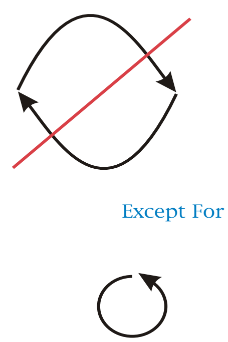
Antisymmetric Relations

- **Definition** *A relation R on A is said to be antisymmetric if*

$$(\forall a, b \in A)(a \; R \; b \wedge b \; R \; a \rightarrow a = b).$$

- The picture for this is:



- **Example** *The $\leq$ relation on $\mathbb{R}$: if $a \leq b$ and $b \leq a$ then $a = b$.*

- **Example** *The subset relation $\subseteq$ on $\mathcal{P}(X)$: if $A \subseteq B$ and $B \subseteq A$ then $A = B$.*

Operations on Relations

- Because relations are sets of ordered pairs, we can combine them using set operations of union, intersection, and complement. These are called the Boolean operations on relations.

- **Example** *Let* $A = \{a, b, c\}$; $R = \{(a, b), (a, c)\}$, *and* $S = \{(c, a)\}$. *Then* $R \cup S = \{(a, b), (a, c), (c, a)\}$; $R \cap S = \emptyset$, *and*

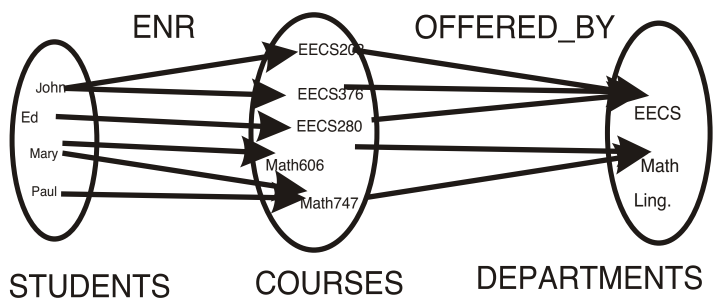
  $$\overline{R} = (A \times B) \backslash R = \{a, a), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}.$$

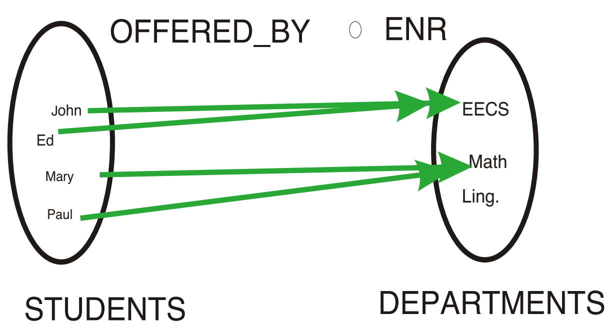
- **Example** *Let A be the set of people. Let B = "brother-of" and S = "sister-of". Then $B \cup S$ = "sibling-of", and $B \cap S = \emptyset$.*

Composing relations

- Because relations are generalizations of functions, it makes sense to ask if we can compose them like functions.
- Consider the "enrolled-in' relation and the "offered-by" relation. The first one is between students and courses, and the second is between courses and departments.
- We can compose the two relations to find out which students participate in which departments.
- Let E stand for the "enrolled-in" relation, and O be the "offered-by" relation. We picture on the next slide the composition $O \circ E$.
- Even though E is the "first" relation, we respect the conventions for functional composition. (Recall that $F \circ G(x) = F(G(x))$.)

Participation in Departments





Defining relational composition

- **Definition** *Let R be a relation between A and B, and S be a relation between B and C. In this case the composition $S \circ R$ can be defined, and is given by the following:*

  $$S \circ R = \{(a, c) \in A \times C \mid (\exists b)((a, b) \in R \text{ and } (b, c) \in S)\}.$$

- This definition says that in order to relate a to c all the way across fom A to C, there has to exist a “bridge element” b in the set B.

- This suggests that there is some connection between the operation of relational composition and the concept of transitivity.

Relational Composition related to Transitivity

- If R is a relation on a set A (a subset of $A \times A$) then we can always compose R with itself. In this case

  $$R \circ R = \{(a, c) \mid (\exists b)((a, b) \in R \text{ and } (b, c) \in R)\}.$$

- Recall that R is transitive iff for all a, b, c, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

- **Theorem** *A relation R on A is transitive if and only if $R \circ R \subseteq R$.*

  The proof is in two parts.

  (i) Assume that R is transitive. Let $(a, c) \in R \circ R$. We show $(a, c) \in R$. Because $(a, c) \in R \circ R$, there is a b so that $(a, b) \in R$ and $(b, c) \in R$. By transitivity of R, $(a, c) \in R$.

  (ii) Conversely, assume $R \circ R \subseteq R$. We must show that R is transitive. Applying the definition of transitivity, let $(a, b) \in R$ and $(b, c) \in R$. Then $(a, c) \in R \circ R$. Since $R \circ R \subseteq R$, we get $(a, c) \in R$, as we wanted. (QED)

Relational Composition and Boolean Matrix Multiplication

- If you use the Boolean matrix representation of relations on a finite set, you can calculate relational composition using an operation called matrix multiplication. See Chapter 2 for some background.

- Let R be a relation on a finite set A with n elements. The Boolean matrix of R will be denoted $[R]$ and is an $n \times n$ array $[R](i,j)$, where $(i,j) \in A \times A$, and

$$[R](i,j) = \begin{cases} 1 & \text{if } (i,j) \in R, \\ 0 & \text{otherwise.} \end{cases}$$

- **Example** *Let* $A = \{1,2,3\}$ *and let* $R = \{(1,2),(2,3),(3,2),(3,3)\}$. *Then*

$$[R] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Example continued

- Further, let $S = \{(1,3),(3,1)\}$, so that the matrix of S is
$$[S] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

- We get the matrix for $S \circ R$ by taking the matrix product $[R] * [S]$. This is given by the formula
$$([R] * [S])(i,k) = \bigvee_{j=1}^{n} ([R](i,j) \wedge [S](j,k)).$$

- Note the similarity to the relational composition definition
$$(i,k) \in S \circ R \leftrightarrow (\exists j, 1 \le j \le n)((i,j) \in R \wedge (j,k) \in S).$$

Example concluded

- Recall $[R] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $[S] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

- For example, to get the (1,3) entry of $[R] * [S]$) we take row 1 of $[R]$, which is $(0, 1, 0)$, and column 3 of S, which is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and form

$$(0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) = 0,$$

  using the truth tables for $\wedge$ and $\vee$.

- You have to repeat this "row i by column k" for each entry (i, k) of the product.

Closures of relations

- Sometimes you have a relation which isn't reflexive, or isn't symmetric, or isn't transitive.
- For each of these properties, we can add ordered pairs to the relation, just enough to make it have the given property. The resulting relation is called the reflexive closure, symmetric closure, or transitive closure respectively.
- Another way to say this is that for property X, the X closure of a relation R is the smallest relation containing R that has property X, where X can be "reflexive" or "symmetric" or "transitive".
- We denote the reflexive closure of R by $refc(R)$, the symmetric closure of R by $symc(R)$, and the transitive closure by $tc(R)$. Another popular notation, though, for the last is R^+.

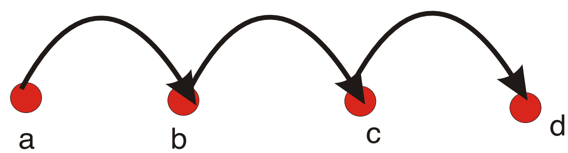
Reflexive and Symmetric Closures

- These are easy, and also are not used a lot.
- **Definition** *The reflexive closure of a relation R on a set A is defined to be $refc(R) = R \cup id_A$.*
- **Example** *If $R = \{(1,2),(2,3),(3,2),(3,3)\}$, then $refc(R) = \{(1,2),(2,3),(3,2),(3,3),(1,1),(2,2)\}$.*
- **Definition** *The symmetric closure of a relation R on a set A is defined to be $symc(R) = R \cup \breve{R}$, where $\breve{R} = \{(y,x) \mid (x,y) \in R\}$.*
- **Example** *If $R = \{(1,2),(2,3),(3,2),(3,3)\}$, then $symc(R) = \{(1,2),(2,1),(2,3),(3,2),(3,3)\}$.*

Transitive Closure

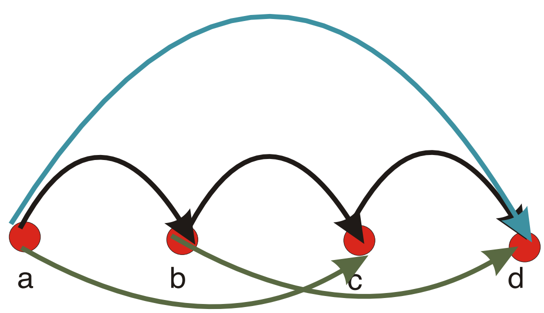
- This is more interesting; adding ordered pairs is an iterative process
- Transitive closure on directed graphs shows where you can go using **some number** of arcs.
- To get the transitive closure, you first add all arrows that traverse (jump) two original arrows; then those that traverse three, and so forth.
- We illustrate on the next slide.

Graphical construction of transitive closure



Original Relation

Original Relation plus 2-jumps



Original Relation plus 2-jumps plus 3-jump

Recursive Definition of Transitive Closure

- Given a binary relation R on a set A, we use the following rules to construct the relation $tc(R)$.

  1. "Basis": if $(x, y) \in R$, then $(x, y) \in tc(R)$.
  2. "Induction": if $(x, y) \in tc(R)$ and $(y, z) \in tc(R)$, then $(x, z) \in tc(R)$.
  3. No pair is in $tc(R)$ unless it is shown there using a finite number of applications of rules 1 and 2.

- **Example** *Using these rules on the example on the last slide, we first use rule 1 three times to put* $(a, b), (b, c)$, *and* (c, d) *into* $tc(R)$.

- *We add the "two-jumps" with two uses of rule 2: once from* (a, b) *and* (b, c) *to get* (a, c), *and then from* (b, c) *and* (c, d) *to get* (b, d).

- *We then use rule 2 from* (a, c) *and* (c, d) *to get* (a, d), *the three-jump.*

- We need rule 3 to insure that $tc(R)$ is the smallest transitive relation containing R. For example, $S = A \times A$ is a much bigger transitive relation containing R.

Proving that $tc(R)$ is what's advertised

- We have to show (i) that $tc(R)$ is transitive, and (ii) that if $R \subseteq S$ and S is transitive then $tc(R) \subseteq S$.
- Proving (i) is easy. If (x, y) and (y, z) are in $tc(R)$ then they got there by some finite number of rule applications. Just one more application of rule 2 puts (x, z) into $tc(R)$.
- The proof of (ii) is a little harder. Let $(x, z) \in tc(R)$. We use induction on the number n of rule applications necessary to get $(x, z) \in tc(R)$, to show that $(x, z) \in S$. The proof continues on the next slide.

Proof continued

Basis: $n = 1$. Then we used rule 1, which says $(x, z) \in R$. Since $R \subseteq S$ we have $(x, z) \in S$, completing the basis case.

Induction step. Assume that whenever (u, v) can be put into $tc(R)$ by using k or fewer rule applications, then $(u, v) \in S$. Suppose we can put (x, z) into $tc(R)$ using $k + 1$ rule applications. The last rule we use is (without loss) rule 2, so we had an $(x, y) \in tc(R)$ using k or fewer applications, and also a (y, z) in $tc(R)$ using k or fewer applications. By inductive hypothesis (twice) $(x, y) \in S$ and $(y, z) \in S$. But S is transitive, so $(x, z) \in S$, as we wanted. (QED)

Characterizing $tc(R)$ other ways

- Since transitivity is connected to composition, it makes sense to see if there's a way to express $tc(R)$ using composition.
- There is a formula to that effect, which leads to a matrix algorithm for calculating transitive closure.
- The formula requires defining powers of a relation inductively.
- **Definition** *Let R be a binary relation on A. For $j \geq 1$ we define the powers R^j of R: put $R^1 = R$ and $R^{j+1} = R^j \circ R$.*
- **Theorem**

$$\begin{aligned} tc(R) &= \bigcup_{j=1}^{\infty} R^j \\ &= R^1 \cup R^2 \cup \ldots \cup R^j \cup \ldots \end{aligned}$$

  You can prove by induction on n that if (x, z) gets into $tc(R)$ by n or fewer rule applications then for some p, $(x, z) \in \bigcup_{j=1}^{p} R^j$. Conversely, if (x, z) is in R^n, then there is a proof using some finite number of steps showing that $(x, z) \in tc(R)$.

A formula for the transitive closure

- if A has only n elements then
$$tc(R) = \bigcup_{j=1}^{n} R^j.$$
. (This can be proved by induction.)

- Let $[R]$ be the matrix of R. Then the matrix of the transitive closure
$$[tc(R)] = \bigvee_{j=1}^{n} [R]^j.$$

- Here we define, for two square Boolean matrices M and N,
$$(M \vee N)(i,j) = M(i,j) \vee N(i,j),$$
i.e., just the elementwise "or" of the two matrices.

- This can be seen to give an $O(n^4)$ algorithm to find the transitive closure. With better bookkeeping, one can derive an $O(n^3)$ algorithm.