EECS 203-1 Homework –6 Solutions

Total Points: 30
Page 182:

16a) Prove that the square of an even number is an even number using a direct proof.

5 points

Assume that the hypothesis of this implication is true, namely, suppose that n is even. Then n = 2k, for some k is an integer. It follows that

$n^2 = (2k)^2$
$\quad = 2^2k^2$
$\quad = 2(2k^2)$. Therefore, n^2 is even (it is twice an integer value).

22) Prove that the product of two rational numbers is rational
3 points

By definition, any rational number can be written as p/q where p and q are integers and $q \neq 0$. Now, let x and y be the 2 rational numbers. Let x = a/b and y = c/d, where a, b, c and d are integers.
On multiplying them, we get:
x * y = (a/b) * (c/d) = (a*c)/(b*d).

Now, if a and c are integers, then a*c is also an integer.
Similarly for b and d. So the product of two rational numbers must be a rational number.

26) Prove or disprove that $2^n +1$ is prime for all nonnegative integers n.
3 points

This can be disproved by a counter example:
For n = 3,
$2^n +1 = 2^3 +1 = 8 + 1 = 9$, and 9 is not a prime number.
Remember it is easy to disprove something by finding just one counter-example.

44) Prove or disprove each of the following statements about the floor and ceiling functions

4 points

b) $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ for all real numbers x and y.
This can be disproved by a counter example:

For x = 0.25 and y = 3.75
L.H.S is
$\lfloor x + y \rfloor = \lfloor 0.25 + 3.75 \rfloor = \lfloor 4.00 \rfloor = 4$
and R.H.S is
$\lfloor x \rfloor + \lfloor y \rfloor = \lfloor 0.25 \rfloor + \lfloor 3.75 \rfloor$
$= 0 + 3$
$= 3$
$\therefore$ L.H.S $\neq$ R.H.S

28) **Do problem 28 on page 183** by a generalization of the proof I did in class, involving prime factorizations. Show (as a lemma) that a positive number j is a perfect square if and only if each of its p-levels, for p a prime number, is even. (The prime numbers start at 2.) The p-level of a number j is the exponent of p in the prime factorization of j. Thus the 2-level of $98 = 2^1.7^2$ is 1, and the 7-level is 2.

Problem 28: Show that $\sqrt{n}$ is irrational if n is a positive integer that is not a perfect square.

10 points

Method 1

We will first prove the following Lemma. The lemma is useful to know in general.

Show that a positive number j is a perfect square if and only if each of its p-levels, for p a prime number, is even.

Proof:

Let j be a positive number, we can write j in terms of it's prime factors,
$j=P_1^a.P_2^b.P_3^c\ldots$ (where P_1, P_2, P_3 … are Prime numbers, and a, b, c …are some integer numbers)

a) We have to prove that if a, b, c… are even numbers, then j is a perfect square.

$\therefore j = P_1^{2x}.P_2^{2y}.P_3^{2z}\ldots$ (where a=2x, b = 2y, c=2z…because a, b, c ...are even integers)

$\therefore j = (P_1^x.P_2^y.P_3^z)^2$

Thus j is a perfect square.

b) We have to prove that if j is a perfect square, then a, b, c…are even numbers.

$\therefore$ j= k^2

We can write k ask = $P_1^x.P_2^y.P_3^z$ … where P_1, P_2, P_3 … are prime numbers,

and x, y, z ..are some integers)
$\therefore$ $j = k^2 = (P_1^x.P_2^y.P_3^z \ldots)^2 = P_1^{2x}.P_2^{2y}.P_3^{2z}\ldots = P_1^a.P_2^b.P_3^c\ldots$
(where a=2x, b = 2y, c=2z...)

Thus a, b, c ...are even numbers.
Hence proved. (Note that in this proof, we assumed (without proof) that every integer has a unique prime factorization.)

Now the proof of "*√n is irrational when n is not a perfect square.*"
This is a proof by contradiction.
Assume √n is rational. Therefore √n = p/q where p and q are integers.
Squaring both sides we get
$n = p^2/q^2$.
$\therefore$ $p^2 = n * q^2$
Since p^2 is a perfect square, the **LHS is a number that has all even p-levels** (by lemma).
Now look at the RHS. q^2 is a perfect square, so the p-levels of q^2 are all even. But n is not a perfect square (by hypothesis) therefore at least one of its p-levels is even. Since multiplication of n and q^2 leads to summing the powers, **at least one of the p-levels of the RHS is odd**. This is a contradiction.
Therefore √n cannot be rational. **Thus √n is irrational**.

Method 2 (Without Using the Lemma)
Given n (> 1) is not a perfect square. Suppose that √n is rational i.e
√n = a/b, for some positive integers a and b, and that b is the smallest positive integer denominator for which this is true.
Then $b^2 < n* b^2 = a^2$,
because n > 1, so 0 < b < a.
Now divide a by b, obtaining quotient q and remainder r, so a = q*b + r, with 0 ≤ r < b.
Now if r = 0, we have a = q*b, and a/b = q, so $n = q^2$, and n is a perfect square, a contradiction. This means that r cannot be zero, and
so 0 < r = a - q*b < b.
Now $n* b^2 = a^2$

$$n* b^2 - q*a*b = a^2 - q*a*b$$
$$b*(n*b-q*a) = a*(a-q*b)$$
$$(n*b-q*a)/(a-q*b) = a/b = \sqrt{n}$$

This contradicts the minimality of b, since 0 < a - q*b < b. This contradiction means that no such integers a and b can exist, and
√n is irrational.

The next two problems refer to the universe of functions from R+ to R+, where R+ is the set of positive real numbers.

5 points

1) Prove that for any f and g, that if $f = O(g)$, then $g = \Omega(f)$.

$\boldsymbol{f = O(g)}$, implies, by definition, $\exists$ C, k (both positive constant) such that

$| f(x)| \leq C \, | g(x)|$, for x>k

$\therefore 1/C^* \, | f(x)| \leq | g(x)|$

$\therefore | g(x)| \geq C' * | f(x)|$, where C' = 1/C is a constant

$\therefore$By definition of Big Omega,

$g(x) = \Omega(f(x))$; i.e.; $\boldsymbol{g = \Omega(f)}$**.**

2) Prove that if $f = \Theta(g)$, then $g = \Theta(f)$

Method 1

$\boldsymbol{f = \Theta(g) \Rightarrow f = O(g)}$ **and** $\boldsymbol{f = \Omega(g)}$

$\boldsymbol{f = O(g) \Rightarrow g = \Omega(f)}$**.** (Proved above)

$\boldsymbol{f = \Omega(g) \Rightarrow g = O(f)}$**.** (Can prove this as above)

$\boldsymbol{g = \Omega(f)}$ **and** $\boldsymbol{g = O(f) \Rightarrow g = \Theta(f)}$**.**

Method 2 (Longer because it just repeats what was proved above)

$\boldsymbol{f = \Theta(g)}$ **implies**

$\boldsymbol{f = O(g)}$ **and** $\boldsymbol{f = \Omega(g)}$

Now $\boldsymbol{f = O(g)}$ implies

$| f(x)| \leq C \, | g(x)|$, for x>k

$\therefore | g(x)| \geq C' * | f(x)|$, where C' = 1/C is a constant

$\therefore$By definition of Big Omega, $\boldsymbol{g = \Omega(f)}$ … (1)

Similarly,

$\boldsymbol{f = \Omega(g)}$ implies

$| f(x)| \geq C \, | g(x)|$

$\therefore | g(x)| \leq C' \, | f(x)|$, where C' = 1/C is a constant

$\therefore$By definition, $\boldsymbol{g = O(f)}$**.** … (2)

From (1) and (2) we can see that $\boldsymbol{g = \Theta(f)}$**.**