Final Exam

EECS 203

Spring 2015

Name (Print): ____

uniqname (Print): ____

Instructions. You have 120 minutes to complete this exam. You may have one page of notes (8.5x11.5 two-sided) but may not use any other sources of information, including electronic devices, textbooks, or notes. Leave at least one seat between yourself and other students. Please write clearly. If we cannot read your writing, it will not be graded.

Honor Code. This course operates under the rules of the College of Engineering Honor Code. Your signature endorses the pledge below. After you finish your exam, please sign on the line below:

I have neither given nor received aid on this examination, nor have I concealed any violations of the Honor Code.

____

Section#	Points
A	/24
B	/5
C	/11
D	/10
E	/16
F	/10
G	/16
H	/8
Total	**/100**

Section A: Multiple choice (24 points)

In this section, each question will have zero or more correct answers. You are to circle each correct answer and leave uncircled each incorrect answer.

[2 points each, -1 per incorrect circle/non-circle, minimum 0 points per problem]

1) Let *p* and *q* be the propositions

    p: You walk to work

    q: You get healthy

    The which, if any, of the following logical expressions is/are equivalent to the statement "Walking to work is sufficient for you to be healthy".

    - $p \rightarrow \neg q$
    - $q \vee \neg p$
    - $p \rightarrow q$
    - $q \rightarrow p$

2) Which of the following are propositions?

    - This statement is false
    - x>3
    - Tea comes with dinner
    - 4+4=9

3) Let F(x,y) be the statement that "x can fool y".

    Circle each logical expression that is equivalent to the statement that "there is someone that no one can fool."

    - $\exists y \forall x \left(\neg F(x, y)\right)$
    - $\neg \exists y \forall x \left(F(x, y)\right)$
    - $\exists y \forall x \left(F(x, \neg y)\right)$
    - $\exists y \forall x (F(x, y) \rightarrow false)$

4) The statement that $\exists y \forall x\ (x^2 \geq y)$ is true if the domain of both variables is restricted to the set:

- Z
- Z^- (negative integers)
- {2, 3, 4}
- (0,1]

5) Circle zero or more of the following which are *tautologies* where A, B and C are *non-empty* subsets of Z.

- $(A \cap B \cap C) \subset (A \cup B)$
- $(A - B) \cap (B - A) = \phi$
- $(B \neq \phi) \rightarrow (|A - B| < |A|)$
- $(A \cap B \neq \phi) \rightarrow ((A - B) \subset A)$

6) Circle each of the following which are a (multiplicative) inverse of 5 modulo 7.

- 3
- 1
- -4
- 10

7) Circle all of the following that are TRUE.

- In RSA, the public key, "n" is generally the product of two very large primes.
- The RSA scheme covered in class makes it difficult to convert between the public key and the private key because to do so requires finding the prime factorization of the encryption exponent "e".
- Perhaps the largest difference between a public and private key system is that a private key system relies upon having shared a "key" securely ahead of time, while a public key scheme does not.
- It would be fairly easy to find a private key in RSA if "p" and "q" are found. But we can share "pq" (p times q) safely because finding p and q from the product of p and q is computationally difficult if p and q are large enough.

8) Which of the following are independent of each other (circle zero or more)

- Flip a coin three times
  A: The first coin is a tail
  B: Two of the the coins are tails and one is heads

- Flip a coin three times.
  A: The first coin is a head
  B: All three coin flips are the same (all heads or all tails)

- Draw two cards from a deck of 52 cards
  A: The first card is a heart
  B: The second card is an ace

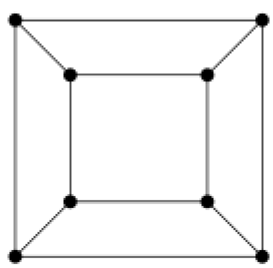
- Roll two dice
  A: The first die is a "6"
  B: The second die is a "6"

9) Consider an application whose runtime can be expressed as $f(n)=2f(n/3)+4n$. Which of the following (if any) would be true of the runtime?

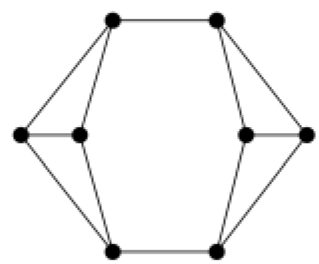
- It is O(n)
- It is $O(n^2)$
- It is O(n log n)
- It is $O(n^{\log(n)})$
- It is O(n!)

10) Consider the graphs W and X. Circle each of the following (if any) which are true

- W and X are isomorphic
- X is bipartite
- W can be colored with 3 colors
- X has an Euler circuit.



W



X

11) The equation $3N^2*\log(N)+4N*\log(N)$ is

- $\Theta(N^2\log(N))$
- $O(N^2)$
- $\Omega(N^2\log(N))$
- $\Theta(N^3)$
- $O(N^3)$

12) The function on the right has a runtime that can be characterized as having a runtime of

```
k:=1
while(k<N)
  k=k*2
  for (j:= 1 to N)
    if(A[i]>A[j])
       A[i]=A[i]+1
```

- $\Theta(N^2\log(N))$
- $O(N^2)$
- $\Omega(N \log(N))$
- $\Theta(N^2)$
- $O(N)$

Section B: Functions (5 points)

Each part has one correct answer. ***(-2 for each wrong or blank answer, minimum 0)***

For each of the following mappings indicate what type of function they are (if any). Use the following key:

i. Not a function
ii. A function which is neither onto nor one-to-one
iii. A function which is onto but not one-to-one
iv. A function which is one-to-one but not onto
v. A function which is both onto and one-to-one

a) The mapping f from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(n) = -2n$.

i ii iii iv v

b) The mapping f from $\{1, 2, 3\}$ to $\{1, 3\}$ defined by $f(n) = n$.

i ii iii iv v

c) The mapping f from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(n) = 1 - n$.

i ii iii iv v

d) The mapping f from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = \frac{1}{x^3}$.

i ii iii iv v

e) The mapping f from $\{1, 3\}$ to $\{1, 2, 3\}$ defined by $f(n) = n$.

i ii iii iv v

Section C: Numeric answers (11 points)

Answer the following questions, clearly showing your work. Partial credit for incorrect answers will be rare though clearly showing your work will certainly help. You may leave your answer in a form that could be entered into a calculator (so 2^4 rather than 16 or even something like C(5,2)-C(3,1) is fine).

1) How many binary strings of length 6 start with 00 and end with a 1? **[3]**

   All strings are of form 00XXX1, where each X can be chosen to be either 0 or 1. So there are 8 different strings.

2) Say you have 3 standard (six-sided) dice. What is the probability of getting a total of 4 or lower? **[3]**

   4/(6^3) = 1/54

3) Say you are getting 10 ice cream sandwiches for 10 students. Say that there are 4 types: Mint, Chocolate, Resse's and Plain. If there are only 2 Mint ice cream sandwiches and only 3 Plain (and plenty of the other two), how many different ways could you select the ice cream sandwiches? **[5]**

   There are (10+4-1 choose 4-1) = (13 choose 3) total ways to get 10 ice cream sandwiches with no constraints. There are (13-3 choose 3) = (10 choose 3) ways to choose 10 ice cream sandwiches that have at least 3 mint (so we must exclude them) and (13-4 choose 3) = (9 choose 3) ways to choose 10 ice cream sandwiches that have at least 4 Plain (so we must exclude them). There are (13-3-4 choose 3) = (6 choose 3) ways to do this with both 4 plain and 3 mint.

   Using inclusion-exclusion we obtain (13 choose 3)–(10 choose 3)–(9 choose 3)+(6 choose 3)

Section D: Probability (10 points)

A given day can be forecast to be rainy, cloudy or clear.

- In Ann Arbor 40% of all days are forecast to be clear, 40% are forecast to be cloudy and 20% are forecast to be rainy.
- On days forecast to be clear, it rains 5% of the time.
- On days forecast to be cloudy it rains 50% of the time.
- On days forecast to be rainy, it rains 80% of the time.

Answer the following questions. *You may leave your answer in a form that could be entered into a calculator (so 2^4 rather than 16 or even something like (0.2*3+0.3)/2 is fine).* <u>Show your work.</u>

1) What is the probability that it rains on any given day? **[3]**

   0.40*0.05 + 0.40*0.50 + 0.20*0.80 = 0.02 + 0.20 + 0.16 = 0.38

2) What is the probability that if it does rain, the forecast was for a clear day? **[7]**

   F: forecast for clear day

   R: it rained

   P[F | R] = P[F]P[R|F]/P(R) = 0.40*0.05/0.38 = 0.02/0.38 = 1/19

Section E: Logic proofs (16 points)

1) Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences. **[9]**

$\neg(p \vee (\neg p \wedge q)) == \neg p \wedge \neg(\neg p \wedge q) == \neg p \wedge (p \vee \neg q) == (\neg p \wedge p) \vee (\neg p \wedge \neg q) ==$

DeMorgans **DeMorgans** **Distributivity** **Negation rule**

$F \vee (\neg p \wedge \neg q) == (\neg p \wedge \neg q)$

Idempotency

2) Prove the following using a truth table: $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$. Be sure to explain how your truth table proves the result. **[7]**

p	q	r	$p \vee q$	$\neg p \vee r$	$(q \vee r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	F	T	T
F	F	F	F	T	F

The highlighted rows are where the left of the implication holds. In all of them, the conclusion is true.

Section F: Proof by induction (10 points)

Use induction to prove the following (you must use induction, any other proof technique will get zero points).

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

Let P(n) be the statement above.

P(1) is true; (1/((1)(1+1)) = 1/2 = 1/(1+1).

Suppose that P(n) is true. We show that P(n+1) is true. Add (1/((n+1)(n+2)) to both sides; the LHS becomes the same with n+1 substituted for n, and the right hand side becomes (n/(n+1)) + (1/((n+1)(n+2)) which can be rewritten as (n^2 + 2n)/(n+1)(n+2) + 1/(n+1)(n+2) = (n^2 + 2n + 1)/(n+1)(n+2) = $(n+1)^2$/(n+1)(n+2) = (n+1)/(n+2), which is the RHS with n+1 replaced in for n. Thus we have proved that P(n+1) is true, completing the inductive step.

Section G: Misc. Problems (16 points)

1) Prove that there exist two powers of 2 that differ by a multiple of 222. That is, $\exists x\ \exists y\ (222|(2^y - 2^x))$ where x and y are positive integers. **[8]**

We prove this by the pigeonhole principle. Let $a_n = 2^n$ mod 222. By definition, $0 <= a_n <= 221$. Since a_n is an infinite sequence but can only take finitely many values, there must be m and n such that $a_n = a_m$ and therefore $2^n == 2^m$ mod 222, and therefore $2^n - 2^m == 0$ mod 222.

2) Let $S_n = \{1, 2, \ldots n\}$ and let a_n denote the number of ***non-empty*** subsets of S_n that contain no two consecutive integers. Find a recurrence relation for a_n. Note that $a_0 = 0$ and $a_1 = 1$. **[8]**

We split S_n into 3 cases:

Case (1): item 1 is not in the subset. We must now choose a non-empty subset of {2..n}. There are a_{n-1} ways to do this.

Case (2): item 1 is in the subset, and there are more elements. We must now choose a non-empty subset of {3..n}. There are a_{n-2} ways to do this.

Case (3): item 1 is in the subset, and no other elements are. There is 1 way to do this.

Total: $a_n = a_{n-1} + a_{n-2} + 1$

Section H: Solving recurrence relations (8 points)

Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

Characteristic equation: $r^2 - 5r + 6$, factored as (r-3)(r-2). So closed-form is $x(2^n) + y(3^n)$. In particular $1 = x(2^0) + y(3^0) = x+y$ and $0 = 2x + 3y$; solving for 2 unknowns with 2 equations is x=3 and y = -2, so the final form is $3(2^n) -2(3^n)$.