EECS 203-1 Homework –7 Solutions

Total Points: 30
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10) Prove that 1.1! + 2.2! + …. + n.n! = (n+1)! -1 whenever n is a positive integer.

4 points

Let P(n) = 1.1! + 2.2! + …. + n.n! = (n+1)! -1

Basis Step: P(1) is true since 1.1! = 1 = (1+1)! -1
= 2! -1 = 2 -1 = 1.

Induction Step: Assume P(n) is true for some n. (Induction Hypothesis)

Then we have to show that P(n+1) is true

1.1! + 2.2! + …. + n.n! + (n+1)(n+1)! = (n+1)! -1 + (n+1)(n+1)!
= (n+1)! [1 + n+1] -1
= (n+1)![n+2] -1
= (n+2)! -1
= ((n+1) +1)! -1

Hence proved!

14) Use mathematical induction to prove that $n! < n^n$ whenever n is a positive integer greater than 1.

4 points

Let $P(n) = n! < n^n$

Basis Step: P(2) is true since $2! = 2 < 2^2 = 4$

Induction Step: Assume P(n) is true for some n (> 1 of course)

Then we have to show that P(n+1) is true

$(n + 1)! = n!\,(n+1) < n^n(n+1)$ [since P(n) is true]
$< (n + 1)^n.(n+1)$ [since $n^n < (n+1)^n$]
$= (n + 1)^{n+1}$

Hence proved!

32) Which amounts of money can be formed using just dimes and quarters? Prove your answer using a form of mathematical induction.

6 points

The amounts of money that can be formed with just dimes and quarters is 10¢, 20¢, 25¢, 30¢…

Thus 10¢ and all amounts of the form (20 + 5n)¢ (where n = 0,1,2,3,…) can be made. This is our claim. We have to prove it. The proof goes like this.

Basis Step: P(0) is true, since we can get 20¢ using 2 dimes. Also P(1) is true, since we can get 25¢ using one quarter.

Inductive Step: Assume P(n) is true for some n ≥ 0. That is we can make an amount of the form (20+5n)¢.

Then we have to show that P(n+1) is true i.e we have to show that an

amount of the form $(20+5(n+1))$¢ = $(20+5n + 5)$¢ can be made using dimes and quarters.
There are 2 cases.
Case 1
Atleast one quarter was used. Replace one quarter with 3 dimes. That is subtract 25¢ and replace it with 30¢. Thus from the initial amount of $(20 + 5n)$¢ we have increased by 5¢ making it $(20+5n+5)$¢ = $(20+5(n+1))$¢. Thus $P(n+1)$ is true in this case.
Case 2
No quarters were used and the amount of $(20 + 5n)$¢ was made entirely using dimes. Since $n \geq 0$, atleast 2 dimes were used. Replace 2 dimes with a quarter to get an amount with value $(20+5n+5)$¢ = $(20+5(n+1))$¢. Thus $P(n+1)$ is true in this case as well.
These 2 cases make up all the possibilities. So $P(n)$ is true $\Rightarrow P(n+1)$ is true.
Hence Proved!

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48) Find the flaw with the following "proof" that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

Basis step: $a^0 = 1$ is true by the definition of a^0.
Inductive step: Assume $a^k = 1$ for all nonnegative integers k with $k \leq n$.
Then note that
$a^{n+1} = a^n . a^n / a^{n-1} = 1.1 / 1 = 1.$

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4 points

The flaw lies in the induction step. This proof stated uses the strong induction hypothesis. The proof that $P(n+1)$ is true should not depend on the value of n i.e the **proof should hold whatever n we choose in the statement** *"Assume $a^k = 1$ for all* ***nonnegative*** *integers k with* $\boldsymbol{k \leq n}$*."*
Look at the case when $n = 0$. The proof involves $a^{n-1} = a^{-1}$ ($n=0$) in this case. Clearly the induction hypothesis is valid only for the non-negative powers of a. Therefore $P(0)$ definitely does not imply $P(1)$ and the proof breaks down here.

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12) Show that $f_{n+1} f_{n-1} – f_n^2 = (-1)^n$ whenever n is a positive integer.
(where f_n is the n^{th} Fibonacci number)

6 points

By definition $f_n = f_{n-1} + f_{n-2}$ (1)
Let $P(n) = f_{n+1} f_{n-1} – f_n^2 = (-1)^n$
Basis Step : P(1) is true since $f_2 . f_0 – (f_1)^2 = -1 = (-1)^1$
$= -1.$
Inductive Step: Assume P(n) is true for some n. i.e
$f_{n+1} f_{n-1} – f_n^2 = (-1)^n$
Then we have to show that P(n+1) is true

L.H.S $= f_{n+2} f_n – f_{n+1}^2$ Now, $f_{n+2} = f_{n+1} + f_n$ from (1)
$= (f_{n+1} + f_n) f_n – f_{n+1}^2$
$= f_{n+1} f_n + f_n^2 - f_{n+1}^2$
$= f_{n+1}(f_n - f_{n+1}) + f_n^2$
$= -[f_{n+1}(f_{n+1} - f_n) - f_n^2]$ $f_{n+1} = f_n + f_{n-1} \Rightarrow f_{n+1} - f_n = f_{n-1}$
$= -[f_{n+1} f_{n-1} - f_n^2]$
$= -1(-1)^n$ Since P(n) is true
$= (-1)^{n+1}$.

Hence proved!

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22) Give a recursive definition of

6 points

a) The set of odd positive integers
1€ S
if x € S , then (x+2) € S

b) The set of positive integer powers of 3
3^1 € S
$3^{(x+y)}$€S if 3^x € S and 3^y € S.

c) The set of polynomials with positive integer coefficients
(Here we call any element belonging to S as p(*x*))
1 € S
x € S
If c € ⊆, p(*x*) € S, then c.p(*x*) € S (where we call any element belonging to S as p(*x*))
If p(*x*) € S, q(*x*) € S, then p(*x*) + q(*x*) € S
If p(*x*) € S, q(*x*) € S, then p(*x*).q(*x*) € S