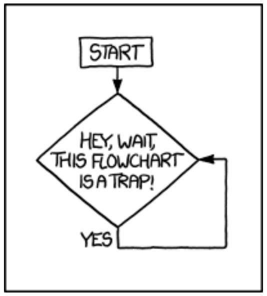
Predicate Logic & Quantification



EECS 203: Discrete Mathematics
Lecture 3 Spring 2016
(Sections 1.4 and start on 1.5)

Things you should do…

- Homework 1 due today at 3pm
  - Via gradescope. Directions posted on the website.
- Group homework 1 posted, due Tuesday.
  - Groups of 1-3. We suggest 3.
  - In LaTeX

Warmup Question

- "Neither the fox nor the lynx can catch the hare if the hare is alert and quick."
  - F: the fox can catch the hare
  - L: the lynx can catch the hare
  - A: the hare is alert
  - Q: the hare is quick

- (A) $\neg(F \vee L) \rightarrow (A \wedge Q)$
- (B) $(A \wedge Q) \rightarrow \neg F \wedge \neg L$
- (C) $\neg F \wedge \neg L \wedge A \wedge Q$
- (D) $(\neg A \vee \neg Q) \rightarrow (F \vee L)$

Warmup Question

- The expression $(p \rightarrow q) \rightarrow (\neg q \rightarrow p)$ can *only* be satisfied by the truth assignment

a. p= T, q = F
b. p= F, q = T
c. This is not satisfiable
d. None of the above

Relational (First-Order) Logic

- In ***propositional logic***,
  - All we have are propositions and connectives, making compound propositions.
  - We learn about deductions and proofs based on the structure of the propositions.
- In ***first-order logic***,
  - We will add objects, properties, and relations.
  - We will be able to make statements about what is true for some, all, or no objects.
- And that comes now.

Propositions & Predicates

- Proposition:
  - A declarative statement that is either true or false.
  - E.g. "A nickel is worth 5 cents."
  - "Water freezes at 0 degrees Celsius at sea level."
- Predicate:
  - A declarative statement with *some terms unspecified.*
  - It *becomes* a proposition when terms are specified.
  - These terms refer to *objects*.

A "truth table" for quantifiers

	$\forall x\, P(x)$	$\exists x\, P(x)$
True when:	P(x) true for every x in the domain of discourse	P(x) true for at least one x in the domain of discourse
False when:	P(x) false for at least one x in the domain of discourse	P(x) false for every x in the domain of discourse

Examples: English → Quantifications

"Everyone will buy an umbrella or a raincoat"

$\forall x\, (B(x,\text{umbrella}) \vee B(x,\text{raincoat}))$

"Everyone will buy an umbrella or everyone will buy a raincoat"

"No one will buy both a raincoat and umbrella"

Examples: English → Quantifications

"Everyone will buy an umbrella or a raincoat"

$\forall x\ (B(x,umbrella) \vee B(x,raincoat))$

"Everyone will buy an umbrella or everyone will buy a raincoat"

$\forall x\ B(x,umbrella) \vee \forall x\ B(x,raincoat)$

"No one will buy both a raincoat and umbrella"

$\neg\exists x(B(x,umbrella) \wedge B(x,raincoat))$

quantified variable

the scope of the variable

Examples: English → Quantifications

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variable | scope | variable | scope

This has the potential to cause confusion so we'll try to avoid it!

Examples: English → Quantifications

- "*Everyone has a car or knows someone with a car.*"
  - Let C(x) be "x has a car"
  - Let K(x,y) be "x knows y"

(A) $\exists x\exists y\ [C(x) \vee (K(x,y) \wedge C(y))]$

(B) $\exists y\forall x\ [C(x) \vee (K(x,y) \wedge C(y))]$

(C) $\forall x\exists y\ [C(x) \vee (K(x,y) \wedge C(y))]$

(D) $\forall x\forall y\ [C(x) \vee (K(x,y) \wedge C(y))]$

Nested Quantifiers

P(x,y) : "person x loves person y"

$\forall \mathbf{x}\exists y\ P(x,y)$ means:

"For *every* x (in the domain) there is *at least one* y (in the domain), *that can depend on x and may be equal to x*, such that P(x,y) is true."

"Everyone loves someone (e.g. his/her mother)"

$\exists y\forall \mathbf{x}\ P(x,y)$ means:

"There is *at least one* y such that for *every* x (including the case y=x), P(x,y) is true."

"There's one guy/gal that everyone loves (e.g. Santa)"

Defining Limits

- In calculus, the limit $\lim_{x \to a} f(x) = L$
  - Is defined to mean:

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \; [0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon]$$

  - As close as you want $f(x)$ to be to L $(\forall \varepsilon > 0)$,
  - there is a margin for x around a $(\exists \delta > 0)$,
  - so that for any x within that margin around a,
  - $f(x)$ will be as close as you wanted to L.

- The limit is an essential concept for calculus.

- Two statements involving quantifiers and predicates are ***logically equivalent*** if they have the same truth value, regardless of the domain of discourse or the meaning of the predicates.
  - $\equiv$ denotes logical equivalence.

- Need new equivalences involving quantifiers.

Negating Quantifiers

- $\neg \forall x \, P(x) \equiv \exists x \, \neg P(x)$
  - There is an x for which P(x) is false.
  - If P(x) is true for every x then $\exists x \, \neg P(x)$ is false.

- $\neg \exists x \, P(x) \equiv \forall x \, \neg P(x)$
  - For every x, P(x) is false.
  - If there is an x for which P(x) is true then $\forall x \, \neg P(x)$ is false

- This is really just DeMorgan's Laws, extended.
  - $\neg(p \wedge q) \equiv \neg p \vee \neg q$
  - $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Be Careful with Equivalences

- It's true that:
  - $\forall x \, [P(x) \wedge Q(x)] \equiv [\forall x \, P(x)] \wedge [\forall x \, Q(x)]$
- But it's not true that:
  - $\forall x \, [P(x) \vee Q(x)] \equiv [\forall x \, P(x)] \vee [\forall x \, Q(x)]$
- Why not?

- Likewise, it's true that:
  - $\exists x \, [P(x) \vee Q(x)] \equiv [\exists x \, P(x)] \vee [\exists x \, Q(x)]$
- But it's not true that:
  - $\exists x \, [P(x) \wedge Q(x)] \equiv [\exists x \, P(x)] \wedge [\exists x \, Q(x)]$

Be Careful With Translation to Logic

- "*Every student in this class has studied calculus.*"
  - S(x) means "x is a student in this class".
  - C(x) means "x has studied calculus".
- Is this correct? $\forall x\ [\ S(x) \wedge C(x)\ ]$
  - (A) Yes
  - (B) No
- How about this? $\forall x\ [\ S(x) \rightarrow C(x)\ ]$
  - (A) Yes
  - (B) No

Be Careful With Translation to Logic

- "*Some student in this class is a math genius.*"
  - S(x) means "x is a student in this class".
  - G(x) means "x is a math genius".
- Is this correct? $\exists x\ [\ S(x) \rightarrow G(x)\ ]$
  - (A) Yes
  - (B) No
- How about this? $\exists x\ [\ S(x) \wedge G(x)\ ]$
  - (A) Yes
  - (B) No

Hard Problem

- Prove: $\forall x\, P(x) \vee \forall x\, Q(x) \equiv \forall x \forall y\, [P(x) \vee Q(y)]$
  - We can rename a bound variable: $\forall x\, Q(x) \equiv \forall y\, Q(y)$

– Method: to prove $A \equiv B$
  - We might prove $A \rightarrow B$ and $B \rightarrow A$.
    - But that will turn out to be too hard.
  - Instead we will prove $A \rightarrow B$ and $\neg A \rightarrow \neg B$.
    - That will do the trick just as well.

Prove the $A \rightarrow B$ Direction

- Assume that $\forall x\, P(x) \vee \forall x\, Q(x)$ is true.
  - Consider the case where the disjunct $\forall x\, P(x)$ is true.
    - The other case, $\forall x\, Q(x)$, is the same.
  - Then for any value of y, $\forall x\, (P(x) \vee Q(y))$ is true.
    - by the Identity Law, since P(x) is true.
  - This is the definition of $\forall y\, \forall x\, (P(x) \vee Q(y))$.
    - by definition of the universal quantifier.
  - And this is equivalent to $\forall x\, \forall y\, (P(x) \vee Q(y))$.
    - section 1.5, example 3 (pp.58-59).
  - Thus: $\forall x\, P(x) \vee \forall x\, Q(x) \rightarrow \forall x\, \forall y\, (P(x) \vee Q(y))$

$\forall x\, P(x) \vee \forall x\, Q(x) \equiv \forall x \forall y\, [P(x) \vee Q(y)]$

Prove the $\neg A \rightarrow \neg B$ Direction

- Assume that $\forall x\, P(x) \vee \forall x\, Q(x)$ is false.
  - Then: $\neg[\, \forall x\, P(x) \vee \forall x\, Q(x)\,]$

    $\equiv \quad \neg\forall x\, P(x) \wedge \neg\forall x\, Q(x)$

    $\equiv \quad \exists x\, \neg P(x) \wedge \exists x\, \neg Q(x)$
  - Then let (a,b) be such that $\neg P(a)$ and $\neg Q(b)$.
  - Therefore: $\neg P(a) \wedge \neg Q(b)$

    $\equiv \quad \exists x\, \exists y\, [\, \neg P(x) \wedge \neg Q(y)\,]$

    $\equiv \quad \exists x\, \exists y\, \neg[P(x) \vee Q(y)]$

    $\equiv \quad \neg\forall x\, \forall y\, [P(x) \vee Q(y)]$
  - Which is $\neg B$

$\forall x\, P(x) \vee \forall x\, Q(x) \equiv \forall x \forall y\, [P(x) \vee Q(y)]$

- QED. The whole statement is proved.

Exercises.
Start by defining your predicates!

- Every two people have a friend in common.
  (Life isn't facebook! If A is a friend of B, B is not necessarily a friend of A.)
- All my friends think I'm their friend too.
- There are two people who have the exact same group of friends.
- Everyone has two friends, neither of whom are friends with each other.

Additional Exercises

- $M(x)$: "x is male"
- $F(x)$: "x is female"
- $P(x,y)$: "x is the parent of y"
  - "Everyone has at least one parent"

Additional Exercises

- $M(x)$: "x is male"
- $F(x)$: "x is female"
- $P(x,y)$: "x is the parent of y"
  - "Someone is an only child"

Additional Exercises

- M(x) : "x is male"
- F(x) : "x is female"
- P(x,y) : "x is the parent of y"
  - "Bob has a niece"

Additional Exercises

- M(x) : "x is male"
- F(x) : "x is female"
- P(x,y) : "x is the parent of y"
  - "I do not have any uncles" (rephrased: "any sibling of my parent is female")

Additional Exercises

- M(x) : "x is male"
- F(x) : "x is female"
- P(x,y) : "x is the parent of y"
  - "Bob has a niece"
  - "Not everyone has two parents of opposite sexes"
  - "I have a half-brother" (rephrased: "I and my half-brother share one but not two parents")
  - "I do not have any uncles" (rephrased: "any sibling of my parent is female")
  - "No one's parents are cousins" (this is one is rather long...)

So far…

- You can
  - Express statements as compound propositions
  - Prove that two compound propositions are equivalent
  - Express statements as quantified formulae (with predicates and universal & existential quantifiers)
- Next:
  - Formal proofs, rules of inference
  - Proof methods
  - Strategies for designing proofs

Start on Inference and Proofs

Section 1.5

Definition

- An **argument** for a statement S is a sequence of statements ending with S.
- We call S the **conclusion** and all the other statements the **premises**.
- The argument is **valid** if, whenever all the premises are true, the conclusion is also true.
  - Note: A valid argument with false premises could lead to a false conclusion.
- **Proofs** are **valid arguments** that establish the truth of mathematical statements.

Simple Example

- Premises:
  - "If you're a CS major then you must take EECS 203 before graduating."
  - "You're a CS major."
- Conclusion:
  - (Therefore,) "You must take EECS 203 before graduating."

- This is a valid argument (why?).

Inferences

- Basic building block of logical proofs is an ***inference***
  - Combine two (or one or more) known facts to yield another

$p \rightarrow q$ ← premises
p ← premises
$\therefore q$ ← conclusion

Based on the tautology:
$((p \rightarrow q) \wedge p) \rightarrow q$

$p \vee q$ ← premises
$\neg p \vee r$ ← premises
$\therefore q \vee r$ ← conclusion

Based on the tautology:
$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

Note: (crossed out)
$p \rightarrow q$
q
$\therefore p$

This is **not** a valid inference because
$((p \rightarrow q) \wedge q) \rightarrow p$
is ***not*** a tautology!

The Basic Rules of Inference

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

Based on the tautology: $((p \to q) \land p) \to q$

"modus ponens"
lit.: mode that affirms

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Based on the tautology: $((p \to q) \land \neg q) \to \neg p$

"modus tollens"
lit.: mode that denies

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Based on the tautology: $((p \to q) \land (q \to r)) \to (p \to r)$

"hypothetical syllogism"

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

Based on the tautology: $((p \lor q) \land \neg p) \to q$

"disjunctive syllogism"

The Basic Rules of Inference

$$\begin{array}{l} p \\ \hline \therefore p \lor q \end{array}$$

Based on the tautology: $p \to p \lor q$

"Addition"

$$\begin{array}{l} p \land q \\ \hline \therefore p \end{array}$$

Based on the tautology: $(p \land q) \to p$

"Simplification"

$$\begin{array}{l} p \\ q \\ \hline \therefore p \land q \end{array}$$

Based on the tautology: $((p) \land (q)) \to (p \land q)$

"Conjunction"

$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \hline \therefore q \lor r \end{array}$$

Based on the tautology: $((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$

"Resolution"

- **Modus ponens** $\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$
  - "If you have access to ctools, you can download the homework."
  - "You have access to ctools."
  - (Therefore,) "you can download the homework."
- **Modus tollens**
  - "If you have access to ctools, you can download the homework."
  - "You cannot download the homework."
  - (Therefore,) "you do not have access to ctools."
- **Hypothetical syllogism**
  - "If you are registered for this course, you have access to ctools."
  - "If you have access to ctools, you can download the homework."
  - (Therefore,) "if you are registered for this course, you can download the HW."
- **Resolution**
  - "If it does not rain today, we will have a picnic."
  - "If it does rain today, we will go to the movies."
  - (Therefore,) "today, we will have a picnic or go to the movies."

Common fallacies

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

Not a tautology: $((p \to q) \land q) \to p$

When $p = F, q = T$:
LHS: $(F \to T) \land T \equiv T$
RHS: $p = F$
Together: $T \to F \equiv F$

"Fallacy of affirming the conclusion"

$$\begin{array}{l} p \to q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

Not a tautology: $((p \to q) \land \neg p) \to \neg q$

When $p = F, q = T$:
LHS: $(F \to T) \land T \equiv T$
RHS: $\neg q = F$
Together: $T \to F \equiv F$

"Fallacy of denying the hypothesis"

Showing that an argument is valid

- Is this argument valid? How would we show its validity?

- Premises :
  i. "If Jo has a bacterial infection, she will take antibiotics."
  ii. "Jo gets a stomach ache when and only when she takes antibiotics and doesn't eat yogurt."
  iii. "Jo has a bacterial infection."
  iv. "Jo doesn't eat yogurt."
- Conclusion:
  – "Jo gets a stomach ache."

Step 1: Convert to propositions

- Premises :
  i. "If Jo has a bacterial infection, she will take antibiotics." — i. $B \rightarrow A$
  ii. "Jo gets a stomach ache when and only when she takes antibiotics and doesn't eat yogurt." — ii. $S \leftrightarrow (A \wedge \neg Y)$
  iii. "Jo has a bacterial infection." — iii. B
  iv. "Jo doesn't eat yogurt." — iv. $\neg Y$
- Conclusion:
  – "Jo gets a stomach ache." — S

B: "Jo has a bacterial infection."
A: "Jo takes antibiotics."
S: "Jo gets a stomach ache."
Y: "Jo eats yogurt."

Step 2: Start with premises

i.	$B \rightarrow A$	premise
ii.	$S \leftrightarrow (A \wedge \neg Y)$	premise
iii.	B	premise
iv.	$\neg Y$	premise

B: "Jo has a bacterial infection."
A: "Jo takes antibiotics."
S: "Jo gets a stomach ache."
Y: "Jo eats yogurt."

Step 3: Use inferences to make conclusion

i.	$B \rightarrow A$	premise
ii.	$S \leftrightarrow (A \wedge \neg Y)$	premise
iii.	B	premise
iv.	$\neg Y$	premise
1.	A	modus ponens, i, iii
2.	$(A \wedge \neg Y)$	conjunction, iv, 1
3.	$((A \wedge \neg Y) \rightarrow S) \wedge (S \rightarrow (A \wedge \neg Y))$	definition of $\leftrightarrow$, ii
4.	$(A \wedge \neg Y) \rightarrow S$	simplification, 3
5.	S	modus ponens, 2,4

The desired conclusion!

B: "Jo has a bacterial infection."
A: "Jo takes antibiotics."
S: "Jo gets a stomach ache."
Y: "Jo eats yogurt."