EECS 203-1 Homework –6 Solutions

Total Points: 30
Page 182:

16a) Prove that the square of an even number is an even number using a direct proof.

5 points

Assume that the hypothesis of this implication is true, namely, suppose that n is even. Then n = 2k, for some k is an integer. It follows that
\[ n^2 = (2k)^2 = 2^2k^2 = 2(2k^2). \]
Therefore, \( n^2 \) is even (it is twice an integer value).

22) Prove that the product of two rational numbers is rational

3 points

By definition, any rational number can be written as \( p/q \) where \( p \) and \( q \) are integers and \( q \neq 0 \). Now, let \( x \) and \( y \) be the 2 rational numbers. Let \( x = a/b \) and \( y = c/d \), where \( a, b, c \) and \( d \) are integers.

On multiplying them, we get:
\[ x * y = (a/b) * (c/d) = (a*c)/(b*d). \]

Now, if \( a \) and \( c \) are integers, then \( a*c \) is also an integer.
Similarly for \( b \) and \( d \). So the product of two rational numbers must be a rational number.

26) Prove or disprove that \( 2^n + 1 \) is prime for all nonnegative integers \( n \).

3 points

This can be disproved by a counter example:
For \( n = 3 \),
\[ 2^n + 1 = 2^3 + 1 = 8 + 1 = 9, \]
and 9 is not a prime number.
Remember it is easy to disprove something by finding just one counter-example.
Prove or disprove each of the following statements about the floor and ceiling functions

4 points

b) \[ \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor \] for all real numbers \( x \) and \( y \).

This can be disproved by a counter example:

For \( x = 0.25 \) and \( y = 3.75 \)

L.H.S is

\[ \lfloor x + y \rfloor = \lfloor 0.25 + 3.75 \rfloor = \lfloor 4.00 \rfloor = 4 \]

and R.H.S is

\[ \lfloor x \rfloor + \lfloor y \rfloor = \lfloor 0.25 \rfloor + \lfloor 3.75 \rfloor \]
\[ = 0 + 3 \]
\[ = 3 \]

\[ \therefore \text{L.H.S} \neq \text{R.H.S} \]

28) Do problem 28 on page 183 by a generalization of the proof I did in class, involving prime factorizations. Show (as a lemma) that a positive number \( j \) is a perfect square if and only if each of its \( p \)-levels, for \( p \) a prime number, is even. (The prime numbers start at 2.) The \( p \)-level of a number \( j \) is the exponent of \( p \) in the prime factorization of \( j \). Thus the 2-level of 98 = 2\(^1\).7\(^2\) is 1, and the 7-level is 2.

Problem 28: Show that \( \sqrt{n} \) is irrational if \( n \) is a positive integer that is not a perfect square.

10 points

Method 1
We will first prove the following Lemma. The lemma is useful to know in general.

Show that a positive number \( j \) is a perfect square if and only if each of its \( p \)-levels, for \( p \) a prime number, is even.

Proof:
Let \( j \) be a positive number, we can write \( j \) in terms of it’s prime factors, \( j = P_1^a \cdot P_2^b \cdot P_3^c \ldots \) (where \( P_1, P_2, P_3 \ldots \) are prime numbers, and \( a, b, c \ldots \) are some integer numbers)

a) We have to prove that if \( a, b, c \ldots \) are even numbers, then \( j \) is a perfect square.

\[ \therefore j = P_1^{2a} \cdot P_2^{2b} \cdot P_3^{2c} \ldots \] (where \( a=2x, b=2y, c=2z\ldots\) because \( a, b, c \ldots \) are even integers)

\[ \therefore j = (P_1^x \cdot P_2^y \cdot P_3^z)^2 \]

Thus \( j \) is a perfect square.

b) We have to prove that if \( j \) is a perfect square, then \( a, b, c \ldots \) are even numbers.

\[ \therefore j = k^2 \]

We can write \( k \) as \( k = P_1^x \cdot P_2^y \cdot P_3^z \ldots \) where \( P_1, P_2, P_3 \ldots \) are prime numbers,
and x, y, z are some integers
\[ j = k^2 = (P_1^x P_2^y P_3^z \ldots)^2 = P_1^{2x} P_2^{2y} P_3^{2z} \ldots = P_1^a P_2^b P_3^c \ldots \]
(where a=2x, b = 2y, c=2z..)

Thus a, b, c are even numbers.
Hence proved. (Note that in this proof, we assumed (without proof) that every integer has a unique prime factorization.)

**Now the proof of “√n is irrational when n is not a perfect square.”**
This is a proof by contradiction.
Assume \( \sqrt{n} \) is rational. Therefore \( \sqrt{n} = p/q \) where p and q are integers.
Squaring both sides we get
\[ n = p^2/q^2. \]
\[ \therefore \] p^2 = n * q^2
Since p^2 is a perfect square, the **LHS is a number that has all even p-levels** (by lemma).
Now look at the RHS. q^2 is a perfect square, so the p-levels of q^2 are all even. But n is not a perfect square (by hypothesis) therefore at least one of its p-levels is even. Since multiplication of n and q^2 leads to summing the powers, **at least one of the p-levels of the RHS is odd**. This is a contradiction.
Therefore \( \sqrt{n} \) cannot be rational. **Thus \( \sqrt{n} \) is irrational.**

**Method 2 (Without Using the Lemma)**
Given n (> 1) is not a perfect square. Suppose that \( \sqrt{n} \) is rational i.e \( \sqrt{n} = a/b \), for some positive integers a and b, and that b is the smallest positive integer denominator for which this is true.
Then \( b^2 < n \) \( \Rightarrow b^2 = a^2 \),
because n > 1, so 0 < b < a.
Now divide a by b, obtaining quotient q and remainder r, so a = q*b + r, with 0 ≤ r < b.
Now if r = 0, we have a = q*b, and a/b = q, so n = q^2, and n is a perfect square, a contradiction. This means that r cannot be zero, and so 0 < r = a - q*b < b.
Now \( n \) \( \Rightarrow b^2 = a^2 \)
\[ n^* b^2 - q^* a^* b = a^2 - q^* a^* b \]
\[ b^* (n^* b - q^* a) = a^* (a - q^* b) \]
\[ (n^* b - q^* a)/(a - q^* b) = a/b = \sqrt{n} \]

This contradicts the minimality of b, since 0 < a - q*b < b. This contradiction means that no such integers a and b can exist, and \( \sqrt{n} \) is irrational.
The next two problems refer to the universe of functions from $\mathbb{R}^+$ to $\mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers.

5 points

1) Prove that for any $f$ and $g$, that if $f = O(g)$, then $g = \Omega(f)$.

- $f = O(g)$, implies, by definition, $\exists C, k$ (both positive constant) such that
  $|f(x)| \leq C |g(x)|$, for $x > k$

  $\therefore \frac{1}{C} |f(x)| \leq |g(x)|$

  $\therefore |g(x)| \geq C' \times |f(x)|$, where $C' = \frac{1}{C}$ is a constant

  $\therefore$ By definition of Big Omega, $g(x) = \Omega(f(x))$; i.e.; $g = \Omega(f)$.

2) Prove that if $f = \Theta(g)$, then $g = \Theta(f)$

**Method 1**

- $f = \Theta(g) \Rightarrow f = O(g)$ and $f = \Omega(g)$

- $f = O(g) \Rightarrow g = \Omega(f)$. (Proved above)

- $f = \Omega(g) \Rightarrow g = O(f)$. (Can prove this as above)

- $g = \Omega(f)$ and $g = O(f) \Rightarrow g = \Theta(f)$.

**Method 2** (Longer because it just repeats what was proved above)

- $f = \Theta(g)$ implies

  $f = O(g)$ and $f = \Omega(g)$

  Now $f = O(g)$ implies

  $|f(x)| \leq C |g(x)|$, for $x > k$

  $\therefore |g(x)| \geq C' \times |f(x)|$, where $C' = 1/C$ is a constant

  $\therefore$ By definition of Big Omega, $g = \Omega(f)$ … (1)

Similarly,

- $f = \Omega(g)$ implies

  $|f(x)| \geq C |g(x)|$

  $\therefore |g(x)| \leq C' \times |f(x)|$, where $C' = 1/C$ is a constant

  $\therefore$ By definition, $g = O(f)$. … (2)

From (1) and (2) we can see that $g = \Theta(f)$. 