EECS 203
Homework – 5 Solutions

Total Points: 40

Page 69:

56) Suppose that f is an invertible function from Y to Z and g is an invertible function from X to Y. Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

4 points

If a function is invertible, then it has to be one-to-one and onto i.e it has to be a bijective function.

g : X to Y, $\therefore$ there is a g^{-1} : Y to X which is also bijective.

$\therefore g(g^{-1}(y)) = y$ and $g^{-1}(g(x)) = x$

f : Y to Z, $\therefore$ there is a f^{-1} : Z to Y which is also bijective.

$\therefore f(f^{-1}(z)) = z$ and $f^{-1}(f(y)) = y$

Since f and g are bijective functions (f o g): X to Z is also bijective.

$\therefore (f \circ g)^{-1}$ exists and takes elements in Z to elements in X.

Since the range (Y) of f^{-1} is the domain (Y) of g^{-1}, $(g^{-1} \circ f^{-1})$: Z to X exists and is bijective.

To prove that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$, we need to prove that

$g^{-1} \circ f^{-1}(f \circ g(x)) = x$ and $f \circ g\,(g^{-1} \circ f^{-1}(z)) = z$

Let $g(x) = y$ and let $f(y) = z$

$g^{-1} \circ f^{-1}(f \circ g(x)) = g^{-1}(f^{-1}(f(g(x)))) = g^{-1}(f^{-1}(f(y))) = g^{-1}(y) = x$.

$f \circ g\,(g^{-1} \circ f^{-1}(z)) = f(g(g^{-1}(f^{-1}(z)))) = f(g(g^{-1}(y))) = f(y) = z$.

Thus $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Page 78:

4) What are the terms a_0, a_1, a_2 and a_3 of the sequence $\{a_n\}$, where a_n equals

4 points

a) $(-2)^n$?
Ans: **1, -2, 4, and -8.**

b) 3?
Ans: **3, 3, 3, and 3**.

c) $7+4^n$?
Ans: **8, 11, 23, and 71**.

d) $2^n + (-2)^n$?
Ans: **2, 0, 8, and 0.**

---

6) List the first 10 terms of each of the following sequences

2 points

a) The sequence obtained by starting with 10 and obtaining each term by subtracting 3 from the previous term.

10, 7, 4, 1, -2, -5, -8, -11, -14, -17.

d) The sequence whose n^{th} term is $\lfloor \sqrt{n} \rfloor$

0, 1, 1, 1, 2, 2, 2, 2, 2, 3.

14) What are the values of the following sums, where S ={1, 3, 5, 7}

4 points

a) $\sum_{j \in S} j$

16

b) $\sum_{j \in S} j^2$

84

c) $\sum_{j \in S} (1/j)$

1/1 + 1/3 + 1/5 +1/7 = 1.6761

d) $\sum_{j \in S} 1$

4

---

20) Use the identity 1/(k (k+1)) = 1/k – 1/(k+1) and Exercise 19 to compute

$$\sum_{k=1}^{n} 1/(k(k+1))$$

2 points

1/(k (k+1)) = 1/k – 1/(k+1)

$$\therefore \sum_{k=1}^{n} 1/(k(k+1)) = \sum_{k=1}^{n} 1/k - 1/(k+1)$$

$$= 1/1 - 1/(1+1) + 1/2 - 1/(2+1) + \ldots + 1/n - 1/(n+1)$$

[This is called **Telescopic cancellation** and it is useful in many problems]

= 1 – 1/(n+1)

$$\therefore \sum_{k=1}^{n} 1/(k(k+1)) = 1 - 1/(n+1)$$

Page 90:

14) Is it true that x^3 is $O(g(x))$, if g is the given function? **[**For example, if $g(x) = x+1$, this question asks whether x^3 is $O(x+1)$**]**

6 points

a) $g(x) = x^2$

No

c) $g(x) = x^2 + x^3$

Yes

e) $g(x) = 3^x$

Yes

---

18) **2 points**

Let k be a positive integer. Show that $1^k + 2^k + \ldots + n^k$ is $O(n^{k+1})$

Since each of the terms in the sum of the 1st n terms do not exceed n^k, it follows that

$1^k + 2^k + \ldots + n^k \le n^k + n^k + n^k + \ldots$ n times $= n.n^k = n^{k+1}$

$\therefore |1^k + 2^k + \ldots + n^k| \le |n^{k+1}|$

$\therefore$ Taking **C = 1** and for **n > 1**, by definition

$1^k + 2^k + \ldots + n^k$ is $O(n^{k+1})$ {Note that here k is some constant value}

---

22) **12 points**

For each function in Exercise 1, determine whether that function is $\Omega(x)$ and whether it is $\Theta(x)$?

Note: A function f(x) is $\Theta(g(x))$ if it is both **O**(g(x)) and $\Omega(g(x))$. Here the function g(x) = x (the identity function).

a) $f(x) = 10$

This function is $O(x)$ but **not** $\Omega(x)$ $\therefore$ it is **not** $\Theta(x)$

b) $f(x) = 3x + 7$

This function is $O(x)$ and also $\Omega(x)$ $\therefore$ it is $\Theta(x)$

c) $f(x) = x^2 + x + 1$

This function is **not** $O(x)$, but is $\Omega(x)$ $\therefore$ it is **not** $\Theta(x)$

d) $f(x) = 5\log x$

This function is $O(x)$, but **not** $\Omega(x)$ $\therefore$ it is **not** $\Theta(x)$

e) $f(x) = \lfloor x \rfloor$

This function is $O(x)$ and also $\Omega(x)$ $\therefore$ it is $\Theta(x)$

f) $f(x) = \lceil x/2 \rceil$

This function is $O(x)$ and also $\Omega(x)$ $\therefore$ it is $\Theta(x)$

---

30) **4 points**

Explain what it means for a function to be $\Omega(1)$.

All functions for which there exists real numbers *k* & *C* such that

$|f(x)| \ge C$, for $x > k$ are $\Omega(1)$.

In other words, if a function is $\Omega(1)$, it is actually bounded away from 0 for all sufficiently large values of **x.**
