Recursive definitions of sets

- We saw last time how to define the set of propositional expressions recursively.
- The method extends to other kinds of sets: Let S be the subset of $\mathbb{N}$ defined by the following rules:
  1. $3 \in S$;
  2. if $x \in S$ and $y \in S$ then $x + y \in S$;
  3. No number is in S unless it can be shown to be there using (1) and (2).
- Example:
  - $3 \in S$ (rule 1)
  - $6 = 3 + 3 \in S$ by 1. and rule 2
  - $9 = 6 + 3 \in S$ by 2. and rule 2.

Using induction along with recursive definitions

Proposition *The set S on the last slide is the set of positive multiples of 3.*

Proof: Let P be the set of positive multiples of 3. We show $P \subseteq S$ and $S \subseteq P$. For the first inclusion, we show that every integer of the form $3n$, for $n \geq 1$, is in S. We do this by induction on n.

Basis. When $n = 1$, the number $3 \cdot 1 \in S$ by rule 1.

Induction step. Assume that $3k$ is in S. We want $3(k+1) \in S$. But $3(k+1) = 3k+3$. By inductive hypothesis $3k \in S$, and 3 is already in S, so $3k+3 \in S$ by rule 2.

Proof continued: $S \subseteq P$.

To show this, we rely on rule 3, which says nothing is in S unless you can show it in a finite number of uses of rules 1 and 2. Let n be the number of uses of these rules. We show by induction on n that the integer proved to be in S is in fact a positive multiple of 3.

Basis. We just apply one rule, which has to be rule 1. This rule shows $3 \in S$, and 3 is a positive multiple of 3.

Induction step (strong form). Assume that whenever we show that $p \in S$ by using k or fewer steps, then p is a positive multiple of 3. Consider a "proof" using $k+1$ steps. The last rule used in this proof is rule 2, which says that if x and y are in S, so is $x+y$. Now x was shown in S by $\leq k$ steps, and so was y. By inductive hypothesis (twice), we know that x and y are positive multiples of 3, and therefore so is $x + y$.

Relations
Chapter 6

- Intuitively, relations are properties that hold among things in a world.
- For example:
  - "loves"
  - "uncle-of"
  - "friend-of"
  - "left-of"
  - "small"
- All except the last of these hold between two things. They are called *binary* relations. "Small" is a *unary relation*.
- You can also have ternary relations, etc.

Relations involving more than one world

- For example, students and courses:

$$\begin{gathered} John,\ EECS303 \\ Ed,\ EECS376 \\ John,\ EECS280 \\ Mary,\ Math606 \\ Mary,\ Math747 \\ Paul,\ Math747 \\ John,\ Math606 \end{gathered}$$

- You can look at each row of the table as an ordered pair, and the table as a set of ordered pairs.
- That's the official definition of “binary relation between two sets.”

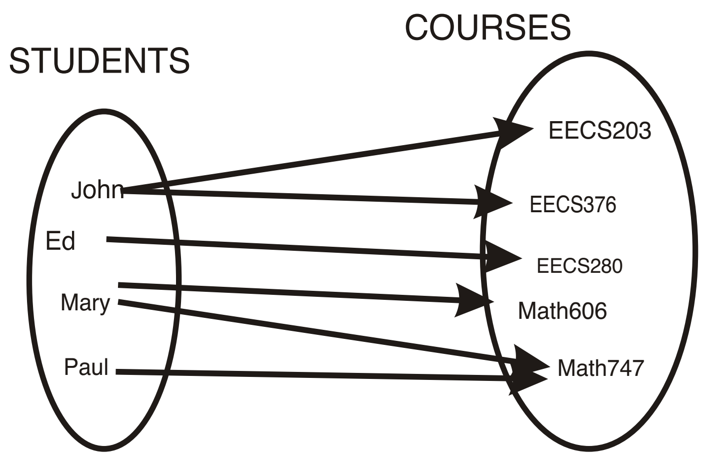
Official Relation Definitions

- **Definition** *Let A, B be sets.*
  - *A binary relation between A and B is a subset R of $A \times B$.*
  - *A binary relation on A is a subset of $A \times A$.*
- **Example**

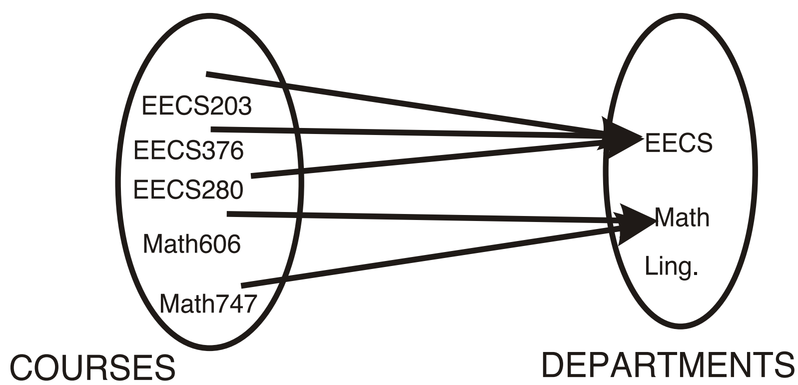
$$S = \{John, Ed, Mary, Paul\}$$
$$C = \{E203, E376, E280, M606, M747\}$$
$$ENROLLED = \{(John, E203), (John, E376), (Ed, E280), (Mary, M606), (Mary, M747), (Paul, M747)\}$$

- The relation $ENROLLED \subseteq S \times C$.
- Notation: we write $a\ R\ b$ to mean $(a, b) \in R$. Thus, $John\ ENROLLED\ E376$.

Picturing Relations



Enrollment - a relation between students
and courses



"Offered-by" - a relation between courses
and departments

Array (matrix) representation of relations

- The "enrolled" information can be presented:

S/C	E203	E376	E280	M606	M747
John	1	1	0	0	0
Paul	0	0	0	0	1
Mary	0	0	0	1	1
Ed	0	0	1	0	0

- This can be seen as a function from $S \times C$ to $\{0, 1\}$:

$$F_{ENR}(s, c) = \begin{cases} 1 \text{ if } (s, c) \in ENR, \\ 0 \text{ otherwise.} \end{cases}$$

Relations as maps to a powerset

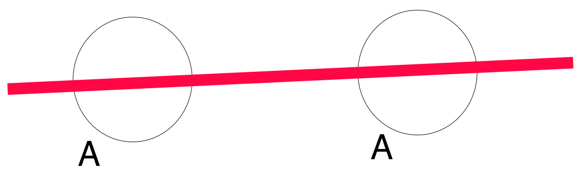
- A relation from A to B can be thought of as a function from A to $\mathcal{P}(B)$.
-
$$\begin{aligned} John &\mapsto \{E203, E376\} \\ Paul &\mapsto \{M606\} \\ Mary &\mapsto \{M606, M747\} \\ Ed &\mapsto \{E280\} \end{aligned}$$
- There are lots of mathematical ways to model relations.

Relations on a set

- The set can be infinite.
- Example: $A = \mathbb{N}^+$; $DIV = \{(m, n) \mid m \text{ divides } n\}$.
- Some ordered pairs: $(1, 10), (3, 6), (4, 20) \ldots$.
- We write $m \mid n$ to indicate that $(m, n) \in DIV$. Thus $1 \mid 10,\ 3 \mid 6,\ 4 \mid 20, \ldots$.
- The "less-than-or-equal-to" relation on $\mathbb{R}$ is another example of a binary relation on an infinite set.

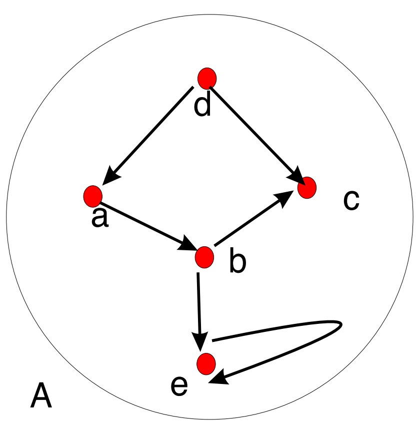
Picturing a relation on a finite set A

Don't need two copies of
the set:



Suppose A = {a,b,c,d,e} and
R = {(d,a), (d,c), (a,b), (b,c),(b,e),(e,e)}.

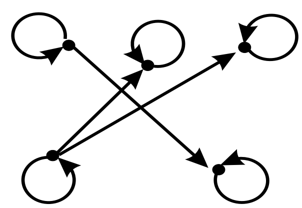
Just work with one copy of A:



Properties of Relations on a set

- **Definition** *A binary relation R on a set A is said to be reflexive if $(\forall a \in A)(a\ R\ a)$.*

- In terms of the graph picture, there is a self-loop at every node:



- **Example** *On every set A, there is the identity relation*

$$id_A = \{(a, a) \mid a \in A\}.$$

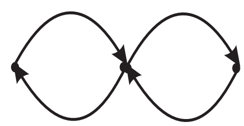
- Other examples:
  - the "less-than-or-equal-to" relation $\leq$ on $\mathbb{R}$;
  - the subset relation on $\mathcal{P}(X)$, because $Y \subseteq Y$ for any set Y;
  - the "divides" relation $\mid$ on $\mathbb{N}^+$.

Symmetric Relations

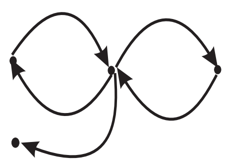
- **Definition** *A binary relation R on a set A is said to be symmetric if*

$$(\forall a, b \in A)(a \; R \; b \rightarrow b \; R \; a).$$

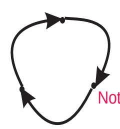
- In terms of pictures:



Symmetric



Not Symmetric



Not Symmetric

- **Example** *The "mutual friend of" relation is symmetric. The "sister" relation isn't. The "sibling" relation is. The identity relation is. The $\leq$ relation isn't.*

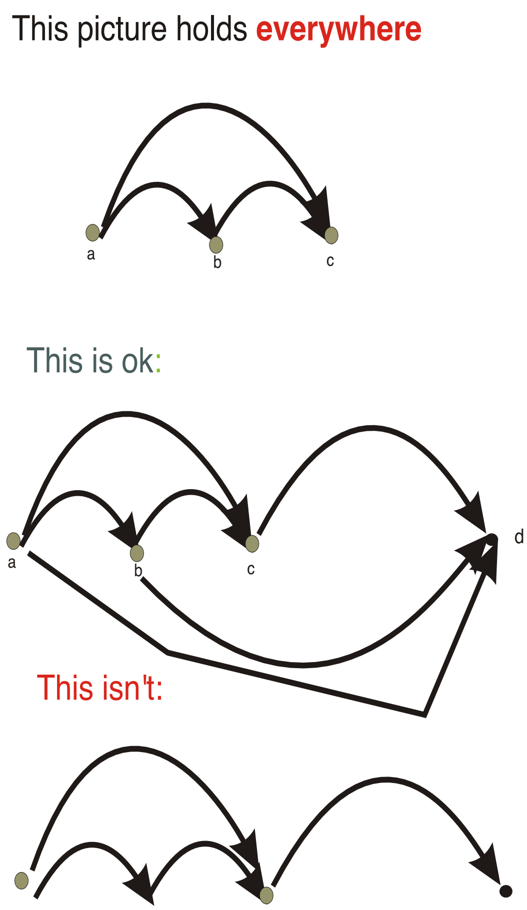
Transitive Relations

- **Definition** *A binary relation R on a set A is said to be transitive if*

$$(\forall a, b, c \in A)(a \; R \; b \wedge b \; R \; c \rightarrow a \; R \; c)$$

.

- The pictures:



Examples of Transitive Relations

- The $\leq$ relation on $\mathbb{N}$.
- The divisibility relation on $\mathbb{N}^+$.
- The subset relation $\subseteq$ on $\mathcal{P}(X)$.
- The relation of logical equivalence on the set of propositional expressions.
- NOT the "father-of" relation on the set of people.
- NOT the "acquainted-with' relation on the set of people.
- How about the "sister-of" relation?