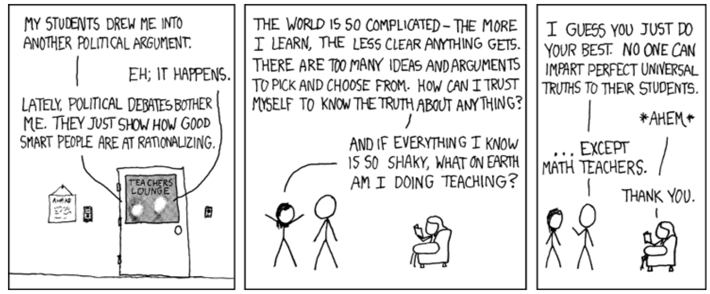
Proving logical equivalencies (1.3)

One thing we'd like to do is prove that two logical statements are the same, or prove that they aren't.



Vocabulary time

In order to discuss the idea of logical equivalencies, it is helpful to define a number of terms.

- A ***tautology*** is a proposition that is always true (e.g. $\neg p \vee p$)
- A ***contradiction*** is a proposition that is always false (e.g. $\neg p \wedge p$)
- A ***contingent*** proposition is one that is neither a tautology nor a contradiction.

Indicate if each of the following propositions is a tautology, a contradiction, or a contingent proposition.

Proposition	**Tautology, Contradiction or Contingent?**
$p \rightarrow p$	
$p \rightarrow \neg p$	
$p \leftrightarrow \neg p$	
$(p \vee q) \vee \neg p$	

More vocabulary:

- A proposition is ***satisfiable*** if it is not a contradiction.
- Two compound propositions p and q are ***logically equivalent*** if they have the same truth table, i.e., if $p \leftrightarrow q$ is a tautology. We write $p \equiv q$ if p and q are logically equivalent.

Example

Consider the following statements:

1. If my nose is cold then I am unhappy.
2. If I'm happy then my nose isn't cold.
3. My nose is not cold or I'm unhappy.

Are (1), (2), and (3) equivalent statements? Let's try to write each of these as a compound proposition using **p = "my nose is cold"** and **q = "I am unhappy"**. Then fill in the truth table for each one.

p	q			
T	T			
T	F			
F	T			
F	F			

"Proof by truth table[1]" is a quite reasonable way of proving things, but it doesn't work well for a large proposition. As noted last time, truth tables grow quickly (if you have n variables, you need 2^n rows!).

A better way: use an "algebra" for logical propositions, analogous to algebra for arithmetic:

- Why is x*1 = x? — 1 is the identity for multiplication
- Why is x*(y+z) = xy + xz? — multiplication distributes over addition
- Why *isn't* x+y*z = (x+y)*(x+z)? — addition doesn't distribute over multiplication

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Taken from page 27 of Rosen. *You need to memorize, by name: Commutative, Associative, Distributive, and De Morgan's.*

Let's prove one of De Morgan's Laws by perfect induction.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T			
T	F			
F	T			
F	F			

There are also a number of useful logical equivalencies found on page 28. I've listed three of them here:

- $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
- $p \rightarrow q \equiv \neg p \vee q$ *(We'll call this one the "definition of implication")*

[1] More formally called "perfect induction" among other things.

Example: using the rules of logic, prove $\neg(p \rightarrow q) \equiv p \wedge \neg q$ (This is example 6 in section 1.3)

$\neg(p \rightarrow q)$	$\equiv \neg(\neg p \vee q)$	*Definition of implication*
	$\equiv \neg(\neg p) \wedge \neg q$	*De Morgan's Law (2nd one…)*
	$\equiv p \wedge \neg q$	*Double negation*

Let's use the rules of logic to prove: $(p \wedge q) \rightarrow (p \vee q)$

Satisfiability (1.3)

One thing that is surprisingly important, is the notion of satisfiability. That is, figuring out if the truth table for a given logical equation is ever true. It seems like this should be easy, but it turns out this is the "gold standard" for hard problems[2]. In fact, if you can efficiently solve "SAT" as it is known, there are a huge number of problems that can be efficiently solved. The text uses Sudoku puzzles as an application of satisfiability. I've posted some slides that walk you through an example.

And back to logical puzzles

We jumped into logical puzzles without any deep justification. Now let's consider the following problem:

- You meet Alice and Bob on the island.
  - Alice says "Bob is a truthteller"
  - Bob says "Alice and I are different types"

Let p: "Alice=truthteller" q: "Bob=truthteller"

	Alice said	Bob said	Island rule
p	q	$p \oplus q$	$(p \leftrightarrow q) \wedge (q \leftrightarrow (p \oplus q))$
T	T	F	F
T	F	T	F
F	T	T	F
F	F	F	T

OK, before we move on, let's do a quick "quiz"

1. What is a "proposition"?
   a. A declarative statement that is true.
   b. A declarative statement.
   c. A declarative statement that follows from axioms by a chain of logical deductions.
   d. A declarative statement that is either true or false.

2. To the right is a truth table for proposition X. What is X?
   a. $p \rightarrow (q \vee r)$
   b. $r \rightarrow (q \wedge p)$
   c. $q \rightarrow (\neg p \wedge r)$
   d. none of the above

p	q	r	X
F	F	F	T
F	F	T	F
F	T	F	T
F	T	T	F
T	F	F	T
T	F	T	F
T	T	F	T
T	T	T	T

3. Translate "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years of age." into a logical expression. Assign letters to the component propositions and then write the logical expression.

And one from last Spring's quiz #1

Problem III. **(6, -2 per incorrect circle/non-circle, min 0 points)** Zero or more of the following compound propositions are satisfiable, but which?

(A) $p \wedge (p \rightarrow \neg q) \wedge q$

(B) $(p \rightarrow q) \wedge (q \rightarrow \neg p)$

(C) $p \oplus q$

(D) $(\neg q \rightarrow p) \leftrightarrow (q \wedge (p \leftrightarrow q))$

First-order logic (1.4)

Propositional logic is a great way to understand *valid logical inferences* but it's not very *expressive*. There are simple statements we can't make.

- **"*x* is a prime number"** not T or F until we know what *x* is!
- **"everyone I know with a bicycle has no car"** We could consider this a single T/F proposition but we lose the *structure* of the statement. How do we say "everyone"?
- **"all my ancestors had brown eyes"** Says something about parents, parents of parents, parents of parents of parents, … *indefinitely*.

We need objects, predicates, and quantifiers!

Propositions and Predicates

Proposition:

- A declarative statement that is either true or false.
- E.g. "A nickel is worth 5 cents."
- "Water freezes at 0 degrees Celsius at sea level."

Predicate:

- A declarative statement with *some terms unspecified.*
- It *becomes* a proposition when terms are specified.
- These terms refer to *objects*.

Let's look at an example: "*I'm taller than both my parents, but they're the same height*."

- Let T(x,y) be "x is taller than y"
- $T(I,mom) \wedge T(I,dad) \wedge \neg T(mom,dad) \wedge \neg T(dad,mom)$

The logic explicitly describes **objects**: I, mom, dad and **relations** among objects: T(x,y).

- Once the variables in a relation are bound to objects, the relation *becomes* a proposition: T(I,dad) is a proposition with a truth value.

You try it.

- Let the predicate H(x) be "x is happy".
    - How would you write "*If you're happy, I'm happy*"?

- Let R(x) be "rain is forecast for x". Let B(x,y) be "x will buy a y"
    - How would you write: "*If rain is forecast for Wednesday I'll buy an umbrella, otherwise I'll buy a latte"*

Quantification

How would we write "Everyone will buy an umbrella"?

- You could go with: B(Aaron,umbrella) $\wedge$ B(Abby,umbrella) $\wedge$ B(Abner,umbrella) $\wedge$ B(Abu,umbrella), etc.

But that doesn't seem reasonable. Heck, there may be an infinite number of things to list. And even if it is finite, we might not be sure who is on the list. So, as good mathematicians, we just create some notation. And just to make it confusing (well really so we don't alias with other symbols) we'll use some weird symbols to do this.

Universal quantifier: ∀ ("for all" "every" "any")

- $\forall x$ B(x,umbrella) "everyone will buy an umbrella"
- $\forall x$ P(x) is sort of like $P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots$

Existential quantifier: ∃ ("there exists" "at least one" "some")

- $\exists x$ B(x,umbrella) "someone will buy an umbrella"
- $\exists x$ P(x) is sort of like $P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots$

Domain of Discourse

So what does "all" mean? Strictly speaking, x can be **anything**: The number 7, the color blue, your left sock, etc.

But if we explicitly specifying the domain, we can focus on something more specific. For example: people, numbers, students in this class, integers, etc.

Keep the domain as general (and clear) as possible.

- $\forall x \;\; x^2 \geq 0$ [implicit domain: **R** = real numbers]
- $\forall x \in \mathbf{R} \;\; x^2 \geq 0$ [explicit domain: **R**]

Are the following statements true?

- $\exists x \in \mathbf{R} \;\; x^2 < 0$ $\exists x \in \mathbf{R} \;\; x^2 < 1$
- $\forall x$ where x is a Pokémon, x cannot speak.

	$\forall x$ P(x)	$\exists x$ P(x)
True when:	P(x) true for every x in the domain of discourse	P(x) true for at least one x in the domain of discourse
False when:	P(x) false for at least one x in the domain of discourse	P(x) false for every x in the domain of discourse