L14: Permutations, Combinations and Some Review

EECS 203: Discrete Mathematics

Last time we did a number of things

- Looked at the sum, product, subtraction and division rules.
  - Don't need to know by name.
- Spent a while on the Pigeonhole Principle
  - Including the generalized version.
  - Worked a few complex examples.
    - They were tricky!
- Started on Combinations and Permutations.

Review: Pigeonhole Principle

THE PIGEONHOLE PRINCIPLE If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

- "Simple" problem:
  - Prove that if you have 51 unique numbers 1 to 100 there exists a pair in that 51 which sum to 100.

Review: Pigeonhole Principle

THE PIGEONHOLE PRINCIPLE If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

- "Old" problem
  - Say we have five distinct points (x_i, y_i) for i= 1 to 5. And say all x and y values are integers. Now draw lines connecting each pair of points. Prove that the midpoint of at least one of those lines has an x,y location where both x and y are integers.

And a tricky one

- Claim: Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing
- Example: Seq of 3^2+1 numbers (3,1,0,2,6,5,4,9,8,7) has increasing subsequence of length 3+1 (0,2,6,9)
- Proof using PP
  - What are the pigeons?
  - What are the holes?

Permutations & Combinations

- $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$
- Permutations
  - $P(n,k)$ = Number of ways to choose k things *(order counts!)* out of n things

Permutations & Combinations

- $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$
- Permutations
  - $P(n,k)$ = Number of ways to choose k things *(order counts!)* out of n things
  - Example. n=3. Three things: {brush teeth, floss, gargle}

    $P(n,1)$ = #ways to do one thing: 3

    (brush), (floss), (gargle)

    $P(n,2)$ = #ways to do two things in order: 6

    (brush, floss), (brush, gargle), (floss, brush), (floss, gargle), (gargle, brush), (gargle, floss)

    $P(n,3)$ = #ways to do three things in order: 6

    (brush, floss, gargle), (brush, gargle, floss), (floss, brush, gargle), (floss, gargle, brush), (gargle, brush, floss), (gargle, floss, brush)

Permutations & Combinations

- $n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot3\cdot2\cdot1$
- Permutations
  - P(n,k) = Number of ways to choose k things *(order counts!)* out of n things
  - $P(n,k) = n\cdot(n-1)\cdot\ldots\cdot(n-k+1) = \dfrac{n!}{(n-k)!}$

n choices for first thing

n-1 choices for second thing

n-k+1 choices for k^{th} thing

Permutations & Combinations

- $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$
- Permutations
  - P(n,k) = Number of ways to choose k things *(order counts!)* out of n things
- Combinations
  - C(n,k) = Number of ways to choose a set of k things (order doesn't matter) out of n things

Permutations & Combinations

- $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$
- Permutations
  - P(n,k) = Number of ways to choose k things *(order counts!)* out of n things
- Combinations
  - C(n,k) = Number of ways to choose a set of k things (order doesn't matter) out of n things
  - Example: n=3. Three things: {brush, floss, gargle}

    C(n,1) = #ways to choose one thing: 3

    {brush}, {floss}, {gargle}

    C(n,2) = #ways to choose two things: 3

    {brush, floss}, {brush, gargle}, {floss, gargle}

    C(n,3) = #ways to choose three things: 1

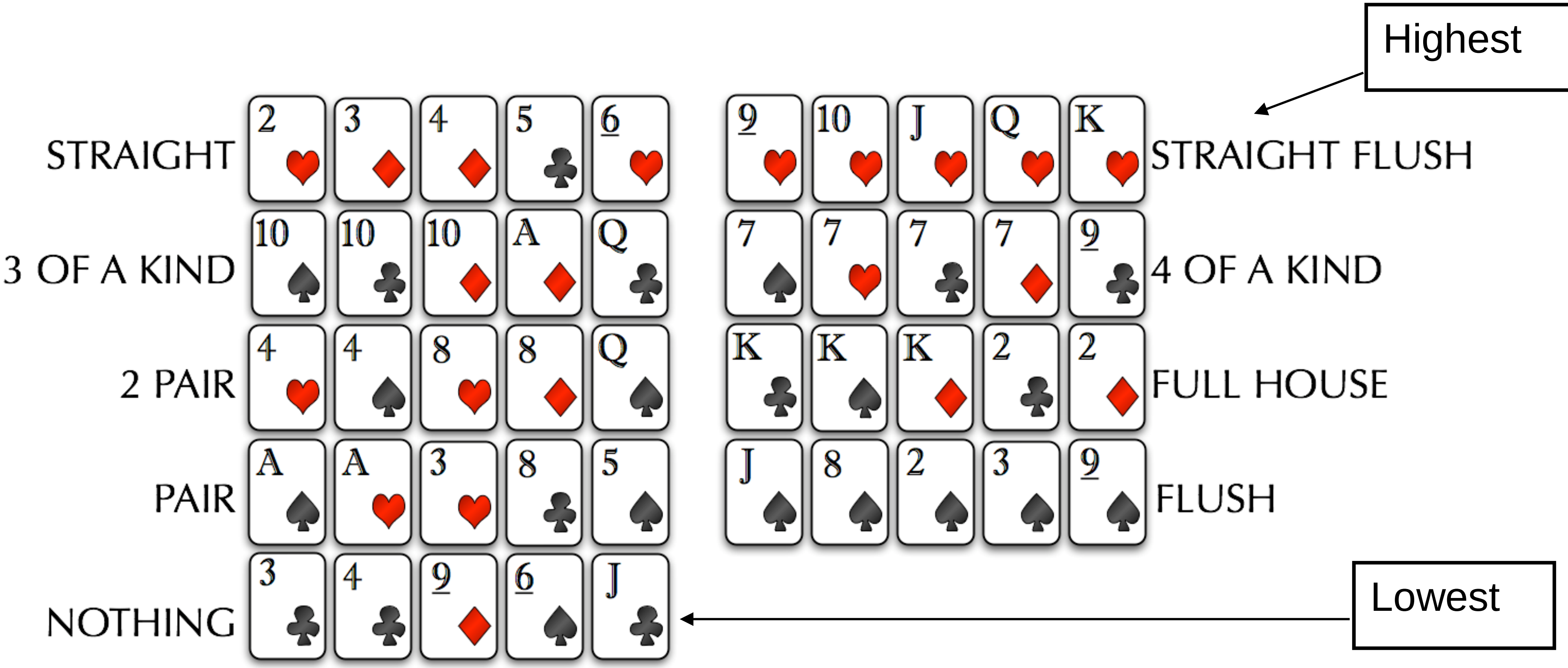
    {brush, floss, gargle}

Permutations & Combinations

- $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$
- Permutations
  - P(n,k) = Number of ways to choose k things *(order counts!)* out of n things
- Combinations
  - C(n,k) = Number of ways to choose a set of k things (order doesn't matter) out of n things
  - C(n,k) = $\dfrac{P(n,k)}{k!} = \dfrac{n!}{(n-k)!\,k!} = \dbinom{n}{k}$

read *"n choose k"*

Poker Hands



Number of different hands: $\binom{52}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5!} = 2{,}598{,}960$

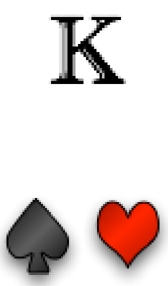
Problem 2: How many ways to make a pair?

- Select a hand with one pair in stages:
  - Stage 1: Pair of what? Choose a number or face card

K

Problem 2: How many ways to make a pair?

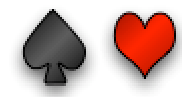
- Select a hand with one pair in stages:
  - Stage 1: Pair of what? Choose a number or face card
    13 choices
  - Stage 2: Choose which suits (from stage 1)



Problem 2: How many ways to make a pair?

- Select a hand with one pair in stages:
  - Stage 1: Pair of what? Choose a number or face card
    13 choices
  - Stage 2: Choose which suits (from stage 1)
    6 choices
  - Stage 3: Choose third card (different than stage 1)

K



Problem 2: How many ways to make a pair?

- Select a hand with one pair in stages:
  - Stage 1: Pair of what? Choose a number or face card
    13 choices
  - Stage 2: Choose which suits (from stage 1)

    6 choices

  - Stage 3: Choose third card (different than stage 1)
    (52-4) = 48 choices
  - Stage 4: Choose fourth card (different than stages 1&3)

  - Stage 5: Choose fifth card (different than stages 1,3,4)

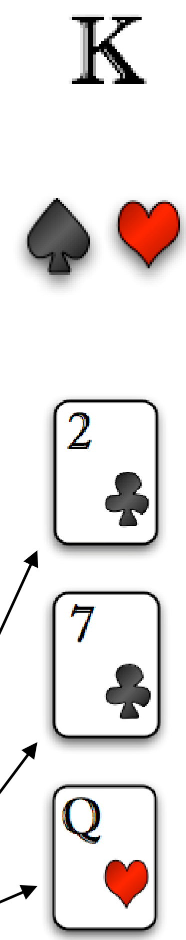
Problem 2: How many ways to make a pair?

- Select a hand with one pair in stages:
  - Stage 1: Pair of what? Choose a number or face card
    13 choices
  - Stage 2: Choose which suits (from stage 1)

    6 choices

  - Stage 3: Choose third card (different than stage 1)

    (52-4) = 48 choices
  - Stage 4: Choose fourth card (different than stages 1&3)
    (52-8) = 44 choices
  - Stage 5: Choose fifth card (different than stages 1,3,4)
    (52-12) = 40 choices



3! ways to arrange these

Problem 2: How many ways to make a pair?

- Select a hand with one pair in stages:
  - Stage 1: Pair of what? Choose a number or face card

    13 choices
  - Stage 2: Choose which suits (from stage 1)

    6 choices
  - Stage 3: Choose third card (different than stage 1)

    (52-4) = 48 choices
  - Stage 4: Choose fourth card (different than stages 1&3)

    (52-8) = 44 choices
  - Stage 5: Choose fifth card (different than stages 1,3,4)

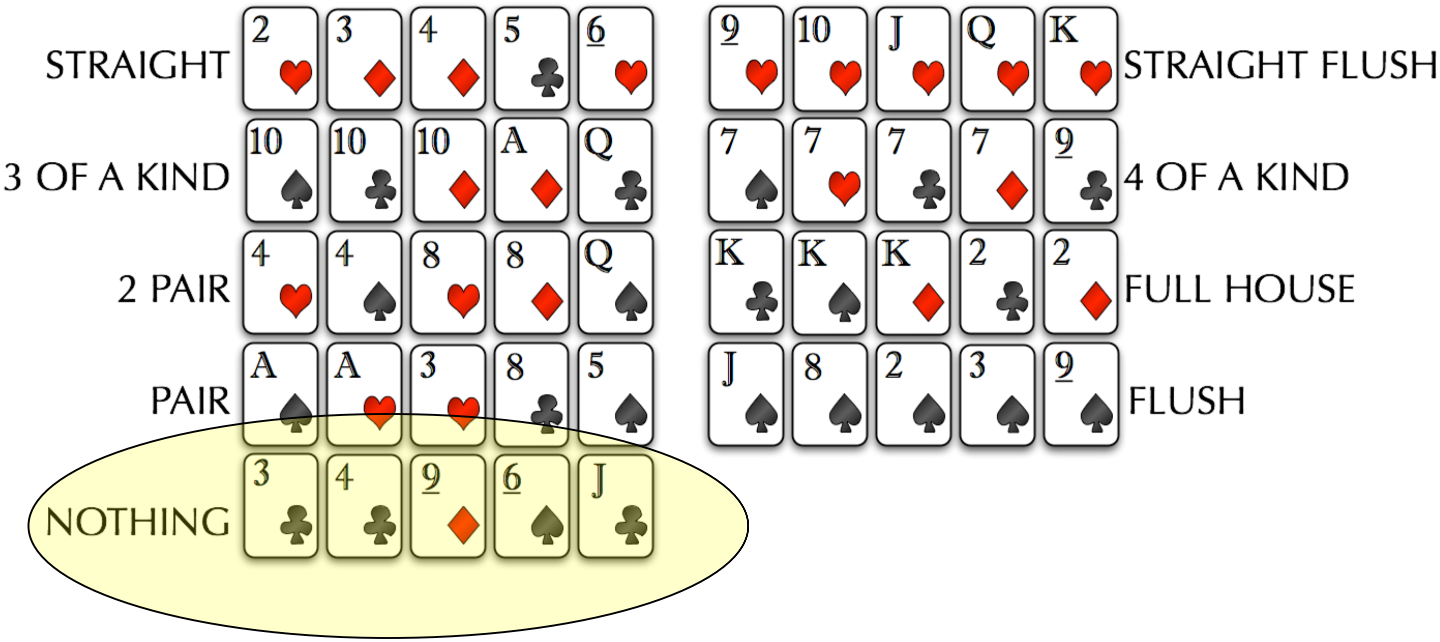
    (52-12) = 40 choices
- Number of ways: $(13 \cdot 6 \cdot 48 \cdot 44 \cdot 40)/3! = 1{,}098{,}240$

  ≈ 40% chance of getting a pair (and nothing better)

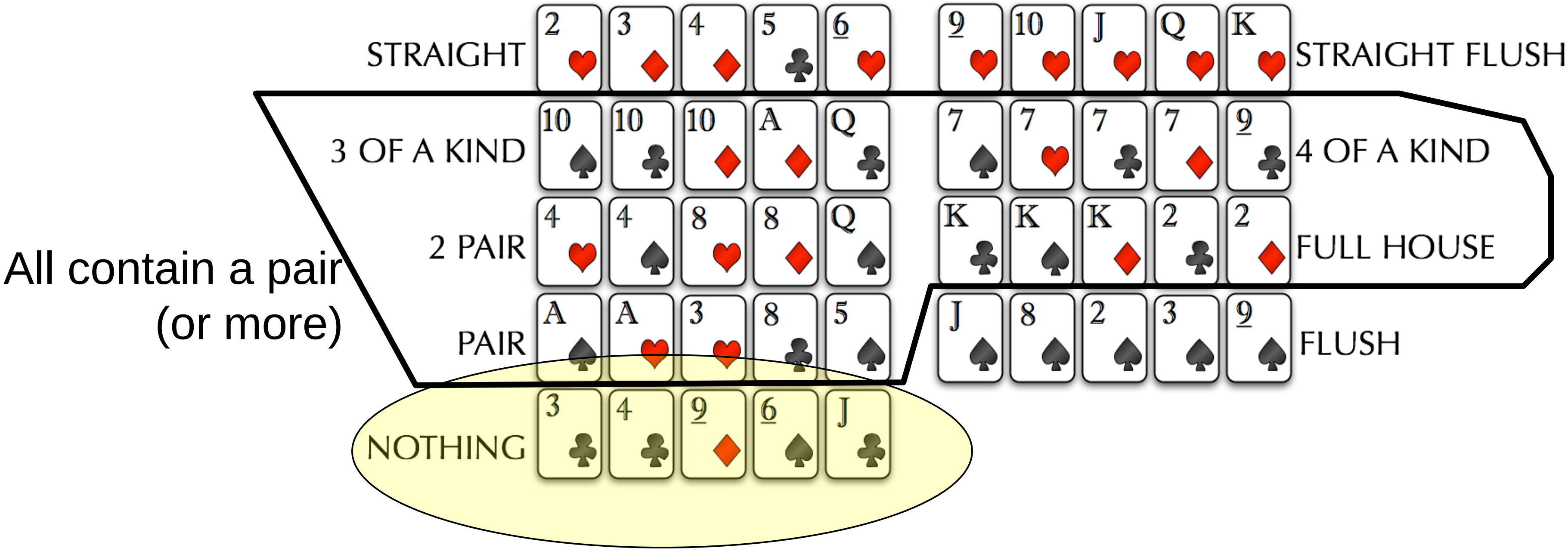
Problem 2: How many ways to make a pair?

- Wikipedia has:
  - C(13,1)*C(2,4)*C(3,12)*4^3.
    - Same number.
    - Can you justify it?

- What are the odds of making nothing?

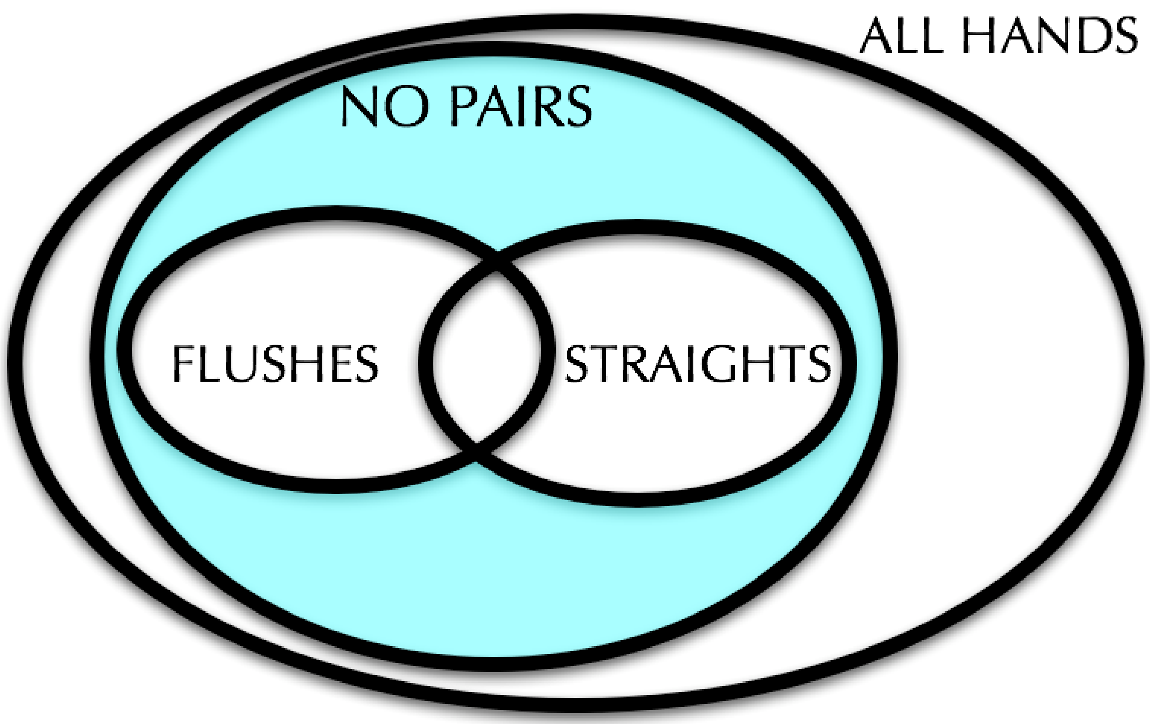


- What are the odds of making nothing?

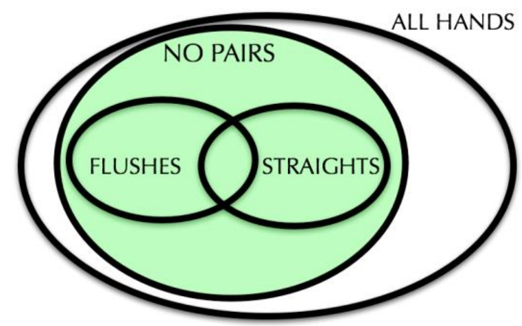


Problem 3: How many ways to make nothing?

- We're counting hands:
  (1) without pairs
  (2) that also do not contain straights or flushes

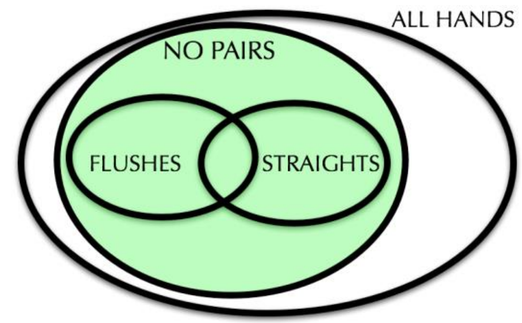


Counting hands without pairs



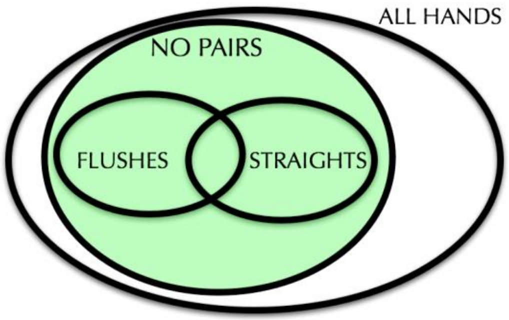
- Pick a hand without a pair
  - Stage 1: ??

Counting hands without pairs



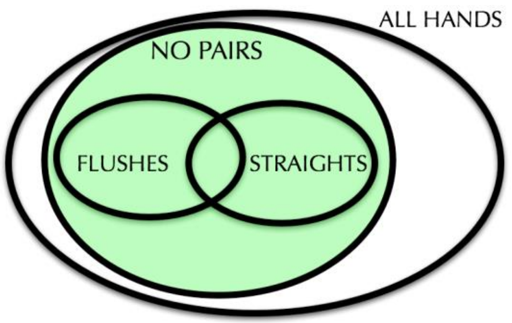
- Pick a hand without a pair
  - 1st card:

Counting hands without pairs



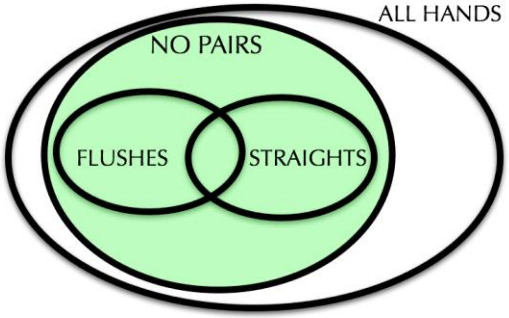
- Pick a hand without a pair
  - 1st card: 52 choices
  - 2nd card: (not same number/face as 1st card)

Counting hands without pairs



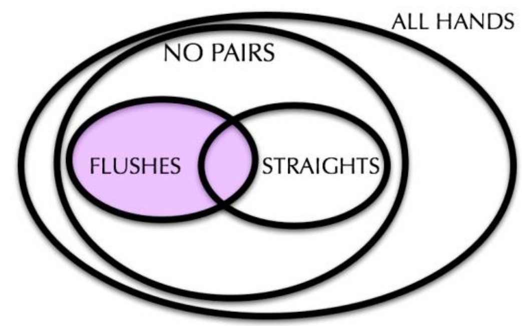
- Pick a hand without a pair
  - 1st card: 52 choices
  - 2nd card: 48 choices (not same number/face as 1st card)
  - 3rd card: 44 choices (different from 1st & 2nd)
  - 4th card: 40 choices (different from 1st,2nd,3rd)
  - 5th card: 36 choices (different from 1st,2nd,3rd,4th)

Counting hands without pairs



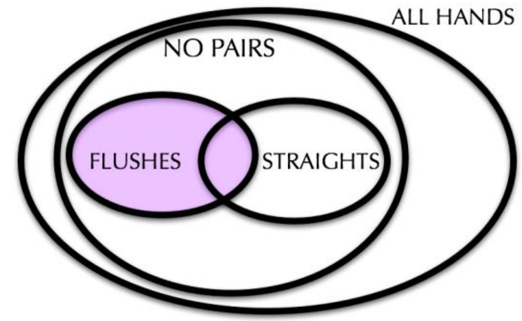
- Pick a hand without a pair
  - 1st card: 52 choices
  - 2nd card: 48 choices (not same number/face as 1st card)
  - 3rd card: 44 choices (different from 1st & 2nd)
  - 4th card: 40 choices (different from 1st,2nd,3rd)
  - 5th card: 36 choices (different from 1st,2nd,3rd,4th)
- Division rule: Each hand could have been chosen in exactly 5! different ways.
- Total = $(52 \cdot 48 \cdot 44 \cdot 40 \cdot 36)/5! = 1{,}317{,}888$
  - Also $C(13,5)*4^5$

Counting flushes (that may be straights too)



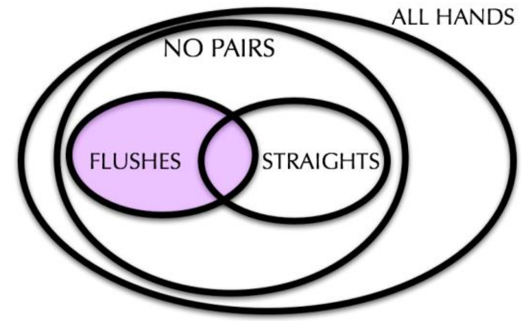
- Stage 1: ?
- Stage 2: ?

Counting flushes (that may be straights too)



- Stage 1: Pick a suit
- Stage 2: Pick which 5 cards in the suit
- Total =

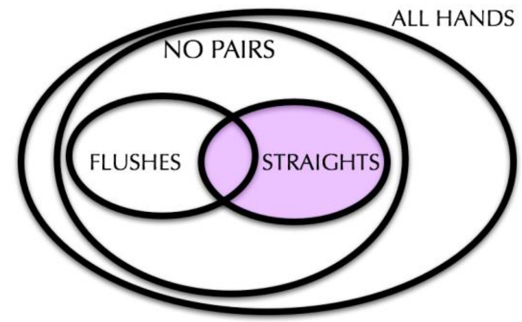
Counting flushes (that may be straights too)



- Stage 1: Pick a suit
  - 4 choices
- Stage 2: Pick which 5 cards in the suit
  - $\binom{13}{5} = \frac{13!}{8!5!} = \frac{13 \times 12 \times 11 \times 10 \times 9}{5 \times 4 \times 3 \times 2}$ = 1,287 choices
- Total = 4 x 1,287 = 5,148

Counting straights (that may be flushes too)

Note: A2345 is a straight!



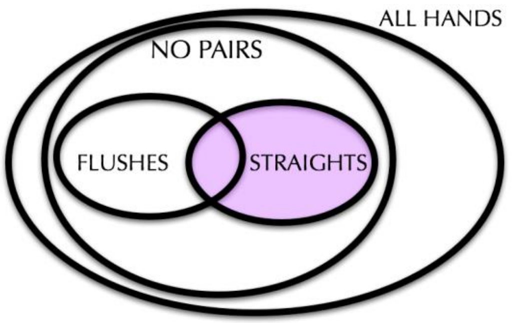
- Stage 1: ??
- Stage 2: ??

  .

  .

  .

Counting straights (that may be flushes too)



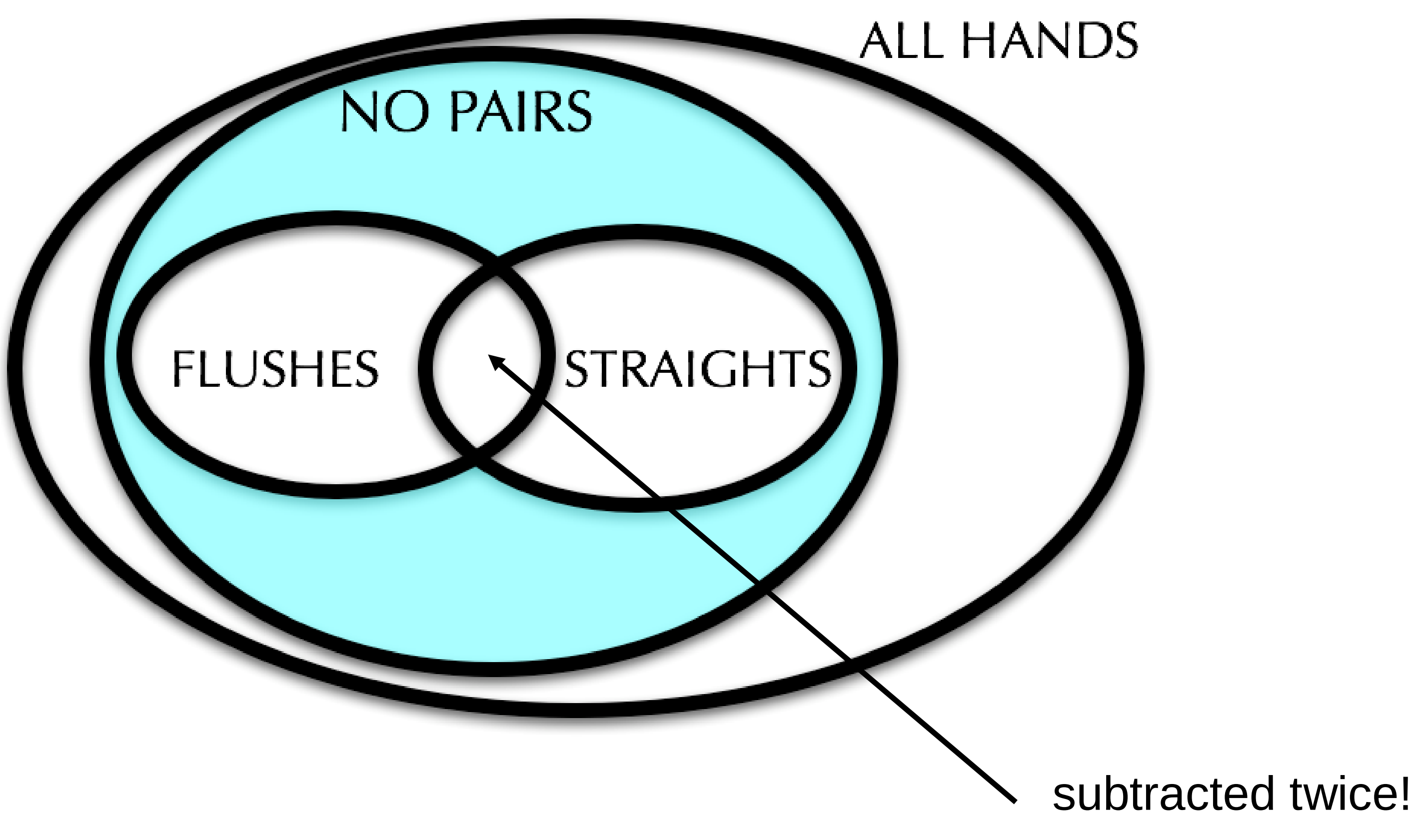
- Stage 1: Pick the lowest card number
  - 10 choices (A,2,3,4,5,6,7,8,9,10)
- Stage 2: Pick the 1st card: 4 choices
- Stage 3: Pick the 2nd card: 4 choices
- Stage 4: Pick the 3rd card: 4 choices
- Stage 5: Pick the 4th card: 4 choices
- Stage 6: Pick the 5th card: 4 choices

- Total = $10 \cdot 4^5 = 10{,}240$

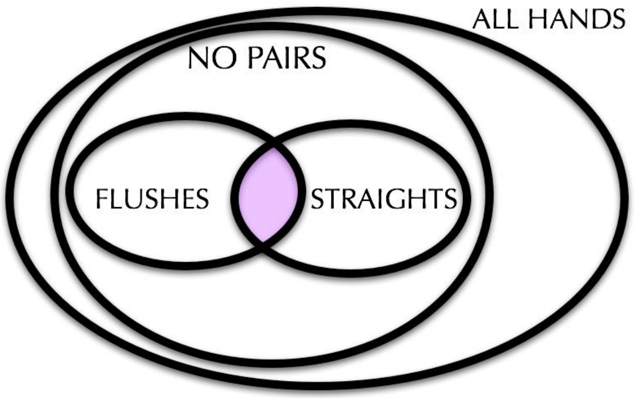
Back to the Venn diagram

Ways to make "nothing"

= (#without pairs) - (#flushes) - (#straights) **+ (#straightflushes)**



Counting straight flushes



- Stage 1: ??

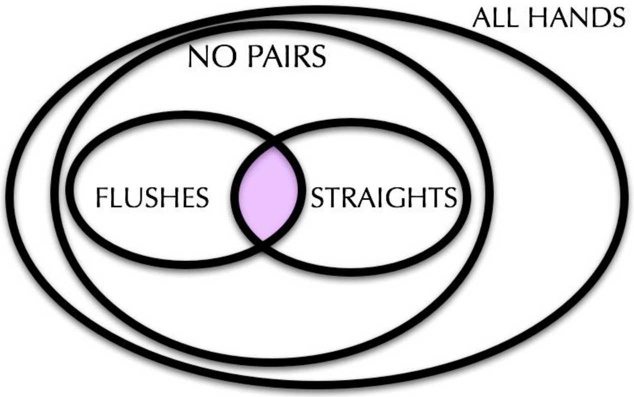
- Stage 2: ??

.

.

.

Counting straight flushes



- Stage 1: Pick a suit
  - 4 choices
- Stage 2: Pick the lowest numbered card (A,2,…,9,10)
  - 10 choices

- Total = 40

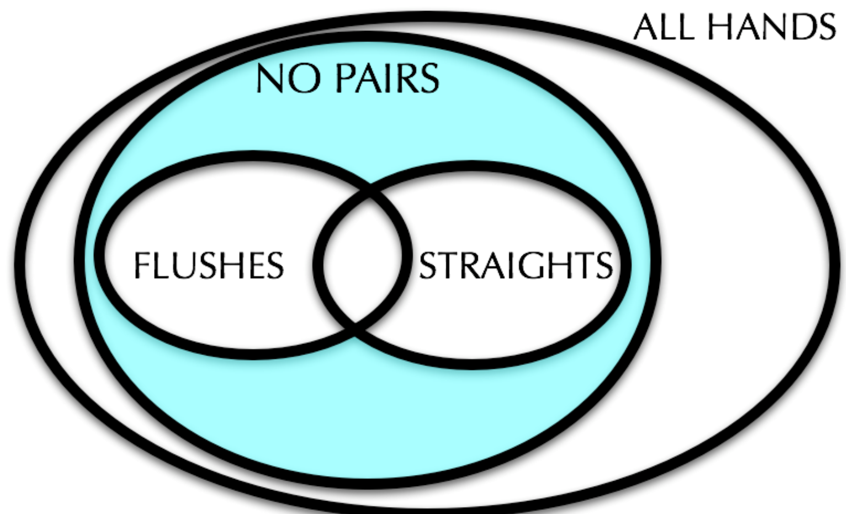
Summing up

Ways to make "nothing"

= (#without pairs) - (#flushes) - (#straights) + (#straightflushes)

= 1,317,888 - 5,148 - 10,240 + 40

= 1,302,540



You get nothing ≈ 52% of the time

Quiz

- How many cards must be selected from a standard deck of 52 cards to guarantee that at least 3 hearts are selected?

A) 9
B) 52
C) 3
D) 42
E) I have no idea

Counting Recap

- ***k-permutation***: a *sequence* of k things (selected from n things)
- ***k-combination***: a *set* of k things (selected from n things)

Repetitions ***not*** allowed!

Counting Recap

- ***k-permutation***: a *sequence* of k things (selected from n things)
- ***k-combination***: a *set* of k things (selected from n things)

- P(n,k) = number of k-permutations

$$P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

- C(n,k) = number of k-combinations

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Permutations and Combinations with repetitions

- So far *today*, we have assumed that we can select items **without repetitions.**
  - We did look at selecting permutations with repetitions last time (dice)
  - We've not looked at combinations with repetitions in any formal way.
- But first, we turn to Binomial Coefficients and Pascal's Triangle

The Binomial Theorem

- $\binom{n}{k}$ often called a *binomial coefficient*

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

…

$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$

$= \textbf{?}$

The Binomial Theorem

- $\binom{n}{k}$ often called a *binomial coefficient*

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

…

$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$

$= \mathbf{?}x^5 + \mathbf{?}x^4y + \mathbf{?}x^3y^2 + \mathbf{?}x^2y^3 + \mathbf{?}xy^4 + \mathbf{?}y^5$

number of ways to pick
3 ***y***'s out of 5 possibilities
Or equivalently, number
of ways to pick 2 **x**'s out
of 5 possibilities

The Binomial Theorem

- $\binom{n}{k}$ often called a *binomial coefficient*
- Binomial Theorem: for any x and y

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

number of ways to choose k ***x***s
out of the n factors (x+y)

Proving things with the binomial theorem

- Binomial Theorem: for any x and y

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Theorem: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

  – In other words ?

- Proof: Set x = 1, y=1

Proving things with the binomial theorem

- Binomial Theorem: for any x and y

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- Theorem: $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$
  - In other words?

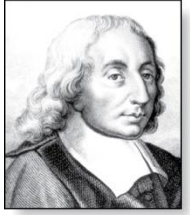
- Proof: Set x = -1, y=1

Quick Exercise

What is $$\sum_{k=0}^{n} \binom{n}{k} (-0.5)^k (1.5)^{n-k}$$?

(a) 2^n
(b) 1.5^n
(c) 1
(d) 2.5^n

Blaise Pascal
(1623-1662)



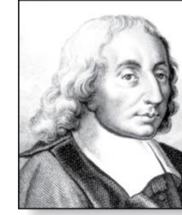
Pascal’s Identity

Pascal’s Identity: If *n* and *k* are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Pascal's Identity

Blaise Pascal
(1623-1662)



Pascal's Identity: If *n* and *k* are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof (*combinatorial*): Let T be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of T containing k elements. Each of these subsets either:

- contains a with $k - 1$ other elements, or
- contains k elements of S and not a.

There are

- $\binom{n}{k-1}$ subsets of k elements that contain a, since there are $\binom{n}{k-1}$ subsets of $k - 1$ elements of S,
- $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S.

Hence,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Pascal's Triangle

The n^{th} row in the triangle consists of the binomial coefficients of C(n,k), k = 0,..,n

$$\binom{n}{k-1}+\binom{n}{k}=\binom{n+1}{k}$$

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$$
$$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$$
$$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$$

By Pascal's identity:

$$\binom{6}{4}+\binom{6}{5}=\binom{7}{5}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

By Pascal's identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

Review problem

- If I have 9 books and plan on taking 4 on the plane with me, how many different sets of books could I bring?

Problem: Counting Bagels

- A bagel shop has 8 kinds of bagels.

  How many ways to buy a dozen bagels?

Problem: Counting Bagels

- A bagel shop has 8 kinds of bagels.
  How many ways to buy a dozen bagels?

  = Number of solutions to: (natural numbers only)

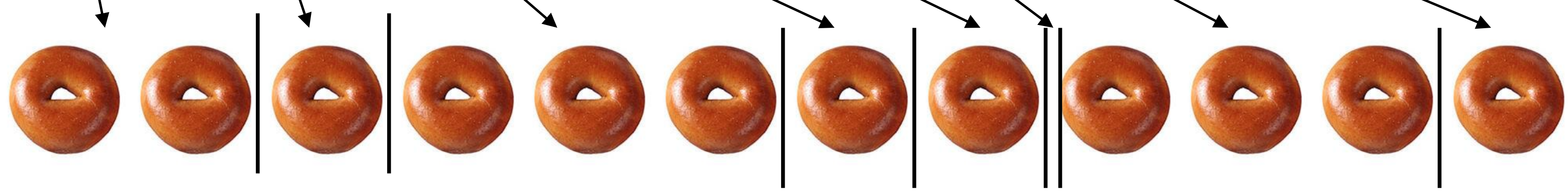
  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$

Problem: Counting Bagels

- A bagel shop has 8 kinds of bagels.
  How many ways to buy a dozen bagels?

  = Number of solutions to: (natural numbers only)

  $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$$



bit string with 12+7 bits. bagel = ‘0’, bar = ‘1’

The Stars ‘n’ Bars Theorem

- The number of ways to choose **k objects** each of **n different types** ***(with repetition)*** is $\binom{k+n-1}{k}$
  - Example. k=2, types = {apple,orange,pear}
  - ★★|| 2 apples
  - ★|★| 1 apple, 1 orange
  - ★||★ 1 apple, 1 pear
  - |★★| 2 oranges
  - |★|★ 1 orange, 1 pear
  - ||★★ 2 pears

Stars = #Objects; Bars = #Types-1

Problem: Counting Bagels (with *lower bounds*)

- A bagel shop has 8 kinds of bagels.
  How many ways to buy a dozen bagels
  with at least 1 of each kind?

  = Number of solutions to: (natural numbers only)

  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$

  $\mathbf{x_i \geq 1}$ for $1 \leq i \leq 8$

Problem: Counting Bagels (with *lower bounds*)

- A bagel shop has 8 kinds of bagels.

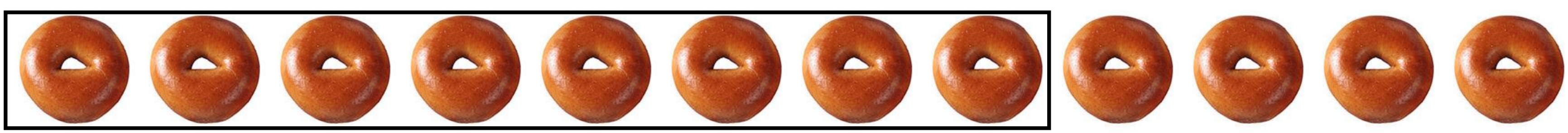
  How many ways to buy a dozen bagels

  with at least 1 of each kind?

  = Number of solutions to: (natural numbers only)

  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = \mathbf{4}$

  $\cancel{x_i \geq 1 \text{ for } 1 \leq i \leq 8}$



eight bagels already determined

k = 4; n = 8

Problem: Counting Bagels (with *upper bounds*)

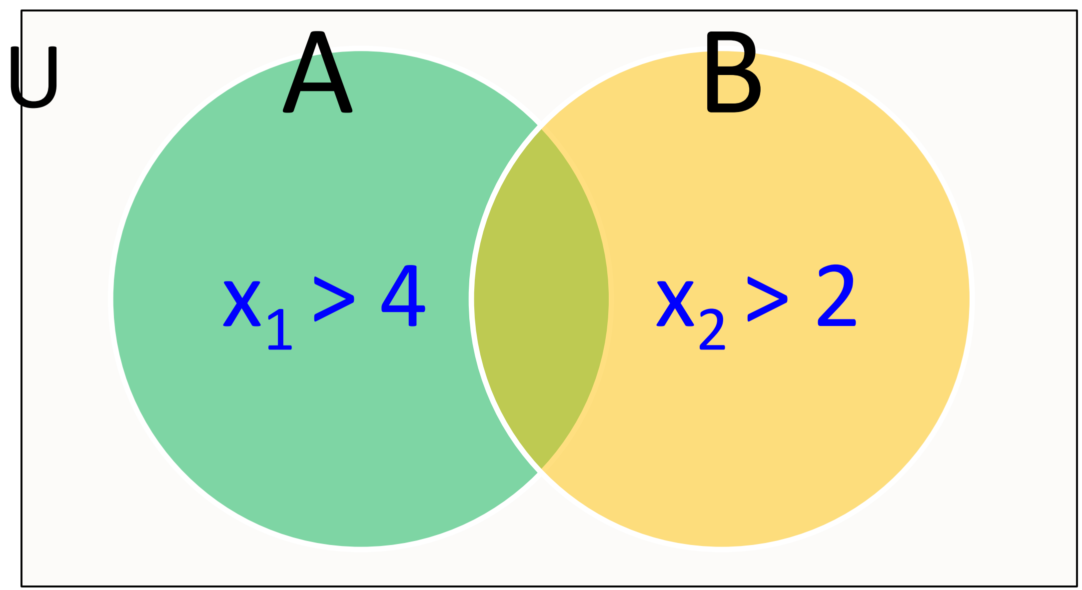
- A bagel shop has 8 kinds of bagels.

  How many ways to buy a dozen bagels with at most 4 onion and at most 2 poppy seed?

  = Number of solutions to: (natural numbers only)

  $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$$

  $$x_1 \leq 4,\ x_2 \leq 2$$



$$|\overline{A \cup B}| = |U| - |A \cup B|$$

$$= |U| - |A| - |B| + |A \cap B|$$

aka: inclusion-exclusion princple

Problem: Counting Bagels (with *upper bounds*)

= Number of solutions to: (natural numbers only)

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 12$

$x_1 \leq 4,\ x_2 \leq 2$

- All Solutions: 12 stars, 7 bars $= \binom{19}{12}$
- Solutions with $x_1 > 4$: 7 stars, 7 bars $= \binom{14}{7}$
- Solutions with $x_2 > 2$: 9 stars, 7 bars $= \binom{16}{9}$
- Solutions with $x_1 > 4$ and $x_2 > 2$: 4 stars, 7 bars $= \binom{11}{4}$

- Solutions with $x_1 \leq 4$ and $x_2 \leq 2$: (inclusion-exclusion principle)

$$\binom{19}{12} - \binom{14}{7} - \binom{16}{9} + \binom{11}{4}$$

Stars = #Objects; Bars = #Types-1