EECS 203-1
Definitions and Theorems for Midterm 2

- Functions; understand intuitive definition. Be able to define surjective, bijective, and injective functions.

  A function $f : A \to B$ is *injective* if $f(x) = f(y) \to x = y$ for all $x, y \in A$. It is *surjective* if $(\forall b \in B)(\exists a \in A)(f(a) = b)$. It is *bijective* if it is both injective and surjective.

- Definition of functional composition.

  Given $g : A \to B$ and $f : B \to C$, the *composition* $f \circ g : A \to C$ is given by $(f \circ g)(x) = f(g(x))$.

- Sequences.

  A *sequence* is a function $a : \{1, \ldots, n\} \to A$, or from $\mathbb{N} \to A$ for some set A.

- Definition $f = O(g)$ for functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$.

  Given two such functions, we say $f = O(g)$ if there are constants k and C such that for all $x \geq k$, $f(x) \leq C \cdot g(x)$.

- First and second principles of mathematical induction (complete statements)

  First principle of mathematical induction: Given a statement $P(n)$ with a free natural-number variable n. If $P(0)$ is true, and $(\forall k)(P(k) \to P(k+1))$ is true, then $P(n)$ is true for all $n \in \mathbb{N}$.

  Second principle: If $P(0)$ is true, and $(\forall k)(((\forall j \leq k)P(j)) \to P(k+1))$ is true, then $P(n)$ is true for all $n \in \mathbb{N}$.

- Def: binary relation from A to B; binary relation on A

  A *binary relation* from A to B is a subset of $A \times B$. A binary relation on A is a subset of $A \times A$.

- Def: reflexive, symmetric, antisymmetric, transitive relations

  Let R be a binary relation on A. R is *reflexive* if $(\forall a \in A)((a, a) \in R)$. R is *symmetric* if $(\forall x, y \in A)((x, y) \in R \to (y, x) \in R)$. R is *transitive* if $(\forall x, y, z \in A)((x, y) \in R \wedge (y, z) \in R \to (x, z) \in R)$. R is *antisymmetric* if $(\forall x, y \in A)((x, y) \in R \wedge (y, x) \in R \to x = y)$.

- Def: composition $S \circ R$ of two relations R and S;

  Given $R \subseteq A \times B$ and $S \subseteq B \times C$, the *composition* $S \circ R$ is the set

  $$\{(a, c) \in A \times C \mid (\exists b \in B)((a, b) \in R \wedge (b, c) \in S)\}.$$

- Know the theorem characterizing transitive relations in terms of composition

  Given $R \subseteq A \times A$, R is transitive if and only if $R \circ R \subseteq R$.

- reflexive, symmetric, transitive closures of a binary relation: The reflexive closure of a relation R on A is $R \cup id_A$. The symmetric closure is the relation $R \cup R^{-1} = R \cup \{(y,x) \mid (x,y) \in R\}$.

- Inductive definition of transitive closure:

  Given a binary relation R on a set A, we use the following rules to construct the relation $tc(R)$.

  1. "Basis": if $(x,y) \in R$, then $(x,y) \in tc(R)$.
  2. "Induction": if $(x,y) \in tc(R)$ and $(y,z) \in tc(R)$, then $(x,z) \in tc(R)$.
  3. No pair is in $tc(R)$ unless it is shown there using a finite number of applications of rules 1 and 2.

- Powers of a relation; formula for transitive closure of a general relation. Let R be a binary relation on A. For $j \geq 1$ we define the powers R^j of R: put $R^1 = R$ and $R^{j+1} = R^j \circ R$.

  Theorem:

  $$\begin{aligned} tc(R) &= \bigcup_{j=1}^{\infty} R^j \\ &= R^1 \cup R^2 \cup \ldots \cup R^j \cup \ldots \end{aligned}$$

- Definition of equivalence relation; definition of partition; definition of the equivalence class of an element.

  An *equivalence relation* on a set A is one which is reflexive, symmetric, and transitive.

  Given a set A, a *partition* of A is a collection Π of subsets of A having the following two properties: (1): Every element of A is in least one set in Π; (2): Every element of A is in at most one set in Π.

  If $\sim$ is an equivalence relation on A, and $x \in A$, the *equivalence class* $[x]$ of x is

  $$\{y \in A \mid x \sim y\}.$$