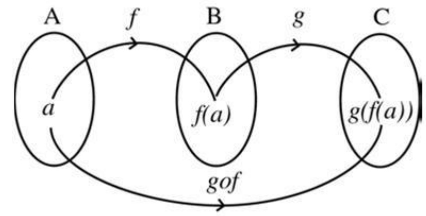
Review from last time



1. What can you say about the sets A and B if we know that
   a. $A \cup B = A$?
   b. $A - B = A$?
   c. $A - B = B - A$?

2. Let f : B→C and g: A→B. (note, this was different/wrong in the notes handed out)
   a. If *f* and *g* are one-to-one, is f ∘ g also one-to-one?
   b. If *f* and *g* are onto, is *f* ∘ *g* also onto?
   c. If *g* and *f* ∘ *g* are onto, is f also onto?

3. Write a formal definition of what it means if function *f* from A→B is onto.

A and B are sets.Define the *symmetric difference*:
$A \oplus B \equiv (A \cup B) - (A \cap B)$

4. Is it true that A ⊕ B ≡ B ⊕ A?

Start on Sequences and Summations (2.4)

A sequence is a function from a subset of the set of integers (usually either the set {0, 1, 2, . . .} or the set {1, 2, 3, . . .}) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

- Consider the sequence where $a_n=2n$. In that case, the sequence, starting with a_1 is 2, 4, 6, 8, etc.
- Consider the sequence where $a_n=\frac{1}{n}$. In that case, what is the sequence?

Recurrence Relations

A recurrence relation expresses the value of an as a function of previous values. For example, the recurrence relation $a_0 = 0, a_1 = 1, a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ defines the *Fibonacci sequence*. That is 0, 1, 1, 2, 3, 5, 8, 13, 21, etc.

The *sequence* is the *solution* to the recurrence relation. Ideally we'd be able to find it in "closed form" or "closed formula" (that is where we could compute a_n directly without finding a_{n-1} first).

- Define the even integers (from lowest to highest) as a recurrence relation.
- Define the factorial function (n!) as a recurrence relation.

Recurrence relations hold a special place in computer science. It is often the case that we develop algorithms and then ask ourselves how efficient the algorithm is. And fairly commonly we find ourselves with algorithms where we split the problem into parts and then re-run the algorithm on the parts (recursive programs do this…). We will in fact spend a fair bit of time near the end of this term learning some general techniques for solving recurrence relations (one algorithm turns out to be nearly the same as solving differential equations).

Summation of a sequence

Sometimes we want to know the sum of a sequence. So the sum of the first N elements ($a_1+a_2+\ldots+a_N$). We generally write that as

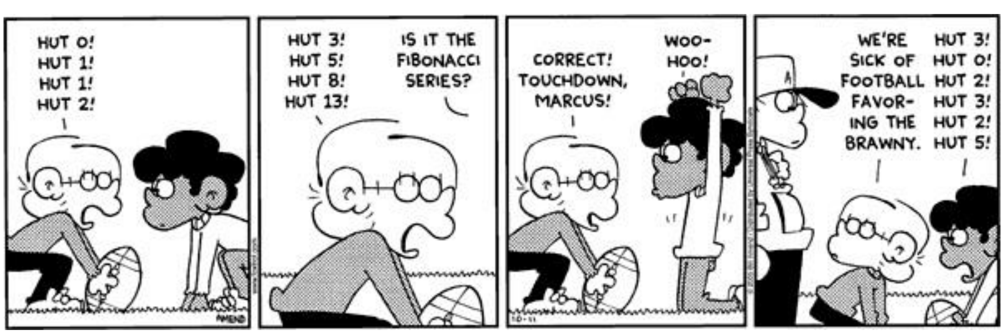
$$\sum_{i=1}^{N} a_i$$

Consider the following summation of the sequence ½, ¼, etc.

$$\sum_{i=1}^{N} \frac{1}{2^i}$$

Can you find a closed-form solution to that? What is the value as N approaches infinity?

Find a closed-form solution for the sum of the sequence 1, 2, 3, etc.



Questions

1. What is the value of $\sum_{i=0}^{5}(-2)^i$?

2. What is the closed form of the sum $\sum_{k=0}^{n} a_{k+1} - a_k$? (This is called a "telescoping series")

3. Find $\displaystyle\sum_{j=0}^{9}(2^{j+1} - 2^j)$

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

An example summation

Sometimes the telescoping sum trick can be useful where it doesn't seem like it would be useful. Let's attack the following summation in two different ways. $\sum_{k=0}^{n} k(k+1)$

Answer one: This is basically $k^2 + k$ summed from 0 to n. By the associative/commutative properties of addition we can write this as $\sum_{k=0}^{n} k(k+1) = \sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} k$. We can start k at 1 rather than 0 (as k and k^2 are both 0 when k is 0) and then just use the formulae above for the sum of k and k^2.

[With a bit of algebra, we can get the sum of k^2 from 1 to n to be $(1/3)n^3 + (1/2)n^2 + (1/6)n$. So what is $\sum_{k=0}^{n} k(k+1)$?]

Answer two: We can try to manipulate this into a telescoping sum. <u>Note</u>: *I'd have never ever come up with this, but it is cool.*

$$\begin{aligned}3k(k+1) &= [3+k-k]k(k+1)\\ &= [(k+2)-(k-1)]k(k+1)\\ &= k(k+1)(k+2)-(k-1)k(k+1)\\ &= a_{k+1}-a_k \qquad \text{when } a_k = (k-1)k(k+1)\end{aligned}$$

And thus we get:

$$\sum_{k=0}^{n} k(k+1) = \frac{1}{3}\sum_{k=0}^{n} 3k(k+1) = \frac{1}{3}\sum_{k=0}^{n}(a_{k+1}-a_k)$$
$$= {}^{1}/_{3}\,(a_{n+1}-a_0)$$
$$= {}^{1}/_{3}\,[n(n+1)(n+2)-(-1)0(1)]$$
$$= n(n+1)(n+2)/3$$

And a very relevant series

Compound interest is important. If you start with a_0 dollars with an interest rate of "r", you get $a_1=(1+r)a_0$.

- What is a_2 in terms of a_1? ________________ in terms of a_0? __________________
- What is a_n in terms of a_0?__________________________

What would it look like if in addition, we saved d dollars a year (so a_0=d and we add d dollars every year)?

Let k=1+r (so if the interest rate is 10%, r=0.1 and k=1.1).

- $a_0 = d$
- $a_1 = ka_0 + d$
- $a_2 = ka_1 + d$

  $\vdots$
- $a_n = ka_{n-1} + d$

- It's easier if you express the recurrence in terms of $k = (1 + r)$ rather than r.
  - $a_0 = d \qquad = d \qquad = d$
  - $a_1 = ka_0 + d \qquad = kd + d \qquad = d(k + 1)$
  - $a_2 = ka_1 + d \qquad = kd(k + 1) + d = d(k^2 + k + 1)$

    $\vdots \qquad \vdots$
  - $a_n = ka_{n-1} + d \qquad = kd(k^{n-1} + \cdots + k + 1) + d$

    $= d(k^n + k^{n-1} + \cdots + k + 1)$
- Each term is the sum of a geometric series, which we can solve in closed form.

$$a_n = d \sum_{i=0}^{n} k^i = d\left(\frac{k^{n+1} - 1}{k - 1}\right)$$

Infinite Sets and Cardinality (2.5)

One interesting thing I've alluded to earlier is that the cardinality of sets isn't always trivial. When dealing with finite sets, it's generally pretty easy to figure out how many elements are in the set. But what about things like **Z**, **N**, **Q**, and **R**? Clearly they are all infinite, but are they the same infinity? Does that question even make sense?

Let's start with some fairly obvious statements.

If $f: A \to B$ *is* 1 to 1 then $|A| \leq |B|$. If $f: A \to B$ is 1 to 1 **and** *onto* then $|A| = |B|$.

Now, let's jump to the definition of a "countable" set. (Recall that a function that is 1 to 1 and onto is set to be a 1 to 1 correspondence or a bijection. Lots of jargon here)

- Countable sets
  - The set S is *countable* if
    - S is finite, OR
    - S is infinite and there is a 1-1 correspondence between S and **N**.
  - If an infinite set S is countable, its cardinality is $|S| = \aleph_0$ = "aleph null".
  - There are *uncountable* infinite sets: $|\boldsymbol{R}| > \aleph_0$.

Let's show that the integers are countable. Can you find a bijection from **Z** to **N**?

Can you find an inverse of your function from N to Z? Do we know one must exist?

Now this should seem a bit strange. Clearly $\boldsymbol{Z} \subset \boldsymbol{N}$, but |Z|=|N|. With finite sets, if $A \subset B$ then cardinality A is always less than B.

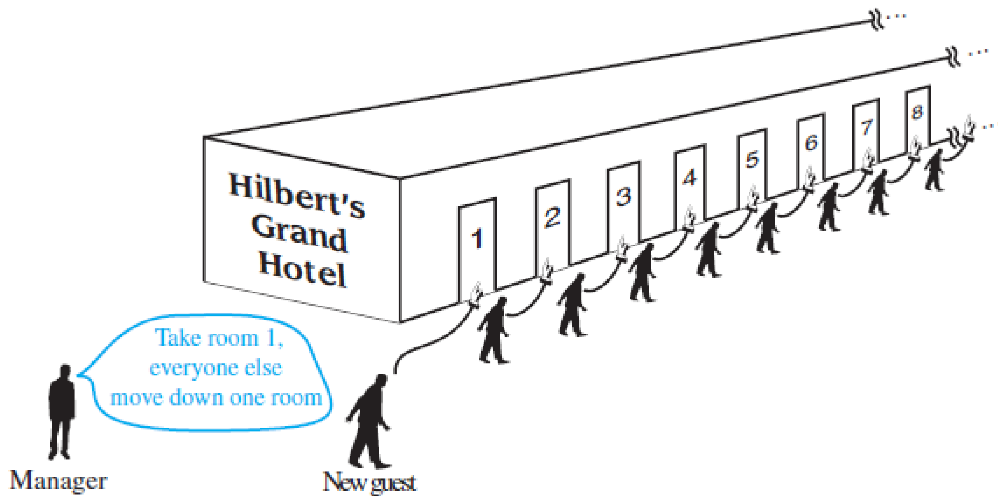
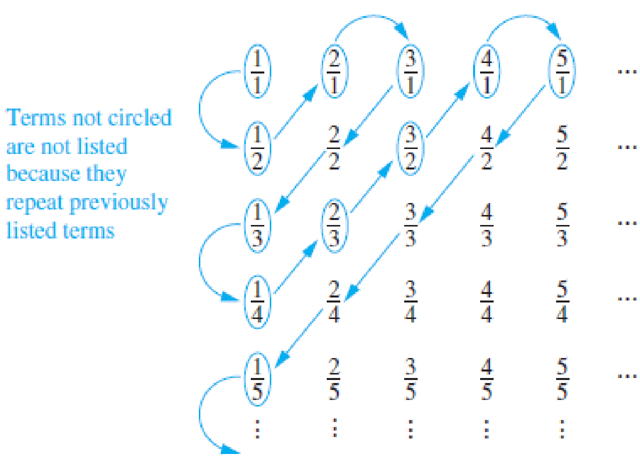


Figure 1:There's always room for one more at Hilbert's Grand Hotel.

How does the above picture relate to our discussion?

Let's consider the positive rational numbers (**Q**⁺). Can you show that *those* are countable? How does the figure below help?



How about **Q**?

OK, so how do we prove something *isn't* countable? I mean it's one thing to show that there *is* a bijection between two sets, but quite another to show that there isn't one.

(Blank space for proof that the reals are not countable)

Some results about cardinality.

If A and B are countable sets, then $A \cup B$ is also countable.

Proof: Suppose that A and B are both countable sets. Without loss of generality, we can assume that A and B are disjoint. (If they are not, we can replace B by $B - A$, because $A \cap (B - A) = \emptyset$ and $A \cup (B - A) = A \cup B$.) Furthermore, without loss of generality, if one of the two sets is countably infinite and other finite, we can assume that B is the one that is finite.

There are three cases to consider: (i) A and B are both finite, (ii) A is infinite and B is finite, and (iii) A and B are both countably infinite.

Case (i): Note that when A and B are finite, $A \cup B$ is also finite, and therefore, countable.

Case (ii): Because A is countably infinite, its elements can be listed in an infinite sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ and because B is finite, its terms can be listed as $b_1, b_2, \ldots, b_m$ for some positive integer m. We can list the elements of $A \cup B$ as $b_1, b_2, \ldots, b_m, a_1, a_2, a_3, \ldots, a_n, \ldots$. This means that $A \cup B$ is countably infinite.

Case (iii): Because both A and B are countably infinite, we can list their elements as $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$, respectively. By alternating terms of these two sequences we can list the elements of $A \cup B$ in the infinite sequence $a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_n, b_n, \ldots$. This means $A \cup B$ must be countably infinite.

We have completed the proof, as we have shown that $A \cup B$ is countable in all three cases. $\triangleleft$

SCHRÖDER-BERNSTEIN THEOREM If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

Now prove that $|(0,1)|=|(0,1]|$.