# Strong Induction

EECS 203: Discrete Mathematics

# Mathematical vs Strong Induction

- To prove that $P(n)$ is true for all positive $n$.
- **Mathematical** induction:

$$\begin{array}{ll} P(1) & \text{Basis step} \\ \forall k \in \mathbf{Z}^+ \; [P(k) \rightarrow P(k+1)] & \text{Inductive step} \\ \hline \forall n \in \mathbf{Z}^+ \;\; P(n) & \text{Conclusion} \end{array}$$

- **Strong** induction:

$$\begin{array}{ll} P(1) & \text{Basis step} \\ \forall k \in \mathbf{Z}^+ \; [P(1) \wedge \cdots \wedge P(k) \rightarrow P(k+1)] & \text{Inductive step} \\ \hline \forall n \in \mathbf{Z}^+ \;\; P(n) & \text{Conclusion} \end{array}$$

# Climbing the Ladder (Strongly)

- We want to show that $\forall n \geq 1\ P(n)$ is true.
  - Think of the positive integers as a ladder.
  - $1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$
- You can reach the *bottom* of the ladder:
  - $P(1)$
- Given *all lower* steps, you can reach the *next*.
  - $P(1) \rightarrow P(2), \quad P(1) \wedge P(2) \rightarrow P(3), \quad \ldots$
  - $\forall k \geq 1\ P(1) \wedge \ldots \wedge P(k) \rightarrow P(k+1)$
- Then, by strong induction:
  - $\forall n \geq 1\ P(n)$

# Is Strong Induction Really Stronger?

- **No**. Anything you can prove with **strong induction** can be proved with regular **mathematical induction**. And vice versa.
  - Both are equivalent to the **well-ordering property**.

- But strong induction **can** simplify a proof.
- How?
  - Sometimes P(k) is not enough to prove P(k+1).
  - But $P(1) \wedge \ldots \wedge P(k)$ is strong enough.

Show that if $n$ is an integer greater than 1, then $n$ can be written as the product of primes.

*Solution:* Let $P(n)$ be the proposition that $n$ can be written as the product of primes.

*BASIS STEP:* $P(2)$ is true, because 2 can be written as the product of one prime, itself. (Note that $P(2)$ is the first case we need to establish.)

*INDUCTIVE STEP:* The inductive hypothesis is the assumption that $P(j)$ is true for all integers $j$ with $2 \leq j \leq k$, that is, the assumption that $j$ can be written as the product of primes whenever $j$ is a positive integer at least 2 and not exceeding $k$. To complete the inductive step, it must be shown that $P(k+1)$ is true under this assumption, that is, that $k+1$ is the product of primes.

There are two cases to consider, namely, when $k+1$ is prime and when $k+1$ is composite. If $k+1$ is prime, we immediately see that $P(k+1)$ is true. Otherwise, $k+1$ is composite and can be written as the product of two positive integers $a$ and $b$ with $2 \leq a \leq b < k+1$. Because both $a$ and $b$ are integers at least 2 and not exceeding $k$, we can use the inductive hypothesis to write both $a$ and $b$ as the product of primes. Thus, if $k+1$ is composite, it can be written as the product of primes, namely, those primes in the factorization of $a$ and those in the factorization of $b$. ◀

# Coin problem

- What is the largest cent-value that **cannot** be formed using only 3-cent and 5-cent stamps?

  - (A) 2
  - (B) 4
  - (C) 7 <= Correct answer
  - (D) 8
  - (E) 11

# Proof for our Coin problem

- Let $P$(k) = "k cents can be formed using 3-cent and 5-cent stamps."

- **Claim**: $\forall n \geq 8\ P(n)$.
- **Proof by strong induction:**
- *Base cases*:
  - $P(8)$: $8 = 3 + 5$
  - $P(9)$: $9 = 3 + 3 + 3$
  - $P(10)$: $10 = 5 + 5$.

# Proof for our Coin problem

- *Inductive step:*
  - Let $k$ be an integer $\geq 11$.

*Inductive hypothesis:* $P(j)$ is true whenever $8 \leq j < k$.

  - $P(k\text{-}3)$ is true.
  - Therefore, $P(k)$ is true. (Add a 3-cent stamp.)
  - This completes the inductive step.