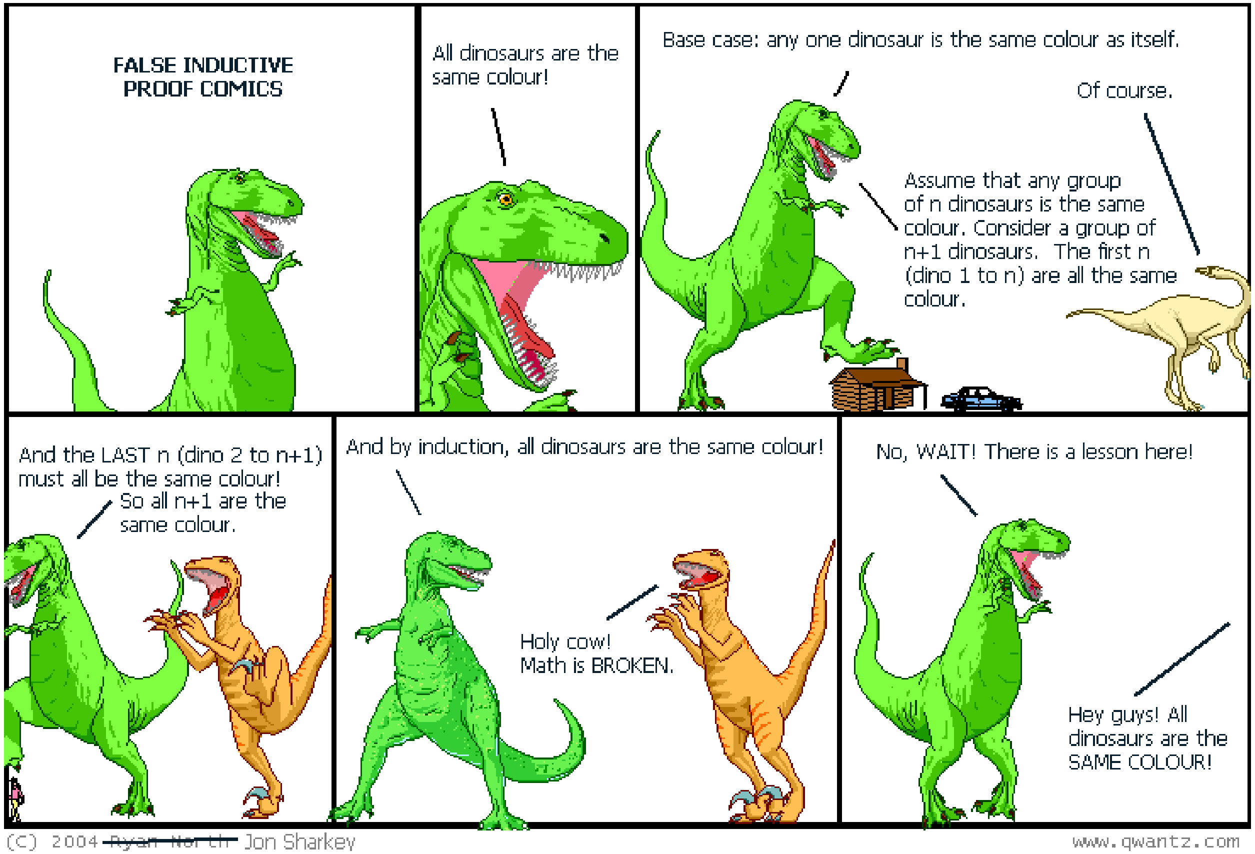
Mathematical Induction



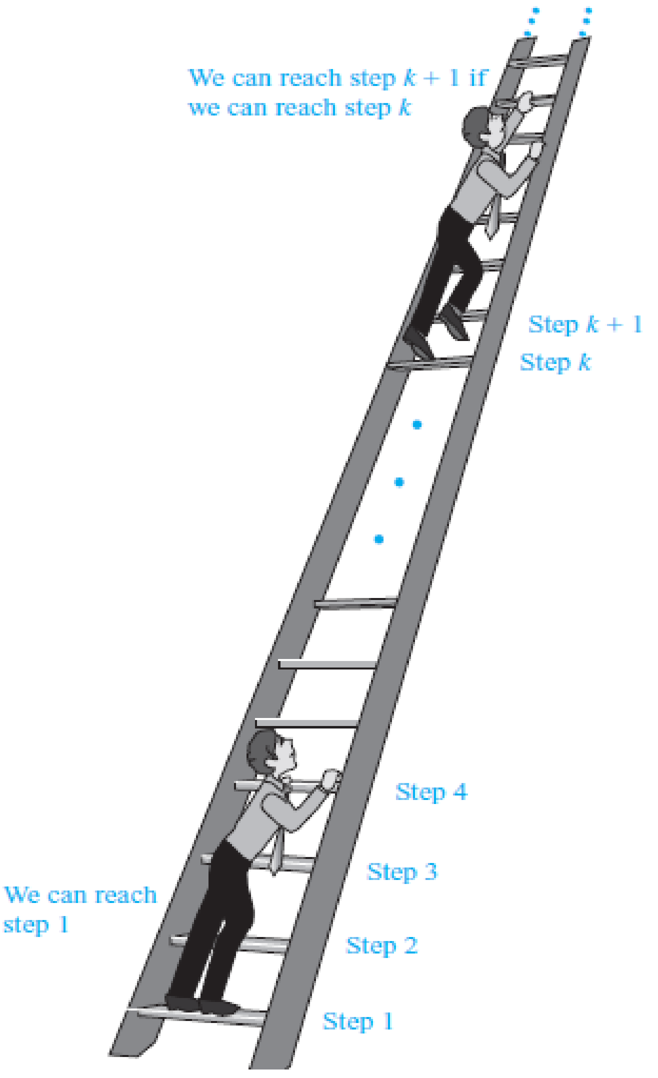
EECS 203: Discrete Mathematics

Lecture 11

Spring 2016

Climbing the Ladder

- We want to show that $\forall n \geq 1\ P(n)$ is true.
  - Think of the positive integers as a ladder.
  - $1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$
- You can reach the *bottom* of the ladder:
  - $P(1)$
- From *each* ladder step, you can reach the *next*.
  - $P(1) \rightarrow P(2),\quad P(2) \rightarrow P(3),\quad \ldots$
  - $\forall k \geq 1\ P(k) \rightarrow P(k+1)$
- Then, by mathematical induction:
  - $\forall n \geq 1\ P(n)$



Mathematical Induction

- How do we prove a universal statement about the positive integers:

$$\forall n \geq 1 \; P(n)$$

- Mathematical induction is an inference rule
  - Base case: P(1)
  - Inductive step: $\forall k \geq 1 \; P(k) \rightarrow P(k+1)$

  - Conclusion: $\forall n \geq 1 \; P(n)$

- The inductive step requires proving an implication.

Simple Math Induction Example

- Prove, for all $n \in \mathbf{N}$,

$$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

- **Proof**: We use induction. Let $P(n)$ be:

$$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

- **Base case**: $P(0)$, because $0 = 0$.

Simple Math Induction Example

- **Inductive step**: Assume that $P(k)$ is true, where k is an arbitrary natural number.

$$0 + 1 + 2 + \cdots + k + (k+1) \quad =$$

Simple Math Induction Example

- **Inductive step**: Assume that *P*(*k*) is true, where *k* is an arbitrary natural number.

$$0 + 1 + 2 + \cdots + k + (k+1) = \frac{k\,(k+1)}{2} + (k+1)$$

$$= \frac{(k+1)\,(k+2)}{2}$$

- The first equality follows from *P*(*k*), and the second by simplification. Thus, *P*(*k*+1) is true.
- By induction, *P*(*n*) is true for all natural numbers *n*. QED
  - *This shows how to write a clear inductive proof.*

Question

- Did you understand the proof by induction?

  - (A) Yes, that's really clever!
  - (B) How can you start with $P(0)$ instead of $P(1)$?
  - (C) When do you use $P(k)$ and when $P(n)$?
  - (D) Isn't it circular to assume $P(k)$, when you are trying to prove $P(n)$?
  - (E) I just don't get it.

Divisibility Example (1)

- Let's try to prove $\forall n \in \mathbf{N}\ \ 3 \mid n^3 - n$.
  - *First, we'll work through the mathematics.*
  - *Then we'll structure a clear proof.*
- **Base case**: $P(0)$ is easy: $3 \mid (0^3 - 0)$
  - Everything divides zero, so it's true.

Divisibility Example (2)

- **Inductive step**: $\forall k \in \mathbf{N}\ P(k) \rightarrow P(k+1)$
  - Assume $P(k)$: $3 \mid (k^3 - k)$ (for arbitrary k)
  - Can we prove $P(k+1)$: $3 \mid ((k+1)^3 - (k+1))$?
- Try multiplying it out:

$$
\begin{aligned}
3 \mid ((k+1)^3 - (k+1)) &\Leftrightarrow 3 \mid (k^3 + 3k^2 + 3k + 1 - k - 1)) \\
&\Leftrightarrow 3 \mid (k^3 + 3k^2 + 2k) \\
&\Leftrightarrow 3 \mid ((k^3 - k) + (3k^2 + 3k))
\end{aligned}
$$

- We know that $3 \mid (k^3 - k)$ and $3 \mid (3k^2 + 3k)$.
- Conclude that $3 \mid ((k+1)^3 - (k+1))$, so $P(k) \rightarrow P(k+1)$.

A Clear Divisibility Proof

- **Theorem**: $\forall n \in \mathbf{N}\ \ 3 \mid (n^3 - n)$
- **Proof**: Use induction on $P(n)$: $3 \mid (n^3 - n)$.
- **Base case**: $P(0)$ is true because $3 \mid (0^3 - 0)$.
- **Inductive step**: Assume $P(k)$, for arbitrary natural number k. Then:

$$
\begin{aligned}
3 \mid (k^3 - k) \quad & \Rightarrow \quad 3 \mid (k^3 - k) + 3(k^2 + k) \\
& \Rightarrow \quad 3 \mid k^3 + 3k^2 + 3k + 1 - k - 1 \\
& \Rightarrow \quad 3 \mid (k+1)^3 - (k+1)
\end{aligned}
$$

- The final statement is $P(k+1)$, so we have proved that $P(k) \rightarrow P(k+1)$ for all natural numbers k.
- By mathematical induction, P(n) is true for all $n \in \mathbf{N}$.
- **QED**

A Clear Divisibility Proof

- **Theorem**: $\forall n \in \mathbf{N}\ \ 3 \mid (n^3 - n)$
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- **Inductive step**: Assume $P(k)$, for arbitrary natural number k. Then:

$$\begin{aligned} 3 \mid (k^3 - k) \quad &\Rightarrow \quad 3 \mid (k^3 - k) + 3(k^2 + k) \\ &\Rightarrow \quad 3 \mid k^3 + 3k^2 + 3k + 1 - k - 1 \\ &\Rightarrow \quad 3 \mid (k+1)^3 - (k+1) \end{aligned}$$

- The final statement is $P(k+1)$, so we have proved that $P(k) \rightarrow P(k+1)$ for all natural numbers k.
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- **QED**

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- Notice: we had to go back and start by assuming P(k) and then showing it implied P(k+1).
  - That first step isn't at all clear unless you've done the math first!

Use induction to show that $\sum_{k=1}^{n}(2n-1) = n^2$

- Theorem:
- Proof:
- Base case:
- Inductive step:

A False "Proof"

- "**Theorem**": All horses are the same color.
- **Proof**: By induction, where $P(n)$ is the proposition that in every set of n horses, all horses are the same color.
- **Base case**: $P(1)$ is true, because in every set of 1 horse, all horses are the same color.
- **Inductive step**: Assume $P(k)$ is true, for arbitrary positive integer k:
  - In any set of k horses, all have the same color.
  - This is called the "*inductive hypothesis*".

False "Proof" (2)

- *Inductive hypothesis* $P(k)$: In any set of k horses, all have the same color.
- Now consider a set of k+1 horses
  - $h_1, h_2, \ldots h_k, h_{k+1}$
- By $P(k)$, the first k horses are all the same color
  - $h_1, h_2, \ldots h_k,$
- Likewise, the last k horses are also the same color
  - $h_2, \ldots h_k, h_{k+1}$
- Therefore, all k+1 horses must have the same color
- Thus, $\forall k \in \mathbf{N}\ P(k) \rightarrow P(k+1)$
- So, $\forall n \in \mathbf{N}\ P(n)$. What's wrong?

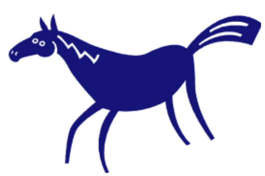
Question

- What’s wrong with that proof by induction?
  - (A) Mathematical induction doesn’t apply to horses.
  - (B) The Base Case is incorrect.
  - (C) The Inductive Step is incorrect.
  - (D) The Base Case and Inductive Step are correct, but you’ve put them together wrong.
  - (E) It’s completely correct: all horses *are* the same color!

The Problem with the Horses

- Inductive step: $\forall k \in \mathbf{N}\ \ P(k) \rightarrow P(k+1)$
  - Now consider a set of k+1 horses
    - $h_1, h_2, \ . \ . \ . \ h_k, \ h_{k+1}$
  - By $P(k)$, the first k horses are all the same color
    - $h_1, h_2, \ . \ . \ . \ h_k,$
  - Likewise, the last k horses are also the same color
    - $h_2, \ . \ . \ . \ h_k, \ h_{k+1}$
  - Therefore, all k+1 horses must have the same color

Start with $k = 1$



Then for $k = 2$

A set of one horse with the same color

Another set of one horse with the same color

The Problem with the Horses

- Inductive step: $\forall k \in \mathbf{N} \; P(k) \rightarrow P(k+1)$
  - Now consider a set of k+1 horses
    - $h_1, h_2, \ldots h_k, h_{k+1}$
  - By $P(k)$, the first k horses are all the same color
    - $h_1, h_2, \ldots h_k,$
  - Likewise, the last k horses are also the same color
    - $h_2, \ldots h_k, h_{k+1}$
  - Therefore, all k+1 horses must have the same color
- The link $P(1) \rightarrow P(2)$ is **false**.
- So, this proof by mathematical induction is invalid.

- However, the inductive step for $k \geq 2$ is actually correct!

The Problem with the Horses

- Inductive step: $\forall k \in \mathbf{N}\ \ P(k) \rightarrow P(k+1)$
  - Now consider a set of k+1 horses
    - $h_1, h_2, \ldots\ h_k,\ h_{k+1}$
  - By $P(k)$, the first k horses are all the same color
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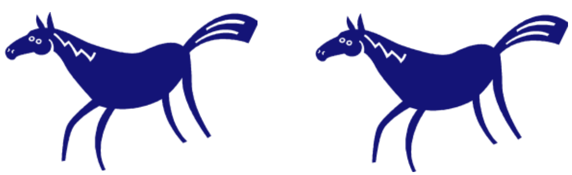
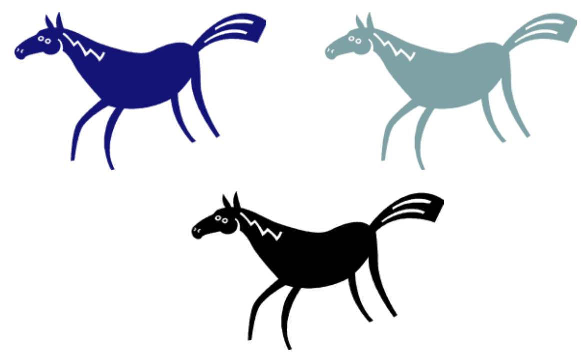
  - Likewise, the last k horses are also the same color
    - $h_2, \ldots\ h_k,\ h_{k+1}$
  - Therefore, all k+1 horses must have the same color

Assume true for $k = 2$



Then for $k = 3$



The Problem with the Horses

- Inductive step: $\forall k \in \mathbf{N} \;\; P(k) \rightarrow P(k+1)$
  - Now consider a set of k+1 horses
    - $h_1, h_2, \; . \; . \; . \; h_k, \; h_{k+1}$
  - By $P(k)$, the first k horses are all the same color
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  - Therefore, all k+1 horses must have the same color

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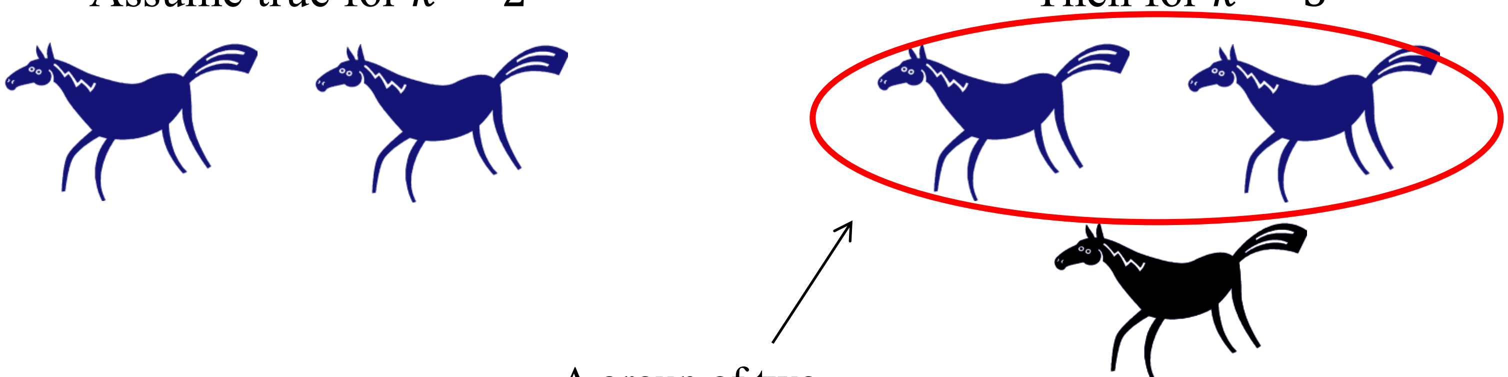
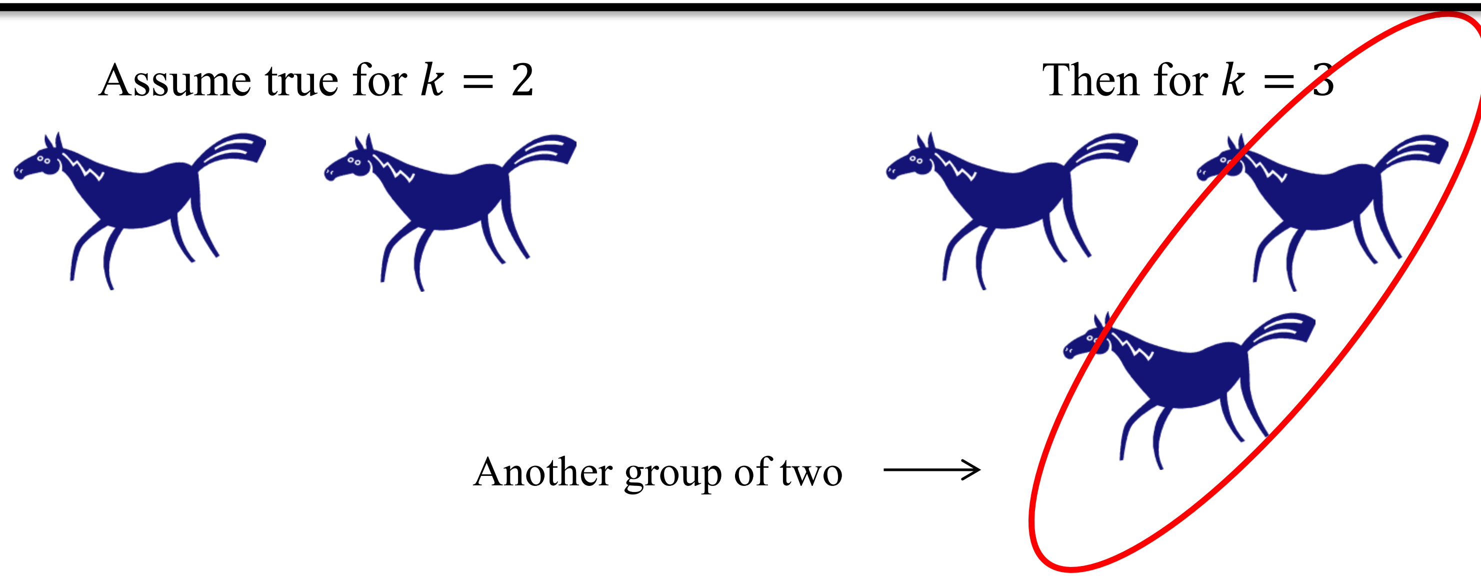
Then for $k = 3$



A group of two

The Problem with the Horses

- Inductive step: $\forall k \in \mathbf{N} \;\; P(k) \rightarrow P(k+1)$
  - Now consider a set of k+1 horses
    - $h_1, h_2, \;...\; h_k, \;\; h_{k+1}$
  - By $P(k)$, the first k horses are all the same color
    - $h_1, h_2, \;...\; h_k,$
  - Likewise, the last k horses are also the same color
    - $h_2, \;...\; h_k, \;\; h_{k+1}$
  - Therefore, all k+1 horses must have the same color



The Problem with the Horses

- Inductive step: $\forall k \in \mathbf{N}\ \ P(k) \rightarrow P(k+1)$
  - Now consider a set of k+1 horses
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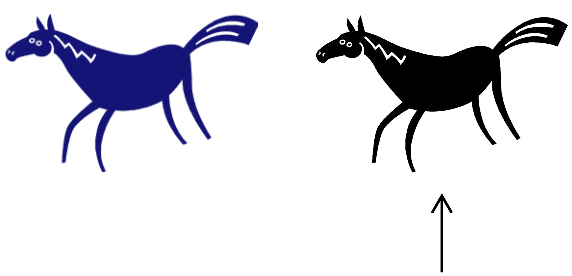
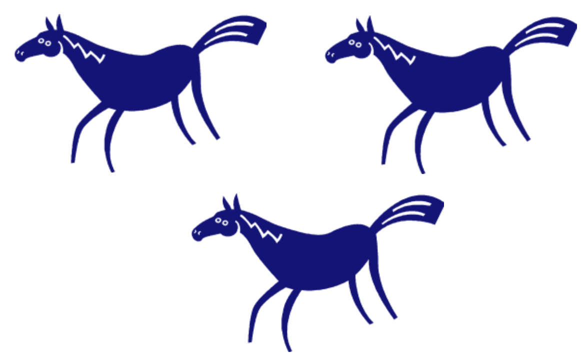
Assume true for $k = 2$



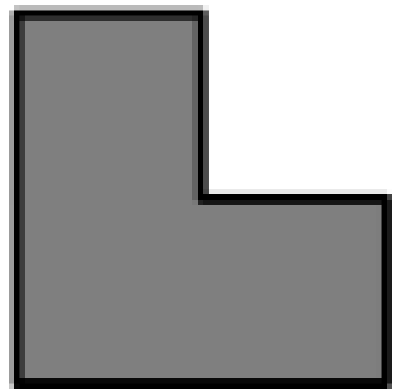
But base step wasn't true!

Then for $k = 3$



Tiling a Checkerboard (1)

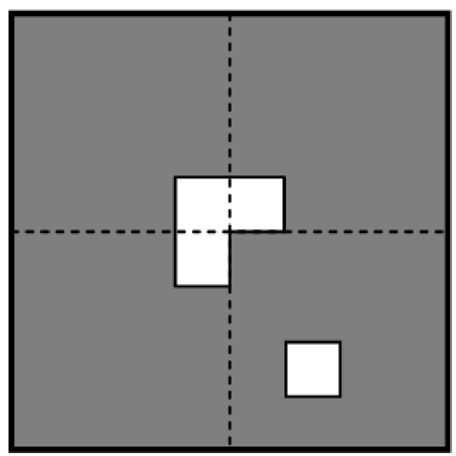
- For $n \geq 1$, consider a $2^n \times 2^n$ checkerboard.
- Can we cover it with 3-square L-shaped tiles? No.



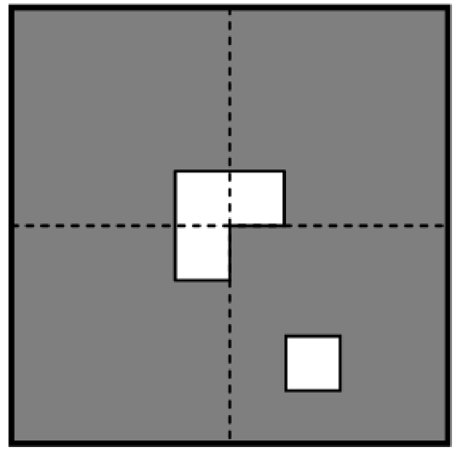
- Remove ***any*** one of the squares. Can we tile it now?
- Prove by induction: Let $P(n)$ be the proposition:
  - A $2^n \times 2^n$ checkerboard, minus any one of the squares, can be tiled with these L-shaped tiles.

Tiling a Checkerboard (2)

- Consider a $2^{k+1} \times 2^{k+1}$ checkerboard, minus any one square.
  - Divide the checkerboard into four $2^k \times 2^k$ quadrants.
- The missing square is in one quadrant.
  - From the other three quadrants, remove the square closest to the center of the checkerboard.



Tiling a Checkerboard (3)



- $P(k)$ says that each quadrant (minus one square) can be tiled. One tile covers the three central squares.
- Thus, $P(k) \rightarrow P(k+1)$, for arbitrary $k \geq 1$.
- By mathematical induction, $\forall n \geq 1\ P(n)$.

The Harmonic Series (1)

- The Harmonic Numbers are the partial sums of the Harmonic Series:

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{j}$$

- We want to prove that:

$$H_{2^n} \geq 1 + \frac{n}{2}$$

- This implies an important property of the Harmonic Series:

$$\lim_{j \to \infty} H_j \quad = \quad \infty$$

The Harmonic Series (2)

- **Theorem**:

$$H_{2^n} \geq 1 + \frac{n}{2} \text{ for all } n \in \mathbf{N}$$

- **Proof**: By induction, with *P*(*n*) being

$$H_{2^n} \geq 1 + \frac{n}{2}$$

- **Base case**: *P*(0) is true because $H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2}$

- **Inductive step**: The inductive hypothesis *P*(*k*) is

$$H_{2^k} \geq 1 + \frac{k}{2}$$

- From that assumption, we want to prove *P*(*k*+1).

The Harmonic Series (3)

- The inductive hypothesis *P*(*k*) is $H_{2^k} \geq 1 + \frac{k}{2}$

$$
\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} \\
&= H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} \\
&= \left(1 + \frac{k}{2}\right) + \frac{1}{2} \\
&= 1 + \frac{k+1}{2}
\end{aligned}
$$

- This demonstrates *P*(*k*+1).

The Harmonic Series (4)

- This proves that $P(k) \rightarrow P(k+1)$ for arbitrary k.
- Therefore, by mathematical induction, P(n) is true for all $n \in \mathbf{N}$. QED.
  - **Theorem**: $H_{2^n} \geq 1 + \frac{n}{2}$ for all $n \in \mathbf{N}$
- Let $j = 2^n$ be some integer. Then $n = \log j$.

$$H_j \;\geq\; 1 + \frac{1}{2}\log j \;\geq\; 1 + \log\sqrt{j}$$

  - Therefore, the Harmonic series diverges, because

$$\lim_{j\to\infty} H_j \;\geq\; \lim_{j\to\infty}\left(1 + \log\sqrt{j}\right) = \infty$$

  - But it diverges *very* slowly.

Prove that 6 divides $n^3 - n$ whenever n is a nonnegative integer.