Equivalence Relations

- You can have a relation which simultaneously has more than one of the properties we have been discussing.
- **Definition** *An equivalence relation on a set A is one which is reflexive, symmetric, and transitive.*
- Instead of a generic name like R, we use symbols like $\equiv$ to stand for equivalence relations.
- This is because an equivalence relation behaves like the identity relation (the equality relation) on A. It lets things be similar without being equal.
- Using this notation, let's recap the three properties. For all $a, b, c \in A$,

  1. $a \equiv a$;
  2. If $a \equiv b$, then $b \equiv a$;
  3. If $a \equiv b$ and $b \equiv c$ then $a \equiv c$.

Examples of equivalence relations

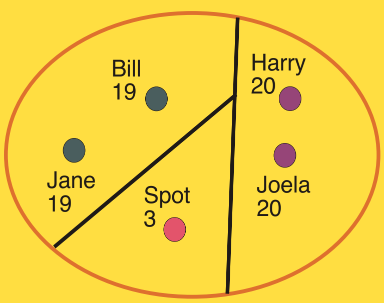
- The identity relation id_A on A.
- The relation "same-age-in years-as" on the set of people.
- The relation "sitting in the same row as" on the set of students in this classroom.
- The relation $\iff$ between propositional expressions.
- The relation of similarity between different triangles sitting in the plane.
- Let A be the set of all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$. Define $f \sim g$ as $f = \Theta(g)$. Then $\sim$ is an equivalence relation on A.

An extended example: congruence mod 5

- Let $\mathbb{Z}$ be the set of integers, positive, negative, and 0.
- We define $m \equiv n \pmod 5$ if $m - n$ is a multiple of 5. This is pronounced "m is congruent to n mod 5."
- Thus $3 \equiv 8$, $0 \equiv -10$, $104 \equiv 89$, etc.
- We prove that $\equiv \pmod 5$ is reflexive, symmetric, and transitive.
  1. Since $m - m = 0 = 0 \cdot 5$, $\equiv$ is reflexive.
  2. If $m - n = 5k$, then $n - m = 5(-k)$, so $\equiv$ is symmetric.
  3. If $m - n = 5k$ and $n - p = 5j$, then $m - p = (m - n) + (n - p) = 5k + 5j = 5(k + j)$, so $\equiv$ is transitive.
- We can define $\equiv \pmod p$ for any positive integer $p \geq 2$. Quite often p is a prime number like 5.

Equivalence relations and partitions

- Equivalence relations "group things as being the same."
- Consider the "same age" relation in the following picture.



Students along with their ages.

- The people in the subgroups are all the same age as each other (and themselves.)
- This collection of subsets is what's called a partition.

From equivalence relations to partitions

- We start with an equivalence relation $\sim$.
- In our example, the relation $\sim$ is the following set of ordered pairs:

$$\{(Bill, Jane), (Jane, Bill), (Harry, Joela), (Joela, Harry)\} \cup id_A.$$

- **Definition** *If $x \in A$, the equivalence class $[x]$ of x is*

$$\{y \in A \mid x \sim y\}.$$

- In our example, $[Bill] = [Jane] = \{Bill, Jane\}$, $[Spot] = \{Spot\}$, and $[Harry] = [Joela] = \{Harry, Joela\}$.
- You can see that these are the subsets indicated in the picture.

ERs to Parts, continued

- **Definition** *Given a set A, a partition of A is a collection Π of subsets of A having the following two properties:*

  1. *Every element of A is in least one set in Π;*
  2. *Every element of A is in at most one set in Π.*

- The sets in Π are called the *blocks* of the partition. We may summarize properties 1 and 2 by saying that each element of A is a member of exactly one block of Π.

- In our example,

$$\Pi = \{\{Bill, Jane\}, \{Spot\}, \{Harry, Joela\}\}.$$

- **Theorem** *The equivalence classes of an equivalence relation on a set A form a partition of A.*

Proof of the theorem.

- Assume that $\equiv$ is a reflexive, symmetric, and transitive relation on A.
- Because $\equiv$ is reflexive, for every $a \in A$, $a \in [a]$. So every element of A is in at least one equivalence class.
- We need to show that every element of A is in at most one equivalence class. This uses symmetry and transitivity. Suppose that a is in $[x] \cap [y]$. We show that $[x] \subseteq [y]$, which will be enough to show $[x] = [y]$, because the reverse inclusion comes from an exactly similar proof.
- Let $z \in [x]$. Then $x \equiv z$. Since $a \in [x]$, $x \equiv a$. Since $a \in [y]$, $y \equiv a$.
- Rearrange the statements $y \equiv a$ and $x \equiv z$ by symmetry to $a \equiv y$ and $z \equiv x$. We then have $z \equiv x \wedge x \equiv a \wedge a \equiv y$. Therefore $z \equiv y$ by transitivity. By symmetry, $y \equiv z$, proving $z \in [y]$. Therefore $[x] \subseteq [y]$. (QED)

Example of equivalence classes

- Recall the "congruence mod 5" relation on $\mathbb{Z}$:

  $$m \equiv n \pmod 5 \text{ if } m - n \text{ is a multiple of } 5.$$

- What are the equivalence classes of this relation?
- There are exactly 5 such classes: $\{[0], [1], [2], [3], [4]\}$.
- For example

  $$[2] = \{2{+}5k \mid k \in \mathbb{Z}\} = \{2, -3, 7, -8, 12, -13, \ldots\}.$$

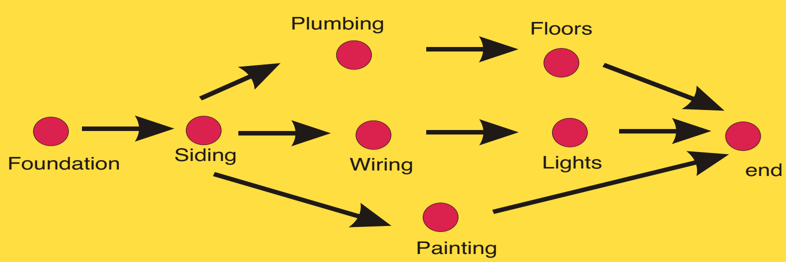
- The equivalence class of n is determined by the non-negative remainder when n is divided by 5. So, $[12] = [-7] = [2]$.

Partial Orderings

- We come to the last major type of relation in the chapter on relations.
- **Definition** *A partial ordering on a set A is a reflexive,* **antisymmetric**, *and transitive relation on A.*
- Using "antisymmetric" instead of "symmetric" completely changes the character of these relations from that of equivalence relations.
- Two standard examples: the relation $\leq$ on $\mathbb{R}$, and the relation $\subseteq$ on $\mathcal{P}(A)$.
- The antisymmetry property: if $x \leq y$ and $y \leq x$ then $x = y$. Similarly for $\subseteq$: if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$.
- The difference between these two is that you can compare any two numbers, but you cannot compare any two subsets. For example, if $A = \{1, 2, 3\}$, $X = \{1, 2\}$, and $Y = \{1, 3\}$, then X is not a subset of Y and Y is not a subset of X. This is why we say "partial".

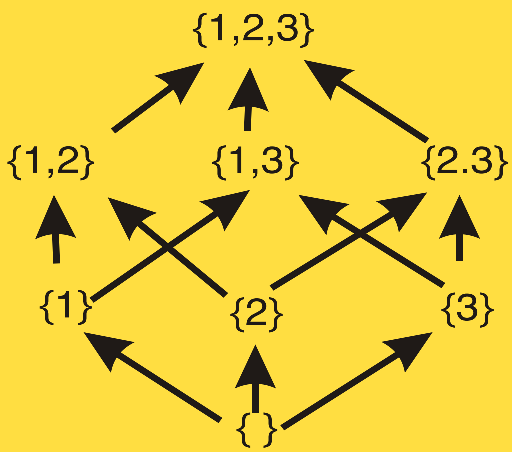
Examples in pictures

- Scheduling tasks in building a house:

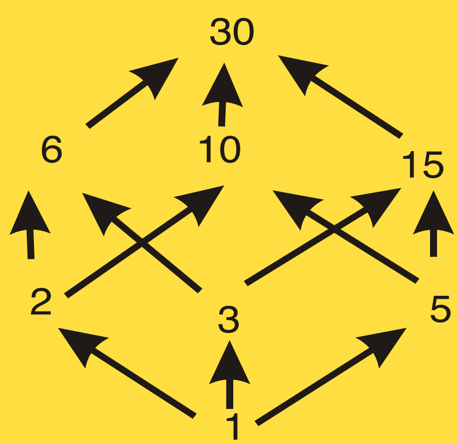


- In these graphs, we omit the self loops and the arrows required by transitivity. The actual relation would be the reflexive, transitive closure of the one pictured here.

Two more examples



Subsets of {1,2,3}



Divisors of 30