EECS 203-1 Homework 9 Solutions

Total Points: 50
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10) Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

4 points

First we need to show that R is reflexive
i.e.: to show that $((a, b), (a, b)) \in R$,
since $ab = ab$, $((a, b), (a, b)) \in R$.
Hence it is reflexive.
To show that R is symmetric.
i.e.: to show that if $((a, b), (c, d)) \in R$, then $((c, d), (a, b)) \in R$
we know that $ad = bc$, and $cb = da$ are the same relation.
Hence it is symmetric.
To show that R is transitive.
i.e.: to show that if $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$,
then $((a, b), (e, f)) \in R$
we know that $ad = bc$, and $cf = de$, multiplying these two equations we get $adcf = bcde \Rightarrow af = be \Rightarrow ((a, b), (e, f)) \in R$
Hence it is transitive.
Thus R is an equivalence relation.

14) Determine whether the relations represented by the following zero-one matrices are equivalence relations.

4 points

a)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The given matrix is reflexive, but it is not symmetric. Hence it **does not represent an equivalence relation.**

c)

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The given matrix is an **equivalence relation**, since it is reflexive(all diagonal elements are 1's), it is symmetric as well as transitive.

16) What are the equivalence classes of the equivalence relations in Exercise 1?
(All the relations are on the set {0, 1, 2, 3})

4 points

a) {(0,0), (1,1), (2,2), (3,3)}
[0] ={0}, [1] = {1}, [2] = {2}, [3] = {3}

d) {(0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)}
The given relation is not an equivalence relation.

22) What is the congruence class $[4]_m$ when m is

2 points

b) 3?
$[4]_3$ = {….-2, 1, 4, 7, 10 ….}

d) 8?
$[4]_8$ = {….-12, -4, 4, 12, 20 ….}

28) A partition P_1 is called a **refinement** of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2.

Suppose that R_1 and R_2 are equivalence relations on a set A. Let P_1 and P_2 be the partitions that correspond to R_1 and R_2, respectively. Show that $R_1 \subseteq R_2$ if and only if P_1 is a refinement of P_2.

4 points

Case 1 ($\Rightarrow$) $R_1 \subseteq R_2$.
This means $(x\ R_1\ y) \rightarrow (x\ R_2\ y)$. (1)
By Theorem proved in class *(An equivalence relation creates a partition)*, an equivalence relation on a set S, creates a partition consisting of the distinct equivalence classes of the elements in S. Take any element $x \in S$ and its equivalence class under R_1 namely $[x]_{R1}$. By definition
$[x]_{R1} = \{y \in S: x\ R_1\ y\} \subseteq \{y \in S: x\ R_2\ y\} = [x]_{R2}$.
Since x was arbitrarily chosen, this holds for every equivalence class under relation R_1.
This means that every block in P_1 is a subset of some block in P_2. That is P_1 is a refinement of P_2.

Case 2 ($\Leftarrow$) P_1 is a refinement of P_2. This means that for any $x \in S$,
$[x]_{R1} \subseteq [x]_{R2}$.
i.e $\{y \in S: x\ R_1\ y\} \subseteq \{y \in S: x\ R_2\ y\}$.
This means that for any $x \in S$, $(x\ R_1\ y) \rightarrow (x\ R_2\ y)$ which is the definition of $R_1 \subseteq R_2$.

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2) Determine whether the relations represented by the following zero-one matrices are partial orders.

2 points

a)
$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The relation represented by this matrix is not transitive; hence it is not a partial order.

c)
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

The relation represented by this matrix is not transitive; hence it is not a partial order.

8) Which of the following pairs of elements are comparable in the poset ($\mathbf{Z}^+$, |)?

4 points

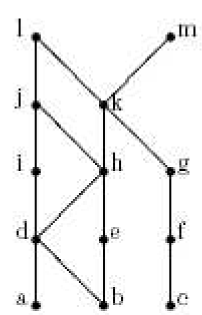
a) 5, 15
Comparable

b) 6, 9
Incomparable

c) 8,16
Comparable

d) 7,7
Comparable

24) Answer the following questions for the partial order represented by the following Hasse Diagram.



8 points

a) Find the maximal elements.
l, m
b) Find the minimal elements
a, b, c
c) Is there a greatest element?
No.
d) Is there a least element?
No.
e) Find all upper bounds of {a, b, c } .
l, k, m
f) Find the least upper bound of { a, b, c } , if it exists.
K
g) Find all lower bounds of {f, g, h}.
None exist
h) Find the greatest lower bound of {f, g, h}, if it exists.
None exists

26) Answer the following questions concerning the poset ({2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72}, |).

8 points

a) Find the maximal elements.
27, 48, 60, 72.
b) Find the minimal elements
9,2
c) Is there a greatest element?
No there is no greatest element.
d) Is there a least element?
No.
e) Find all upper bounds of {2, 9} .
18, 36, 72.
f) Find the least upper bound of {2, 9}, if it exists.
18
g) Find all lower bounds of {60, 72}.
2, 4, 6, 12.
h) Find the greatest lower bound of {60, 72}, if it exists.
12

28) Give a poset that has

3 points

a) a minimal element but no maximal element.
($\mathbf{Z}^+$, ≤)
b) no minimal element but a maximal element.

(Z^-,≤)
c) neither minimal nor maximal element.
(Z , ≤)

32a) Show that there is exactly one greatest element of a poset, if such an element exists

2 points

Suppose that there are two different elements x and y that are greatest.
So $\forall a \in S \quad a \leq x$
And $\forall a \in S \quad a \leq y$
Since $x \in S$ and $y \in S$
We have $x \leq y$ and also $y \leq x$
So $x = y$ because relation ≤ is antisymmetric.
Which is a contradiction because we started out with x and y being different elements. So there can’t be two different elements that are greatest. So if exists then there is only a unique element that is greatest.

34a) Show that the least upper bound of a set in a poset is unique if it exists

2 points

Consider a Poset (P, β)
Assume there are two LUBs u_1, u_2
Since u_1 is a UB then by definition, $u_2 \beta u_1$
Since u_2 is a UB then by definition, $u_1 \beta u_2$
By Antisymmetry $u_1 = u_2$.
Hence the least upper bound of a set in a poset is unique if it exists

Problem 37 page 429 is false! He didn' t say “finite lattice." Give an Infinite lattice, which is a counterexample.
Problem 37: Show that every nonempty subset of a lattice has a least upper bound and a greatest lower bound.

3 points

Consider the poset (Z,≤). This is a lattice and an infinite one.
Consider $S = Z^+ \subset Z$. S = {1,2,3,….}. This is a non empty subset of Z. But does it have a least upper bound. No!
Also consider $T = Z^- \subset Z$. T = {…,-3,-2,-1}. This is a non-empty subset of Z that is doesn’t have greatest lower bound.
Thus you have a counter example.