# Counting—not just for preschool anymore.

As the title implies, it seems odd to have "counting" be on the list of things we cover. Feels like something you should have mastered in Kindergarten or earlier. But counting large numbers of things can actually get tricky. And counting is key to probability. Consider the following question

> When rolling two dice[i], what is the probability of rolling a total of 5 or less?

The easiest way to address this is to simply count. Count the number of ways two dice can be rolled. Count the number of ways they can be rolled and get a total of 5 or less. Then just divide the second number by the first. So what's the answer?

Hopefully the above got you to see that counting can be useful. But at some point just listing all the possibilities gets unwieldy and even unrealistic. What if we rolled 200 dice and wanted to know the odds of getting a 304 or less. Actually listing all the possibilities isn't reasonable (how many are there?). And even figuring out how many there are of 304 or less seems like a real pain. So we'd like some nice techniques for at least properly thinking about counting the options.

We will also look at some really neat applications of counting as well as some cool techniques for quickly counting large sets (like our 200 dice).

- Chapter 6.1 covers the basics of counting
- Chapter 6.2 covers one of those really cool techniques—the pigeonhole principle. It is a trivial statement which leads to some non-intuitive proofs and provides just a neat way to attack certain problems.
- Chapters 6.3-6.5 provide techniques for quickly counting things (binomial coefficients, permutations and combinations), many of which you may have seen elsewhere (Algebra 2 or stats perhaps).

## Basics of counting (6.1)

There are a lot of fairly straightforward rules for counting. We'll touch on a few. You don't need to know these "by name" but you do need to know them.

> THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are $n_1$ ways to do the first task and for each of these ways of doing the first task, there are $n_2$ ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

[i] OK, two six sided dice where each side of each die is labelled "1", "2", … "6" and where the die is fair (each side is equally likely to come up).

We effectively used the product rule above to figure out how many ways the two dice could be rolled. Let's do one more:

1. How many different unames are there assuming that all unames are 8 characters or less, the first character must be a lower-case letter and that the rest can be lower-case letters or a digit.

THE SUM RULE If a task can be done either in one of $n_1$ ways or in one of $n_2$ ways, where none of the set of $n_1$ ways is the same as any of the set of $n_2$ ways, then there are $n_1 + n_2$ ways to do the task.

2. If we need to pick either one of the students (90) or one of the class staff (3) people to go do something, there are _____ ways to pick a single person.

This next one is closely related to the inclusion/exclusion principle.

THE SUBTRACTION RULE If a task can be done in either $n_1$ ways or $n_2$ ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

3. Say we want to know how many 5-bit bit-strings there are that start with a 1 or end with 00.

This next one seems confusing until you see an example.

THE DIVISION RULE There are $n/d$ ways to do a task if it can be done using a procedure that can be carried out in $n$ ways, and for every way $w$, exactly $d$ of the $n$ ways correspond to way $w$.

This one actually comes up for me on occasion when playing board games. In some games, it matters a lot who is on your left and right.

4. Say you have 4 people and want to know how many ways there are to arrange everyone around a table where it doesn't matter what seats they are in, but it does matter what who your neighbors are.

    How many ways are there to seat everyone (assuming exact location does matter)?

    How many of those are equivalent to each other?

I've not covered tree diagrams from chapter 6.1. Please look that over. There is an example on the right.

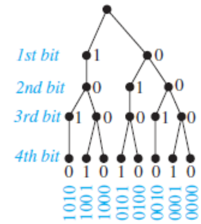


FIGURE 2 Bit Strings of Length Four without Consecutive 1s.

## The Pigeonhole Principle (6.2)

This has got to be one of the most amazingly useful but simple things in math. It says simply:

**THE PIGEONHOLE PRINCIPLE** If $k$ is a positive integer and $k+1$ or more objects are placed into $k$ boxes, then there is at least one box containing two or more of the objects.

Which is a remarkably trivial statement. It's called the pigeonhole principle because the generic objects are traditionally called pigeons and the boxes "pigeonholes[ii]".

So why is this useful? It turns out to show up as a useful way to solve a huge number of fun problems.

1. A function *f* from a set with k+1 or more elements to a set with k elements is not one-to-one.

OK, so that one was trivial. Let's look at something not at all obvious

2. Show that for every number *n* there is a multiple of *n* that has only 1s and 0s in it when written as a decimal number.

That doesn't look to have a darn thing to do with anything related to the pigeonhole principle. But let's consider the n+1 smallest integers that consist of only 1s (so 1, 11, 111, etc.). Now what do we do? (example from page 401)

Let's jump to the more general version:

**THE GENERALIZED PIGEONHOLE PRINCIPLE** If $N$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

3. Among 100 people, what is the minimum number of people that were born in the same month?

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[ii] On useless trivia: the original work referred to drawers or shelves (Schubfachprinzip?)

## Some Elegant Applications of the Pigeonhole Principle

The text has a bunch of wonderful applications of the pigeonhole principle. They aren't obvious, but they do show just how wide the reach of the pigeonhole principle can be.

4. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

   This one seems like the PP could apply, but it's really unclear how. I mean we have 30 days and no more than 45 games. We are looking for 14 games they must play in a row. But I at least couldn't see any way to solve this. Let's get some hints going:

   - Let's define $a_i$ to be the total number of games played this month up to and including day i. So if the team plays 1 game on day 1 and 3 games on day 2, $a_1$=1 and $a_2$=4.
   - Consider the fact that $a_I$ is always increasing (play a game every day)

   Now, given that, what do we need to do?

The next example involves a few definitions (from our text)

Suppose that $a_1, a_2, \dots, a_N$ is a sequence of real numbers.

- A **subsequence** of this sequence is a sequence of the form $a_{i1}, a_{i2}, \dots, a_{im}$, where $1 \le i_1 < i_2 < \cdots < i_m \le N$.

That is, a subsequence is a sequence obtained from the original sequence by including some of the terms of the original sequence in their original order, and perhaps not including other terms.

- A sequence is called **strictly increasing** if each term is larger than the one that precedes it.
- And it is called **strictly decreasing** if each term is smaller than the one that precedes it.

Before we get to the next example, let's just look at these definitions briefly.

5.
   a. Is 1, 2, 2, 3 a subsequence of 1, 2, 3, 2, 4, 3? Is that subsequence strictly increasing?

   b. What is the longest increasing subsequence of 3, 2, 0, 4, 1, 5 that starts with a 0? The longest decreasing subsequence?

6. Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length n + 1 that is either strictly increasing or strictly decreasing.

I spent a long time trying to prove this on my own and got stuck. That said, a good place to start on a problem with this is to start looking at examples. Can you find an algorithm to build a sequence with $n^2$ elements that has no subsequence of length n + 1 that is either strictly increasing or strictly decreasing?

OK, we've got that. Hopefully that gives us a sense of the problem. Now let's add some formalism to the problem and see if that helps.

Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of $n^2 + 1$ distinct real numbers. Associate an ordered pair with each term of the sequence, namely, associate $(i_k, d_k)$ to the term $a_k$, where $i_k$ is the length of the longest increasing subsequence starting at $a_k$, and $d_k$ is the length of the longest decreasing subsequence starting at $a_k$.

The trick here is to show that every $(i_k, d_k)$ must be unique. Why is that? Why does it help?

## Permutations and Combinations (6.3)

When it comes to counting quickly, we generally have a few generic cases.

- "**Selection with replacement, order matters**". We handled the dice problem at the start of lecture this way. We roll 2d6 (2 six-sided dice) and we are checking what *each* die rolls and we allow duplicates (each die could roll the same number). We generally can use the product rule here.

- "**Selection without replacement, order matters**". In this case we can't have duplicates. Say we are drawing numbers from a hat. If the hat has 6 distinct numbers in it, how many ways could we pull numbers? That order matters means that we are counting pulling a 5 then 6 as distinct from a 6 then a 5.

  1. If we are pulling 2 numbers from a hat with 6 distinct values 1 to 6, how many ways are there to pull these (assuming order matters)?

  2. How many of the above possibilities totals out to 5 or less? What % of possible draws are 5 or less?

  In general, we find that if we are pulling r numbers from a hat with n items we get P(n,r) combinations, where $P(n,r) = n(n-1)(n-2) \dots (n-r+1)$ this can also be written as:

  If $n$ and $r$ are integers with $0 \le r \le n$, then $P(n,r) = \dfrac{n!}{(n-r)!}$.

  Notice that if we put the numbers back into the hat after each draw, it's now the same as dice.

- "**Selection without replacement, order doesn't matter**". Still no duplicates, but we think of 5 then 6 and 6 then 5 as the same—we only care what we finish with, not the order they occurred.

  1. If we are pulling 2 numbers from a hat with 6 distinct values 1 to 6, how many ways are there to pull these (assuming order doesn't matter)?

  2. How many of the above possibilities totals out to 5 or less? Can we compute a probability here?

  The number of $r$-combinations of a set with $n$ elements, where $n$ is a nonnegative integer and $r$ is an integer with $0 \le r \le n$, equals

  $$C(n,r) = \frac{n!}{r!\,(n-r)!}.$$