EECS 203 Lecture 20

More Graphs

Admin stuffs

- Last homework due today
- Office hour changes starting Friday (also in Piazza)
  - Friday 6/17: 2-5 Mark in his office.
  - Sunday 6/19: 2-5 Jasmine in the UGLI.
  - Monday 6/20: 10-12 Mark in his office.
  - Monday 6/20: 5-7 Emily in the UGLI.
  - Tuesday 6/21: 10-12 Emily in the Beyster Learning Center.
  - Tuesday 6/21: 1-3 Mark in his office.
  - Wednesday 6/22: 10-12 Emily in the Beyster Learning Center.
  - Wednesday 6/22: 1:30-3 Mark in his office.
  - Thursday: 6/23: 10-12 Emily in the Beyster Learning Center.
  - Thursday 6/23: 1:30-3 Jasmine in the Beyster Learning Center.
- Discussion is still on for Thursday and Friday.
- Exam is Thursday 6/23 from 4-6
  - Room information posted shortly (will be in EECS)

Last time…

- Defined some terms
  - G=(V,E)
  - Directed vs undirected graph
  - What it means to be connected
  - Etc.
- Did Dijkstra’s Algorithm
  - Finds shortest path between a pair of nodes
  - Didn’t analyze runtime though…
- Started on induction proof about connectivity.

Today

- Analyze Dijkstra’s Algorithm
- Deal with induction proof we started on last time.
- Look at a way of finding all-pairs shortest path distances
  - Floyd-Warshall Algorithm
- More terminology
  - Path, cycle, Eulerian path, Eulerian cycle, other graph applications
- (Time allowing)More terminology
  - Trees, minimum-spanning trees (MST), planar graphs
  - Brief overview of ideas associated with these things.

Let’s examine Dijkstra’s Algorithm

- What is the worst-case run time on a graph with |V| nodes?

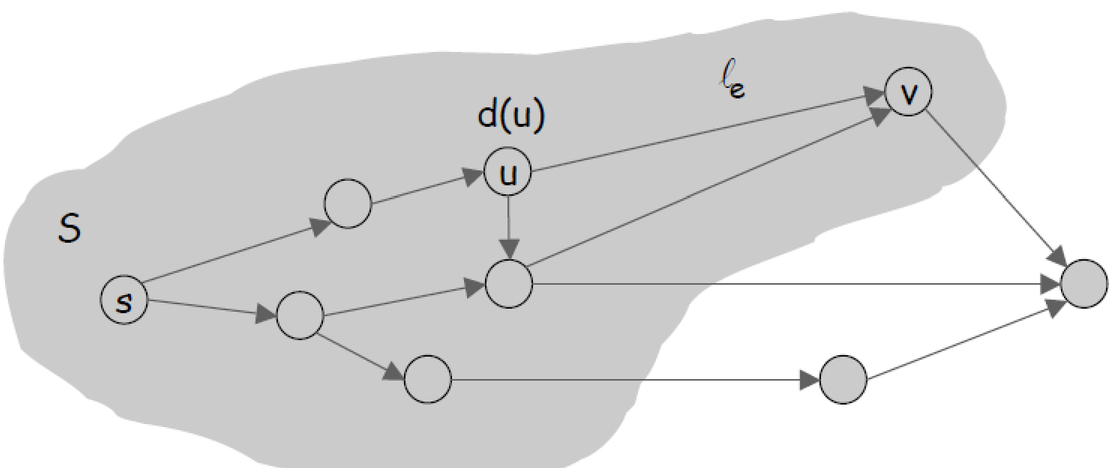
Dijkstra's algorithm.

- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e ,$$

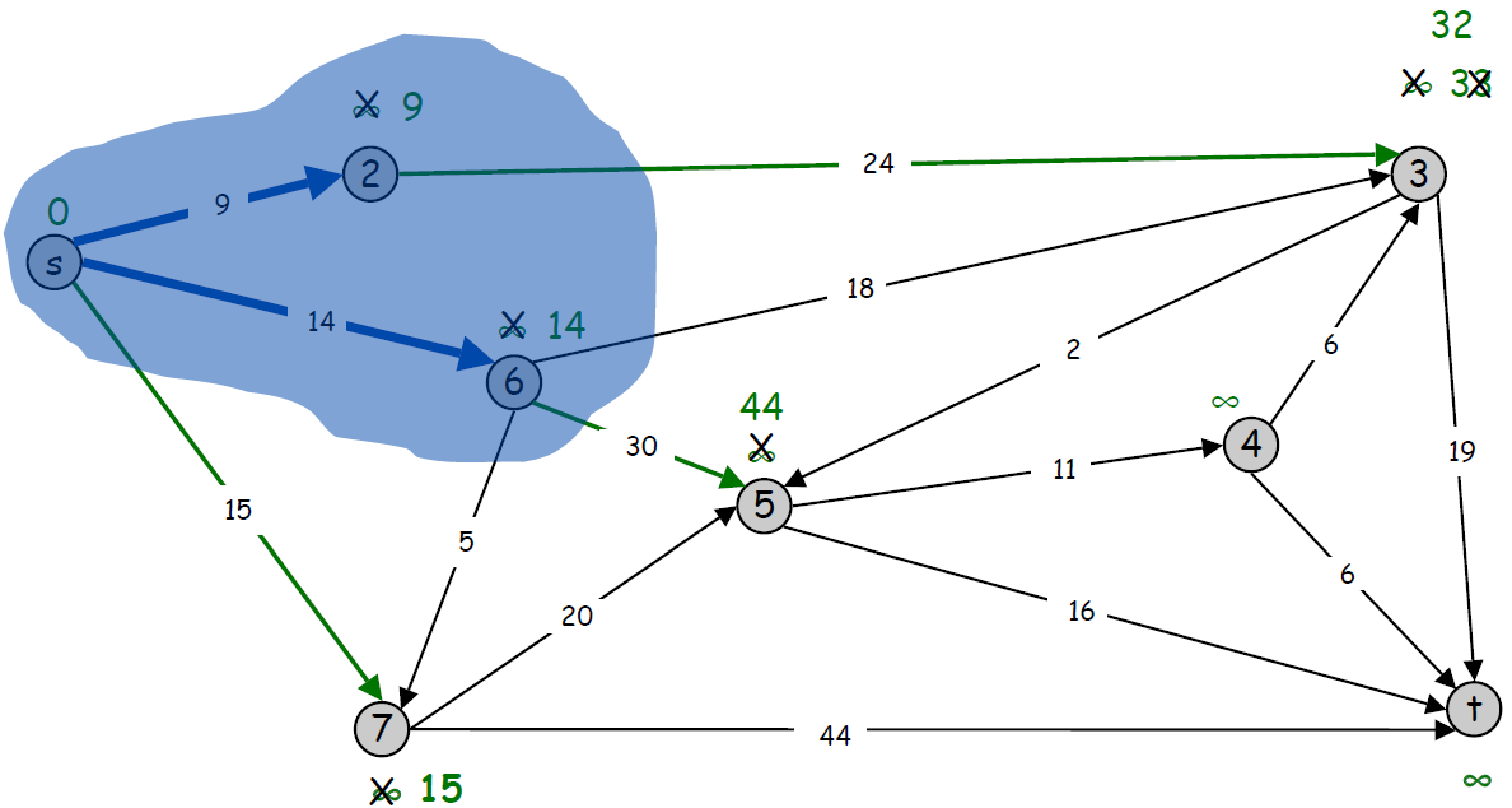
add v to S, and set d(v) = π(v).

shortest path to some u in explored part, followed by a single edge (u, v)



Dijkstra's Shortest Path Algorithm

S = { s, 2, 6 }
PQ = { 3, 4, 5, 7, t }



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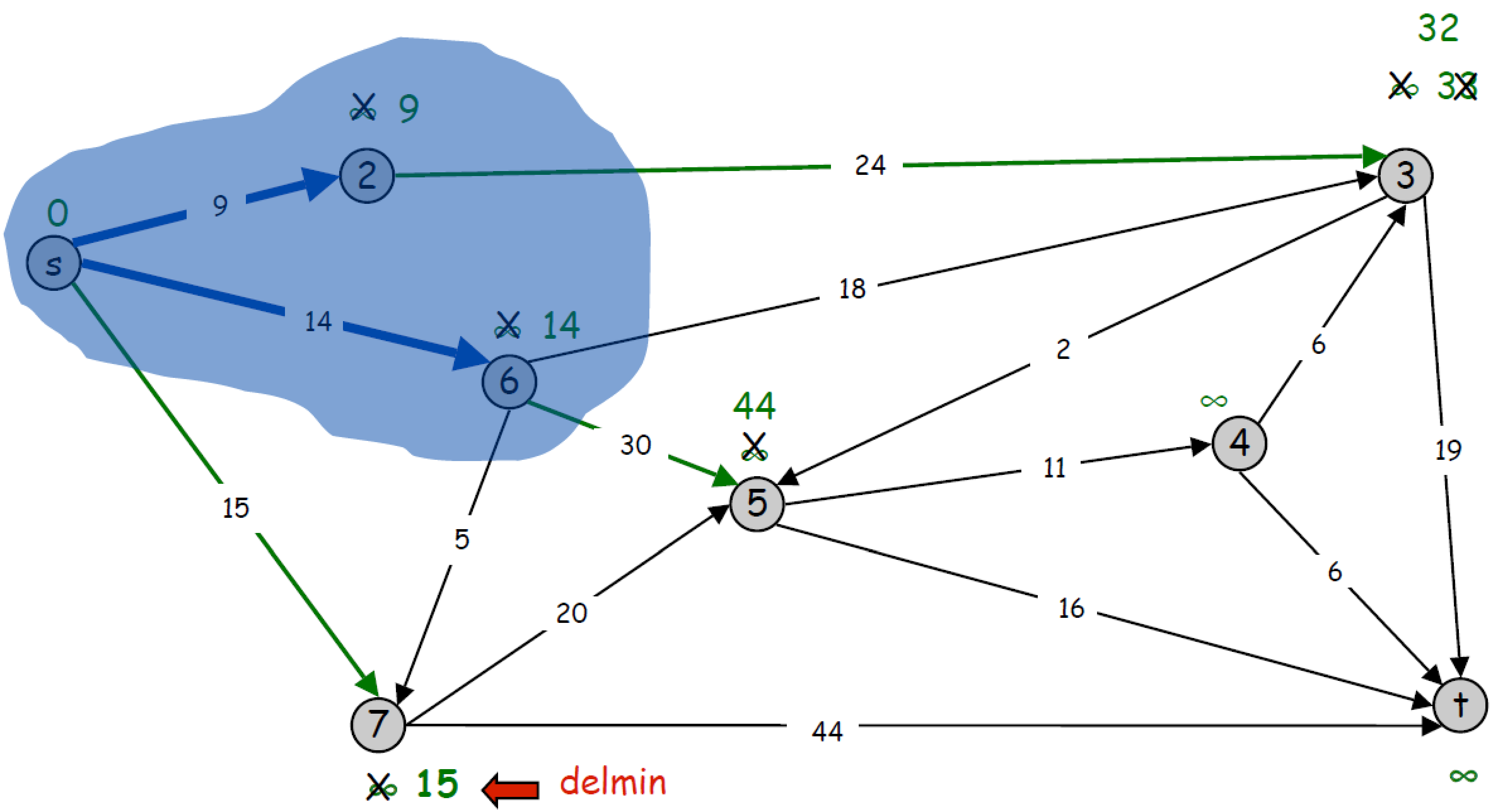

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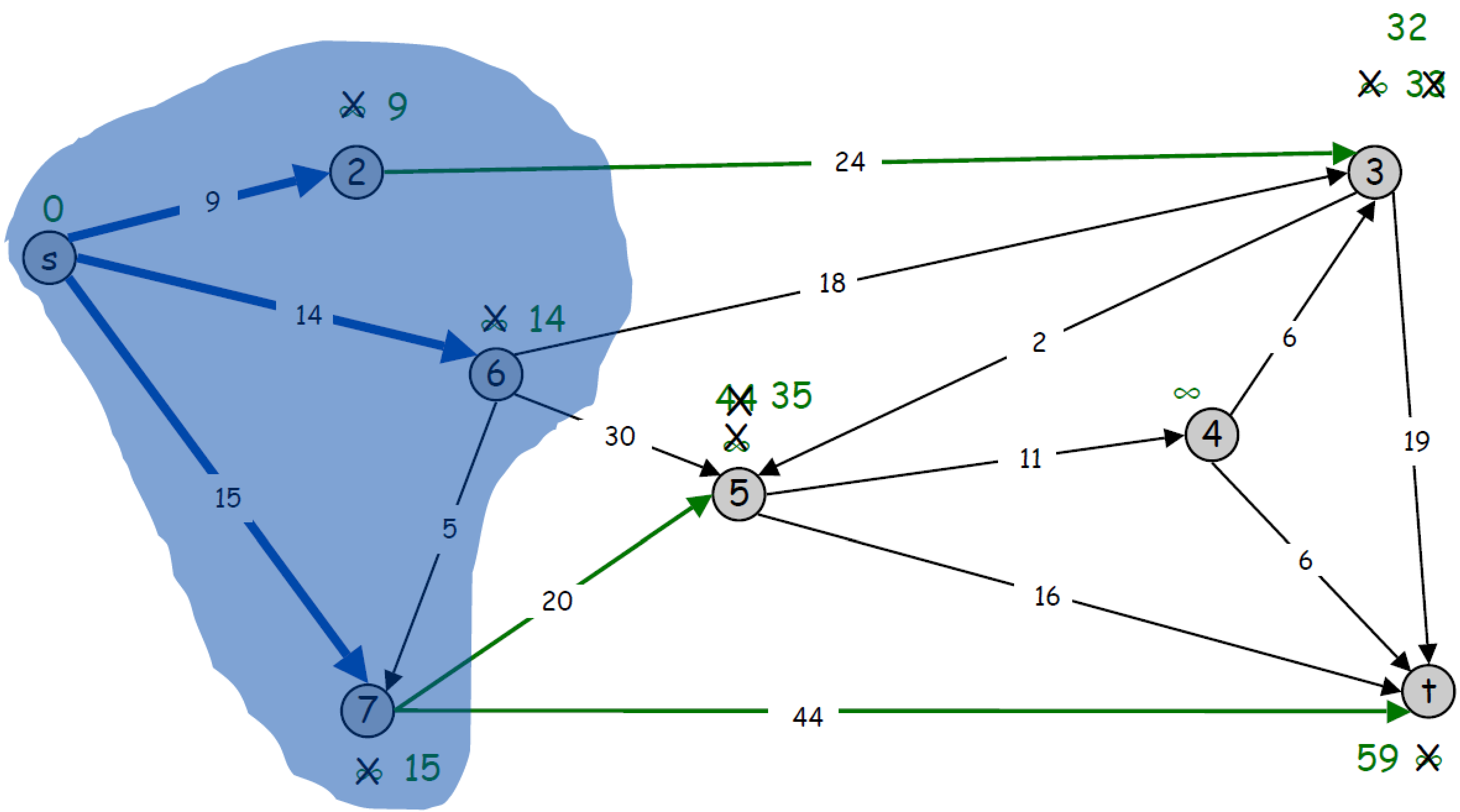

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Theorem 2 *Every connected graph G with $|V(G)| \geq 2$ has at least two vertices x_1, x_2 so that $G - x_i$ is connected for $i = 1, 2$.*

Proof: We proceed by induction on $|V(G)|$. As a base case, observe that if G is a connected graph with $|V(G)| = 2$, then both vertices of G satisfy the required conclusion. For the inductive step, let G be a connected graph with $|V(G)| \geq 2$ and assume that the theorem holds for every graph with $< |V(G)|$ vertices. If $G - x$ is connected for every vertex $x \in V(G)$, then we are done, so we may assume this is not so, and choose $x \in V(G)$ so that $G - x$ has components $H_1, H_2, \ldots, H_m$ where $m \geq 2$. For every $1 \leq i \leq m$ let H_i' be the graph obtained from H_i by adding back the vertex x and all edges with one end x and the other end in $V(H_i)$. So every H_i' is a connected graph with at least two vertices. Furthermore, $|V(H_i')| < |V(G)|$, so by induction, H_i' must have at least one vertex $x_i \neq x$ so that $H_i' - x_i$ is connected. It then follows that $G - x_i$ is connected. Since we have such an x_i for every component (and at least two components), this completes the proof. $\square$

From: http://www.sfu.ca/~mdevos/notes/graph/induction.pdf
There are a few nice observations about the proof there as well as a few nice induction examples.

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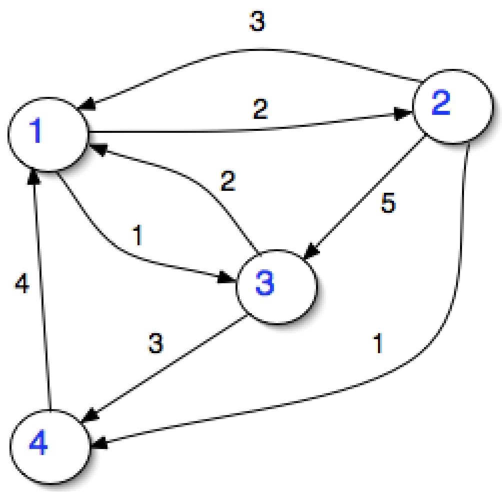
Floyd-Warshall Algorithm

```
let dist be a |V| × |V| array of minimum distances initialized to ∞ (infinity)
for each vertex v
   dist[v][v] ← 0
for each edge (u,v)
   dist[u][v] ← w(u,v)  // the weight of the edge (u,v)
for k from 1 to |V|
   for i from 1 to |V|
      for j from 1 to |V|
         if dist[i][j] > dist[i][k] + dist[k][j]
             dist[i][j] ← dist[i][k] + dist[k][j]
         end if
```

- What is it doing?

- Why does it work?

Floyd-Warshall Algorithm



Dist	1	2	3	4
1	0			
2	2			
3	1			
4	∞			

```
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Graphs

A **graph** G = (V,E) consists of

a non-empty set V of **vertices** (or nodes),

and a set E of **edges**.

Each edge is associated with two vertices (possibly equal), which are its endpoints.

A **path** from *u* to *v* of length *n* is:

a sequence of edges e_1, e_2, . . . e_n in E, such that

there is a sequence of vertices $u=v_0$, v_1, . . . $v_n=v$

such that each e_i has endpoints v_{i-1} and v_i.

A path is a **circuit** when $u=v$.

A graph is **connected** when there exists a path from every vertex to every *other* vertex.

Paths and Circuits

Given a graph G = (V,E):

Is there a path/circuit that **crosses each edge** in E **exactly once**?

If so, G is an **Eulerian** graph,

and you have an Eulerian path, or an Eulerian circuit.

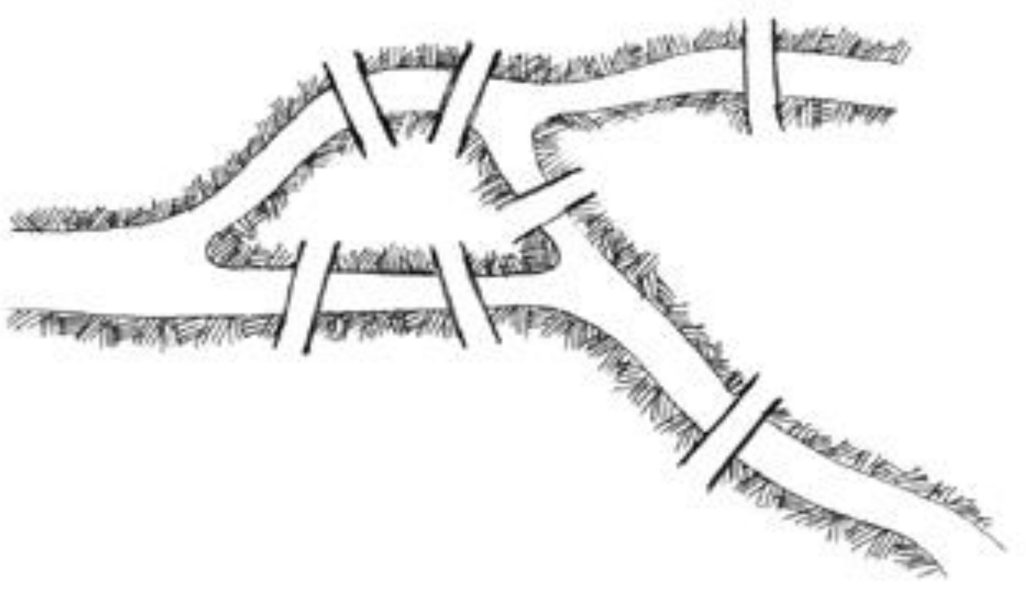
Is there a path/circuit that **visits each vertex** in V **exactly once**?

If so, G is a **Hamiltonian** graph,

and you have a Hamiltonian path or circuit.

Leonhard Euler lived in Königsberg

He liked to take walks. A famous local puzzle was whether you could take a walk that would cross each bridge exactly once.



Euler solved this problem (in 1736) and thus founded graph theory.

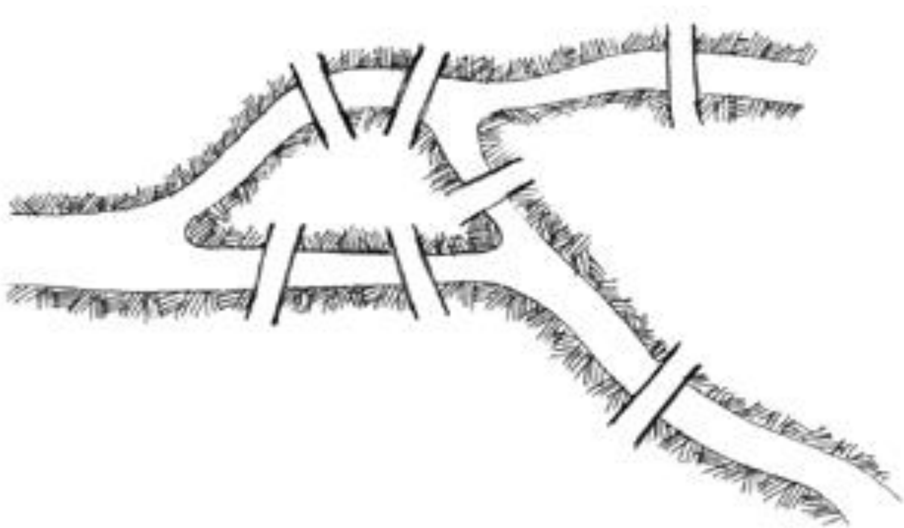
The Graph Abstraction

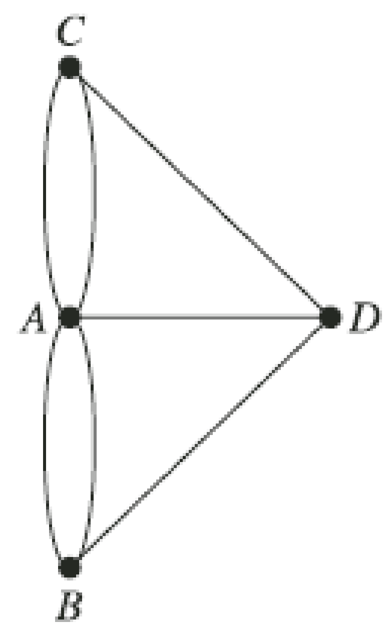
In the physical world:

- Each bridge connects exactly two land-masses.
- Each land-mass may have any number of bridges.

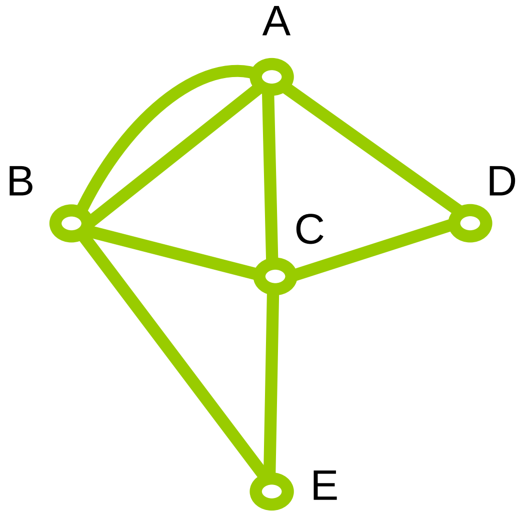
Suggests a graph abstraction:

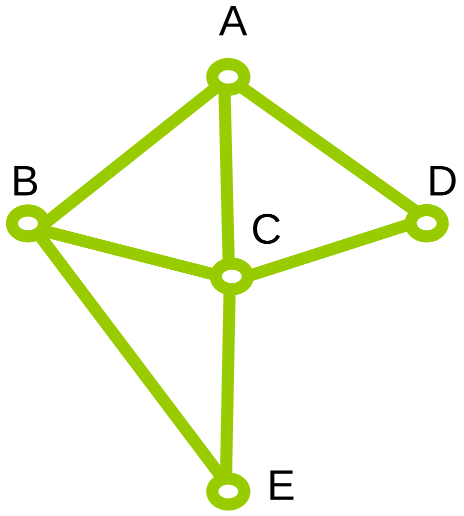
- Represent a bridge by an *edge* in the graph.
- Represent a land-mass by a *verte*

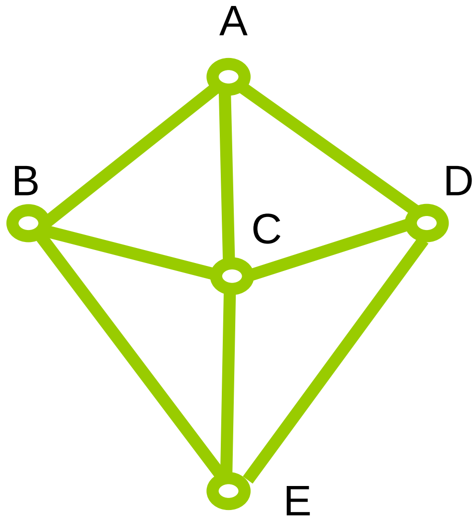




Is there an Euler circuit/path?







Circuit

Path

Not Traversable

Does G have an Euler Path/Circuit?

Theorem: A connected multigraph has an Euler path **iff** it has exactly **zero or two vertices** of odd degree.

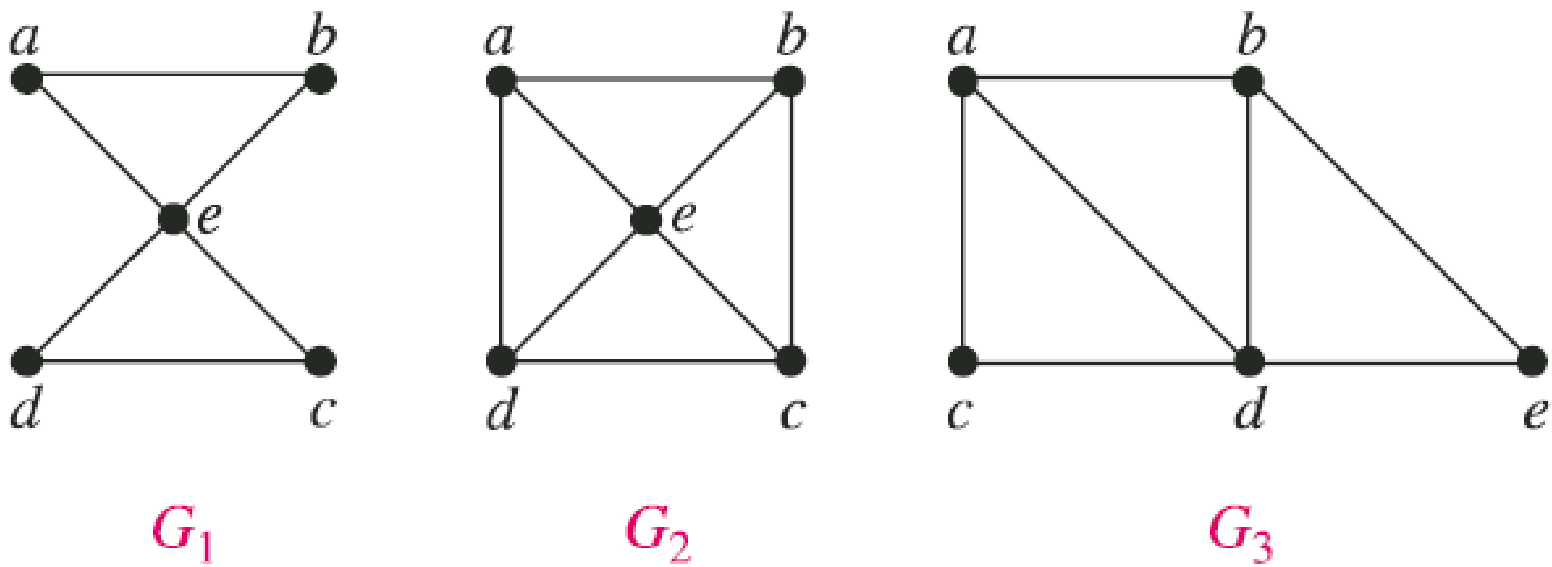
Why?

What if it has only one?

When does it have an Euler circuit?

Examples

Do these graphs have Euler paths/circuits?



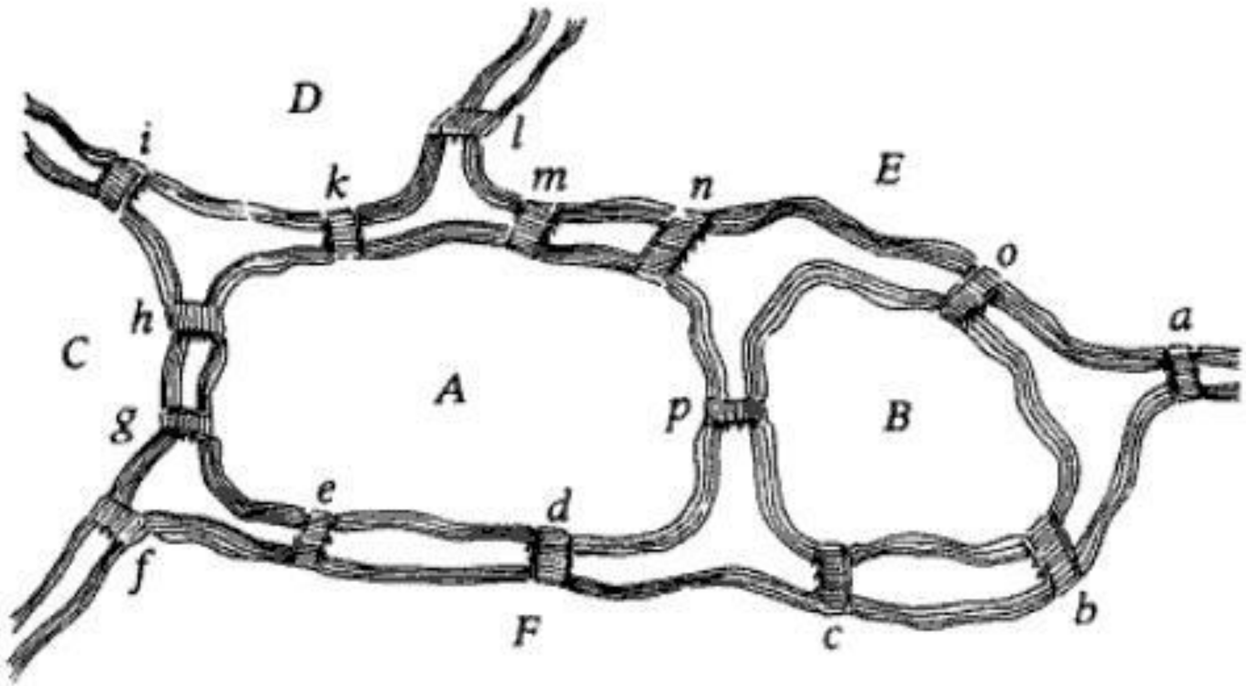
(A) Yes, it has a circuit.

(B) Yes, it has a path (but no circuit).

(C) No, it has neither path nor circuit.

Does this have an Euler path?

In his article on the Königsberg bridges, Euler considered another bridge problem, which is illustrated below.*



Two islands, A and B, are surrounded by water that leads to four rivers. Fifteen bridges cross the rivers and the water surrounding the islands. Is it possible to make a trip that crosses each bridge exactly once?

(A) Yes (B) No

Applications of Euler Paths

Planning routes through graphs that provide efficient coverage of the edges in the graph, without multiple traversals.

- Postal delivery routes
- Snowplowing routes
- Testing network connections
  - Utility transmission network
  - Communication network

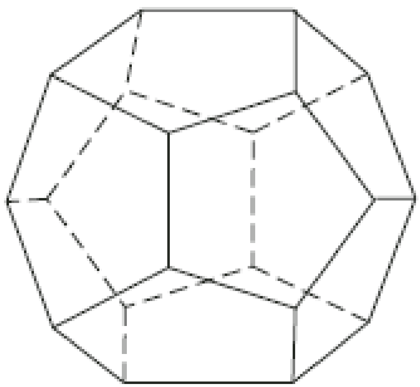
Eulerian Path vs. Hamiltonian Path

- Eulerian:
  - Traverse each edge exactly once

- Hamiltonian
  - Traverse each node exactly once

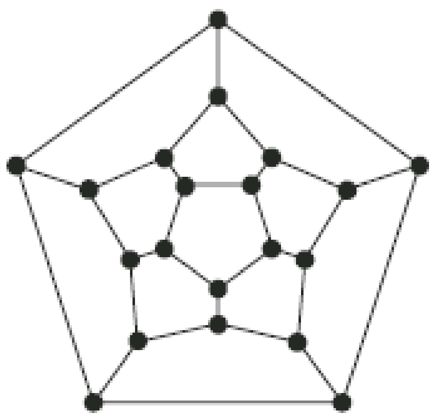
Hamiltonian Paths and Circuits

Given a graph, is there a path that passes through each **vertex** in the graph **exactly once? (**aka a **Hamiltonian path)**



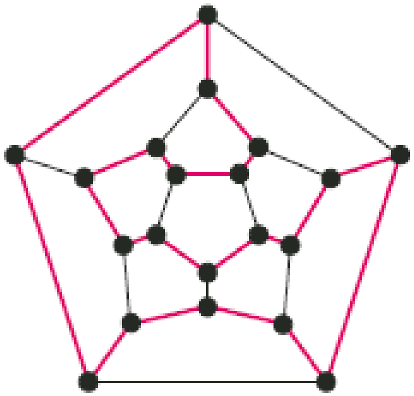
(a)

dodecahedron



(b)

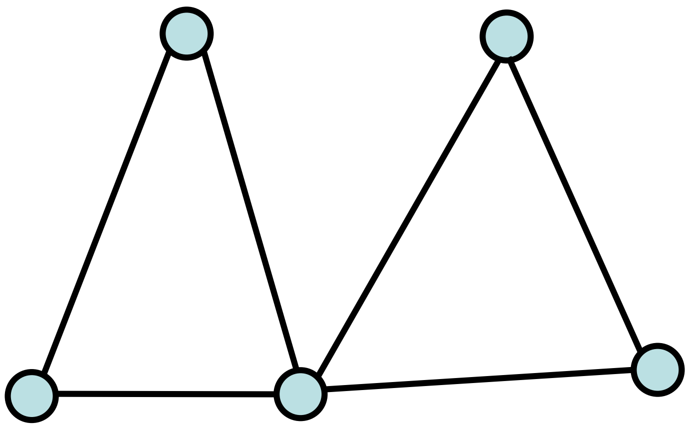
Isomorphic graph



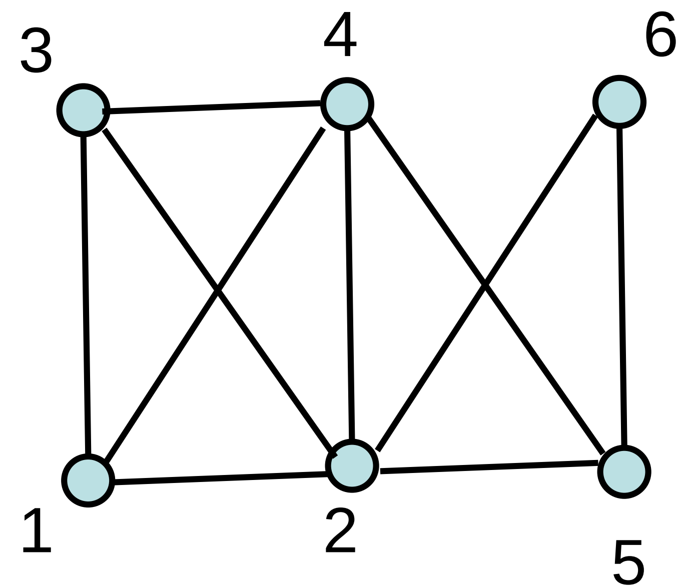
solution

Hamilton circuit

Do these graphs have Hamilton circuits?



G



H

(A) Yes, both.

(B) G does, H doesn’t.

(C) G doesn’t, H does.

(D) No, neither.

Computational Complexity

Deciding whether a graph is **Eulerian** is a simple examination of the degrees of the vertices.

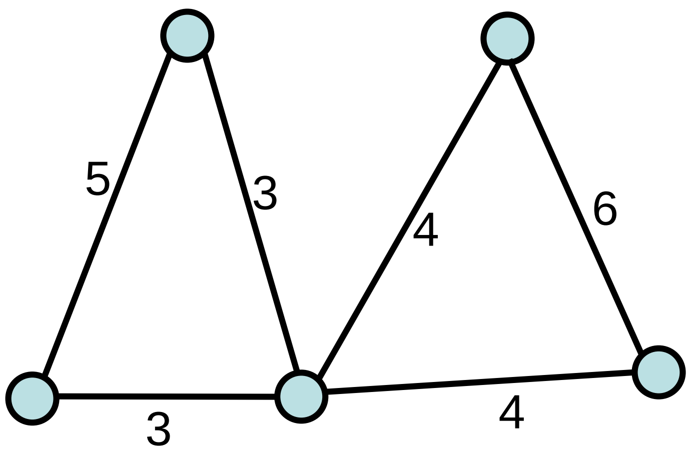
Simple linear-time algorithms exist for finding the path or circuit.

Deciding whether a graph is **Hamiltonian** is, in general, much more difficult (“NP complete”).

Finding a Hamiltonian path or circuit is equally hard.

Weighted graphs

A **weighted graph** has numbers (weights) assigned to each edge.



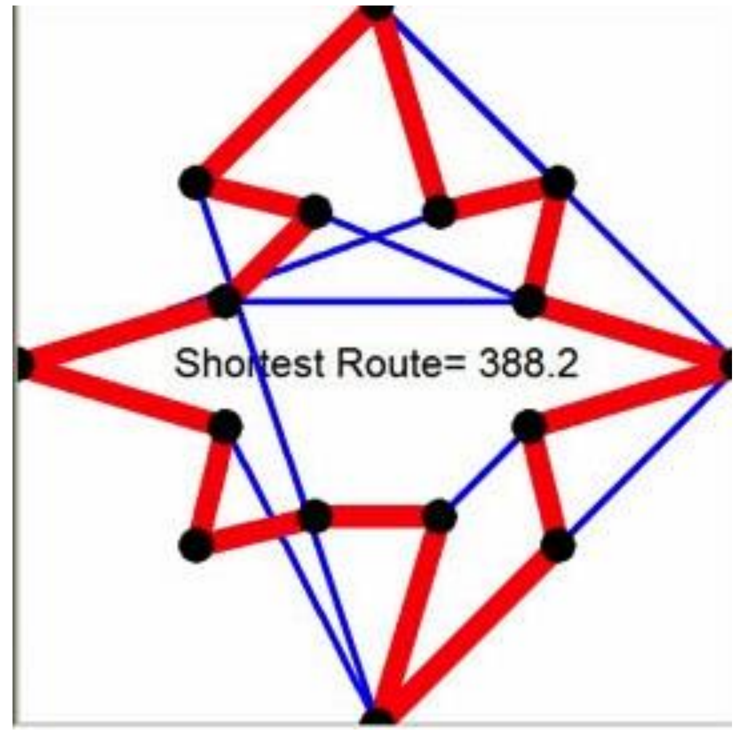
The **length** of a path is the sum of the weights in the path.

Traveling Salesperson's Problem

What is the shortest route in a given map

for a salesperson to visit every city exactly once

and return home at the end?



Mathematically:

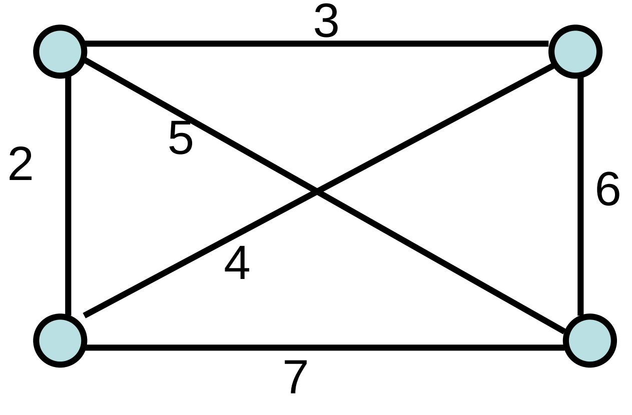
Given an graph with weighted edges, what is the Hamiltonian circuit of least weight?

Applies to both directed and undirected graphs.

The Traveling Salesman Problem

Problem: Given a graph G, find the shortest possible length for a Hamilton circuit.

What is the solution to TSP for the following graph?

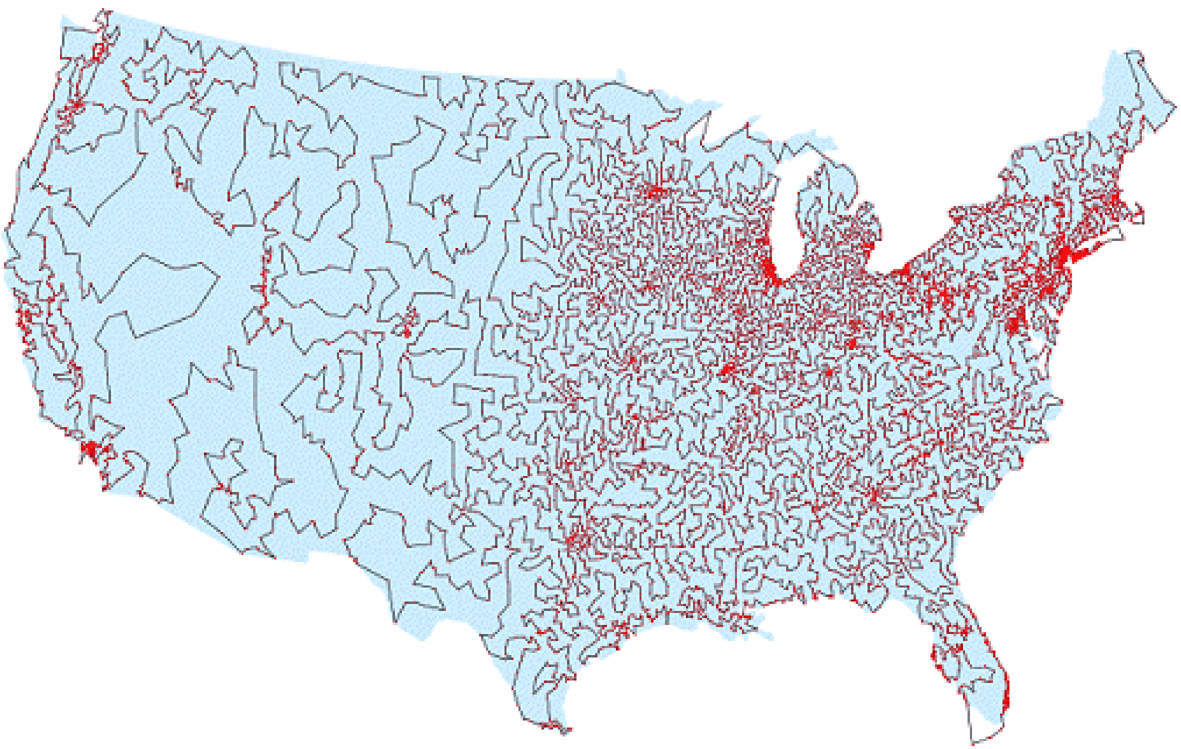


(A) 16 (B) 17 (C) 18 (D) 19 (E) 20

Traveling Salesperson's Problem

The TSP is NP Complete: intractable in general.

- Very large instances have been solved with heuristics.
- An instance with 13,509 cities, solved in 2003.



Applications

The Traveling Salesperson’s Problem and the Hamilton Path/Circuit Problem have many applications.

- Planning and logistics (of course!)
- Manufacture of microchips
- DNA sequencing