\alpha}$, with $\alpha = (\beta - 0.5)^{0.25} > 0$. By Theorem 1 this would imply that the path $(y(t))_{t=1}^{\infty}$ converges to equilibrium with probability 1.

Let $\tilde{y}^i(t+1) \in \operatorname{argmax}_{y^i \in \mathcal{Y}^i} u^i(y^i, f_y^{-i}(t))$. Conditioning on $F_y^{-i}(t) = f_y^{-i}(t)$,

$$0 \leq \varepsilon^i_t = v^i(f_y(t)) - u(y^i(t+1), f_y^{-i}(t))$$

$$= u(\tilde{y}^i, f_y^{-i}(t)) - u(y^i(t+1), f_y^{-i}(t)) = u(\tilde{y}^i, f_y^{-i}(t)) - \bar{U}^i_{k_t}(\tilde{y}^i, f_y^{-i}(t))$$

$$+ \left(\bar{U}^i_{k_t}(\tilde{y}^i, f_y^{-i}(t)) - \bar{U}^i_{k_t}(y^i(t+1), f_y^{-i}(t))\right) + \left(\bar{U}^i_{k_t}(y^i(t+1), f_y^{-i}(t)) - u(y^i(t+1), f_y^{-i}(t))\right)$$

$$\leq \left(u(\tilde{y}^i, f_y^{-i}(t)) - \bar{U}^i_{k_t}(\tilde{y}^i, f_y^{-i}(t))\right) + \left(\bar{U}^i_{k_t}(y^i(t+1), f_y^{-i}(t)) - u(y^i(t+1), f_y^{-i}(t))\right),$$

where the last inequality follows since $y^i(t+1)$ is chosen to maximize $\bar{U}^i_{k_t}(y^i, f^{-i}(t))$, so the random variable $\left(\bar{U}^i_{k_t}(\tilde{y}^i, f_y^{-i}(t)) - \bar{U}^i_{k_t}(y^i(t+1), f_y^{-i}(t))\right)$ is less than or equal to 0. The above analysis did not depend on $f_y^{-i}(t)$; therefore, we have unconditionally

$$0 \leq \varepsilon^i_t \leq \left(u(\tilde{y}^i, F_y^{-i}(t)) - \bar{U}^i_{k_t}(\tilde{y}^i, F_y^{-i}(t))\right) + \left(\bar{U}^i_{k_t}(y^i(t+1), F_y^{-i}(t)) - u(y^i(t+1), F_y^{-i}(t))\right).$$

Applying the derived asymptotic rate of convergence to the two terms of the above bound, we conclude that, with probability 1, $\varepsilon^i_t$ converges to 0 at an asymptotic order of $1/t^{\alpha}$, where $\alpha = (\beta - 0.5)^{0.25}$, establishing the desired result. ∎

In the following proposition we show that every sampled fictitious play process is an $\varepsilon_t$-fictitious play processes and hence the results of section 3.2 apply.

**Proposition 4** *Every sampled fictitious play process is an $\varepsilon_t$-fictitious play processes.*

**Proof:** Let $(y(t))_{t=1}^{\infty}$ be a sampled fictitious play process and $(F_y(t))_{t=1}^{\infty}$ the associated belief path. Let $\varepsilon_t^i = 2\max_{f\in\Delta}|u(f)|$, then $v^i(F_y(t)) - u^i(y^i(t+1), F_y^{-i}(t)) \leq \varepsilon_t^i$ for all $i \in N$ and all $t$, hence $(y(t))_{t=1}^{\infty}$ is an $\varepsilon_t$-fictitious play processes.

## 3.2 Convergence Properties of $\varepsilon_t$-fictitious Play Processes

For an appropriate $\varepsilon_t$-fictitious play process, Theorem 1 only guarantees that the accumulation points of the generated belief path belong to the set of Nash Equilibria. In an algorithmic setting, however, it is desirable to have a guarantee of the uniqueness of the limit point of the sequence of iterates (for example, having this guarantee typically leads to simpler and more reasonable stopping rules). The following theorem leads to conditions under which an $\varepsilon_t$-fictitious play process converges to a Nash Equilibrium (i.e., has a unique limit point which is a Nash Equilibrium). Recall that a set $S$ in a metric space is *perfect* if it is closed and every point of $S$ is an accumulation point of $S$, i.e., if $S$ is its accumulation points, (see, for example, [5]).

**Theorem 3** *Let $\Gamma$ be a finite $n$-player game in strategic form. The set $F$ of limit points of the belief path of any $\varepsilon_t$-fictitious play process is a connected subset of $\Delta$ under the Euclidean norm. Moreover, $F$ is either a singleton or a perfect set.*

**Proof:** Let $(y(t))_{t=1}^{\infty}$ be an $\varepsilon_t$-fictitious play process, and $(f_y(t))_{t=1}^{\infty}$ be the associated belief process. Let $F$ be the set of all limit points of $(f_y(t))_{t=1}^{\infty}$.

Suppose $F$ is disconnected, i.e., there exist two disjoint open sets $A$ and $B$ such that

$$(A \cap F) \neq \emptyset,\ (B \cap F) \neq \emptyset,\ \text{and } F \subset A \cup B. \tag{13}$$

Let $C = \Delta \setminus (A \cup B)$. Then $C$ is a compact subset of $\Delta$ and $C \neq \emptyset$ otherwise $A \cup B \supseteq \Delta$ which implies $\Delta$ is disconnected. We will show that $C$ must contain a limit point of $(f(t))_{t=1}^{\infty}$, resulting in a contradiction.

Let $a \in A \cap F$ and $b \in B \cap F$. Since $A$ and $B$ are open sets we can choose $\epsilon > 0$ such that $B_\epsilon(a) \subset A$ and $B_\epsilon(b) \subset B$, where $B_\epsilon(x)$ denotes the open ball of radius $\epsilon$ centered at point $x$. By definition, both $B_\epsilon(a)$ and $B_\epsilon(b)$ contain an infinite number of points of the sequence $(f_y(t))_{t=1}^{\infty}$. In particular, the sequence $(f_y(t))_{t=1}^{\infty}$ "leaves" the set $A$ infinitely often. More precisely, there exists an infinite subsequence $(f_y(t_i))_{i=1}^{\infty}$ such that $f_y(t_i) \in A$, but $f_y(t_i+1) \notin A$, i.e., either $f_y(t_i+1) \in B$, or $f_y(t_i+1) \in C$. Note that if $f_y(t_i+1) \in B$, then, since $A \cap B = \emptyset$, the line segment connecting $f_y(t_i)$ and $f_y(t_i+1)$ contains at least one point $\tilde{f}_y(t_i+1) \notin A \cup B$. By the convexity of the set $\Delta$, $\tilde{f}_y(t_i+1) \in \Delta$, and hence $\tilde{f}_y(t_i+1) \in C$.

Consider the following infinite sequence: for each $i$, let

$$g_i = \begin{cases} f_y(t_i+1) & \text{if } f_y(t_i+1) \in C, \\ \tilde{f}_y(t_i+1) & \text{otherwise,} \end{cases}$$

where $\tilde{f}_y(t_i+1)$ is an arbitrary point constructed as above. Note that $(g_i)_{i=1}^{\infty} \subset C$, and hence (passing to a subsequence if necessary) $\lim_{i\to\infty} g_i = g \in C$. We will show that $g$ is a limit point of $(f_y(t))_{t=1}^{\infty}$, leading to a contradiction.

By construction,

$$\|f_y(t_i) - g_i\| \le \|f_y(t_i) - f_y(t_i+1)\| = \frac{1}{1+t_i}\|y(t_i+1) - f_y(t_i)\| < \frac{M}{t_i},$$

where

$$M = \max_{f,g\in\Delta} \|f - g\|. \tag{14}$$

Therefore

$$\|f_y(t_i) - g\| \le \|f_y(t_i) - g_i\| + \|g_i - g\| < \frac{M}{t_i} + \|g_i - g\| \to 0 \text{ as } i \to \infty,$$

establishing the desired contradiction and thus proving the first statement of the theorem.

To establish the second claim, assume that the set $F$ contains more than one element. Suppose there exist $d \in F$ which is not an accumulation point of $F$. Then for some $\delta > 0$, $B_\delta(d) \cap F = \{d\}$. Let $A = B_{\frac{\delta}{2}}(d)$, and let $B$ be defined as the complement of the closure of $B_{\frac{\delta}{2}}(d)$. Then the sets $A$ and $B$ are disjoint open sets satisfying conditions (13). This implies that the set $F$ is disconnected, resulting in a contradiction — thus establishing the second claim of the theorem. ∎

Using Theorem 3 we are able to give sufficient conditions under which an $\varepsilon_t$-fictitious play process, and subsequently a standard fictitious play process and sampled fictitious play process, will converge to a Nash Equilibrium.

**Theorem 4** *Let $\Gamma$ be a finite $n$-player game in strategic form. If the set $K = K(\Gamma)$ of Nash Equilibria has a finite set of accumulation points then every $\varepsilon_t$-fictitious play process which converges in beliefs to equilibrium will converge in beliefs to a Nash Equilibrium.*

**Proof:** By Theorem 3, the set $F$ of limit points of any $\varepsilon_t$-fictitious play process is either a singleton, or a perfect set. The case when $F$ is a singleton is trivial. If $F$ is a perfect set, every element of $F$ is an accumulation point of $F$, and hence $F$ is an infinite set. On the other hand, $F \subseteq K$, and hence every point of $F$ is an accumulation point of $K$. Since $K$ has only finitely many accumulation points, we conclude that $F$ must be a singleton. ∎

The following is an immediate corollary of the previous theorem.

**Corollary 1** *Let $\Gamma$ be a finite $n$-player game in strategic form. The image under the utility function $u^i(f)$ of the set of limit points of the belief path of any $\varepsilon_t$-fictitious play process form a closed interval in $\mathbb{R}$.*

**Proof:** Since $u^i(f)$ is a continuous function, the image under $u^i(f)$ of a perfect connected set or singleton is a perfect connected set or singleton (see Munkres[5]), which in $\mathbb{R}$ is a closed interval. By Theorem 3 the set of limit points of any $\varepsilon_t$-fictitious play process is either a singleton or a perfect connected set. ∎

# 4 Conclusions

We have developed a fictitious play algorithm for optimization of complex multi-dimensional problems, where the repeated evaluation of the objective function is undesirable due to the computational difficulties, or where properties of the objective function necessary for the convergence of traditional optimization algorithms cannot be assessed. We have also discussed the issue of implementing a fictitious play-type algorithm in a computationally tractable way by employing Monte-Carlo simulation in place of exhaustive function evaluations.

Implementations of optimization methods often diverge from the existing theory. These divergences can result in computational tractability, faster convergence, or a qualitative improvement, but it is crucial for sound theoretical results to follow. Garcia, Reaume and Smith[3] used sampling because of the extreme difficulty of analytically computing best responses. This paper has not only addressed the issue of sampling but has also given general conditions under which the sequence of belief distributions has a unique limit point, another concern mentioned in Garcia, Reaume and Smith[3].

# A Proofs

## Proof of Proposition 1

**Proof:** First notice the following fact: given $\delta > 0$ there exists an $\varepsilon > 0$ such that for each $\bar{f} \in \Delta$ with $\bar{f} \notin B_\delta(K)$, $u^i(f^i, \bar{f}^{-i}) - u^i(\bar{f}) > \varepsilon$ for some $i \in N$ and some $f \in \Delta$. Simply take

$$\varepsilon = \frac{1}{2} \min_{\bar{f} \in \Delta \setminus B_\delta(K)} \max_{i \in N} \max_{f^i \in \Delta^i} \left( U^i(f^i, \bar{f}^{-i}) - U^i(\bar{f}) \right) \tag{15}$$

which is well-defined and positive since $\Delta \setminus B_\delta(K)$ is a compact set, $N$ is finite, and $\Delta^i$ is compact and $u^i$ is continuous for all $i \in N$.

Now suppose that the belief path $(f(t))_{t=1}^{\infty}$ does not converge to equilibrium. Then for some $\delta > 0$, for any $T$ there exists $t \geq T$ such that $f(t) \notin B_\delta(K)$. Then for $\varepsilon > 0$ defined as in (15), for any $T$ there is a $t \geq T$ for which $f(t)$ is not an $\varepsilon$-equilibrium, which contradicts our hypothesis. Hence the belief path $(f(t))_{t=1}^{\infty}$ converges to equilibrium. ∎

## Proof of Proposition 2

**Proof:** Assume (3) holds but there exists $\delta > 0$ such that

$$\limsup_{T\to\infty} \frac{\#\{1 \leq t \leq T : f(t) \notin B_\delta(K)\}}{T} = c > 0.$$

Define $\varepsilon$ as in (15). Then for each $\bar{f} \in \Delta$ with $\bar{f} \notin B_\delta(K)$, $u^i(f^i, \bar{f}^{-i}) - u^i(\bar{f}) > \varepsilon$ for some $i \in N$ and some $f \in \Delta$. That is,

$$\bar{f} \notin K_\varepsilon \text{ if } \bar{f} \notin B_\delta(K).$$

Therefore,

$$\#\{1 \leq t \leq T : f(t) \notin K_\varepsilon\} \geq \#\{1 \leq t \leq T : f(t) \notin B_\delta(K)\},$$

and so

$$\limsup_{T\to\infty} \frac{\#\{1 \leq t \leq T : f(t) \notin K_\varepsilon\}}{T} \geq \limsup_{T\to\infty} \frac{\#\{1 \leq t \leq T : f(t) \notin B_\delta(K)\}}{T} = c > 0,$$

which contradicts (3). ∎

## Proof of Lemma 1

**Proof:** Let $(y(t))_{t=1}^{\infty}$ be a path, and let $(f_y(t))_{t=1}^{\infty}$ be the associated belief path. We have already noted that for any belief process, convergence to equilibrium implies convergence to equilibrium in Cesaro mean.

To establish the converse, suppose that $(y(t))_{t=1}^{\infty}$ converges in belief to equilibrium in Cesaro mean. Let $\delta > 0$ be chosen arbitrarily. Let

$$M = \max_{f,g\in\Delta} \|f - g\|.$$

Choose $\eta \in (0, \delta/(2\delta + M))$. By (2), there exists an integer $T_0$ such that for every $T \geq T_0$,

$$\#\{1 \leq t \leq T : f_y(t) \notin B_\delta(K)\} < \eta T. \tag{16}$$

We will show that for every $T \geq T_0$, $f_y(T) \in B_{2\delta}(K)$.

Suppose $T \geq T_0$ but $f_y(T) \notin B_{2\delta}(K)$. Then for any integer $t \in [T, T + \lfloor T\delta/(\delta + M)\rfloor]$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$, $f_y(t) \notin B_\delta(K)$. Indeed, if for some $t$ in the above interval $f_y(t) \in B_\delta(K)$, then there exits $f \in K$ such that $\|f_y(t) - f\| < \delta$, which implies

$$2\delta \leq \|f_y(T) - f\| \leq \|f_y(t) - f_y(T)\| + \|f_y(t) - f\| < \|f_y(t) - f_y(T)\| + \delta,$$

i.e., $\|f_y(t) - f_y(T)\| > \delta$.

On the other hand, using expression (4)

$$\|f_y(t) - f_y(T)\| = \frac{1}{t}\left\|\sum_{s=T+1}^{t}(y(s) - f_y(T))\right\| \leq \frac{1}{t}\sum_{s=T+1}^{t}\|y(s) - f_y(T)\|$$
$$\leq \frac{t-T}{t}\cdot M \leq \frac{\delta}{M+\delta}\cdot M \leq \delta,$$

resulting in a contradiction. Hence, for any integer $t \in [T, T + \lfloor T\delta/(\delta + M)\rfloor]$, $f_y(t) \notin B_\delta(K)$.

We conclude that

$$\#\left\{1 \leq t \leq T + \left\lfloor \frac{T\delta}{\delta + M}\right\rfloor : f(t) \notin B_\delta(K)\right\} \geq 1 + \left\lfloor \frac{T\delta}{\delta + M}\right\rfloor > \eta\left(T + \left\lfloor \frac{T\delta}{\delta + M}\right\rfloor\right),$$

contradicting (16). Therefore, $f_y(T) \in B_{2\delta}(K)$ for every $T \geq T_0$. Since $\delta$ was chosen arbitrarily, convergence follows. ∎

## Proof of Theorem 1

**Proof:** Since the payoff functions in $\Gamma$ are identical, the payoff functions $u^i$ in the mixed extension of $\Gamma$ defined by (1) are also identical, and will be defined by $u$. Note that $u$ is a multi-linear function, that is, for any player $i \in N$, if $f^i, g^i \in \Delta^i$ and $\lambda \in [0, 1]$ then

$$u(f^i + \lambda(g^i - f^i), f^{-i}) = u(f^i, f^{-i}) + \lambda(u(g^i, f^{-i}) - u(f^i, f^{-i})). \tag{17}$$

Let $(f_y(t))_{t=1}^{\infty}$ be the belief process associated with an $\varepsilon_t$-fictitious play process $(y(t))_{t=1}^{\infty}$. Also, let $L = \max_{f\in\Delta}|u(f)|$.

Equation (4) implies that

$$u(f_y(t+1)) = u\left(f_y^1(t) + \frac{1}{t+1}(y^1(t+1) - f_y^1(t)), f_y^2(t) + \frac{1}{t+1}(y^2(t+1) - f_y^2(t)),\right.$$
$$\left.\ldots, f_y^n(t) + \frac{1}{t+1}(y^n(t+1) - f_y^n(t))\right)$$

Applying the multi-linearity property (17) repeatedly, we obtain

$$\begin{aligned} u(f_y(t+1)) =& u(f_y(t)) + \frac{1}{t+1}\sum_{i=1}^{n}\left[u(y^i(t+1), f_y^{-i}(t)) - u(f_y(t))\right] \\ &+ \left((3^n - (2n+1)) \text{ terms of the form } \pm\frac{1}{(t+1)^j}u(g) \text{ with } j \geq 2,\ g \in \Delta\right). \end{aligned}$$

Each of the latter terms is bounded below by $-\frac{1}{(t+1)^2}L$, so by (6),

$$u(f_y(t+1)) - u(f_y(t)) \geq \frac{1}{t+1}\sum_{i=1}^{n}\left[v^i(f_y(t)) - u(f_y(t)) - \varepsilon_t^i\right] - \frac{3^n L}{(t+1)^2}. \tag{18}$$

For $t \geq 1$ set

$$a_t = \sum_{i=1}^{n}[v^i(f_y(t)) - u(f_y(t))]. \tag{19}$$

Notice that each term in the above summation is nonnegative, and, in particular, $a_t \geq 0$ for every $t \geq 1$. By (18)

$$\frac{a_t}{t} \leq \frac{t+1}{t}(u(f_y(t+1)) - u(f_y(t))) + 3^n L\,\frac{1}{t(t+1)} + \frac{n\max_i\{\varepsilon_t^i\}}{t}. \tag{20}$$

We will establish that $\sum_{t=1}^{\infty}\frac{a_t}{t} = \lim_{T\to\infty}\sum_{t=1}^{T}\frac{a_t}{t} < \infty$. First, notice that

$$\sum_{t=1}^{\infty}\frac{1}{t(t+1)} = \sum_{t=1}^{\infty}\left(\frac{1}{t} - \frac{1}{t+1}\right) = 1.$$

Next, consider

$$\begin{aligned} &\sum_{t=1}^{T}\frac{t+1}{t}(u(f_y(t+1)) - u(f_y(t))) \\ &= \sum_{t=1}^{T-1}\frac{1}{t(t+1)}u(f_y(t+1)) + \frac{T+1}{T}u(f_y(T+1)) - 2u(f_y(1)), \end{aligned}$$

and hence

$$\sum_{t=1}^{\infty}\frac{t+1}{t}(u(f_y(t+1)) - u(f_y(t))) \leq 4L.$$

Finally, since $\max_i\{\varepsilon_t^i\} \to 0$ at an asymptotic order of $\frac{1}{t^\alpha}$, for some $\alpha > 0$, as $t \to \infty$,

$$\sum_{t=1}^{\infty}\frac{n\max_i\{\varepsilon_t^i\}}{t} < \infty.$$

since

$$\sum_{t=1}^{\infty} \frac{1}{t^\alpha} \leq 1 + \int_2^\infty \frac{1}{t^\alpha}\, \mathrm{d}t < \infty \text{ for } \alpha > 1.$$

Combining these bounds with (20), we conclude that

$$\sum_{t=1}^{\infty} \frac{a_t}{t} < \infty. \tag{21}$$

We next prove that

$$\lim_{T\to\infty} \frac{a_1 + a_2 + \cdots + a_T}{T} = 0. \tag{22}$$

Indeed, for $T \geq 1$ set $b_T = \sum_{t=T}^{\infty}(a_t/t)$. By (21), $\lim_{T\to\infty} b_T = 0$. Therefore

$$\lim_{T\to\infty} \frac{b_1 + b_2 + \cdots + b_T}{T} = 0,$$

which, via algebraic manipulation, implies

$$\lim_{T\to\infty} \left( \frac{a_1 + a_2 + \cdots + a_T}{T} + b_{T+1} \right) = 0.$$

Since $\lim_{T\to\infty} b_{T+1} = 0$, (22) follows.

Suppose $\varepsilon > 0$ is arbitrary. From (19) we conclude that if $f_y(t) \notin K_\varepsilon$, then $a_t \geq \epsilon$. Therefore,

$$\#\{1 \leq t \leq T : f_y(t) \notin K_\varepsilon\} \leq \frac{1}{\varepsilon}\,(a_1 + a_2 + \cdots + a_T),$$

and so

$$\lim_{T\to\infty} \frac{\#\{1 \leq t \leq T : f_y(t) \notin K_\varepsilon\}}{T} \leq \frac{1}{\varepsilon} \lim_{T\to\infty} \frac{a_1 + a_2 + \cdots + a_T}{T} = 0.$$

Therefore (3) is satisfied, and the proof follows from Lemma 1. ∎

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