Nyquist Sampling Theorem

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EECS206 F02
Lect 20

• Special case of sinusoidal signals
• Aliasing (and folding) ambiguities
• Shannon/Nyquist sampling theorem
• Ideal reconstruction of a cts time signal

Q. Is there a minimum sampling rate necessary for good reconstruction?

• For simplicity specialize to a sinusoid
  \[ x(t) = A \cos(2\pi f_0 t + \phi) \]
  where \( f_0 < f_{max} = 1 \text{ KHz} \)

NOTE:
- frequency \( f_0 \) is unknown
- frequency \( f_{max} \) is known

• After sampling at frequency \( f_s = 1/T_s \)
  \[ x[n] = x(nT_s) = A \cos(2\pi f_0 T_s n + \phi) \]

Sampling of Sinusoid

\[ x(t) = A \cos(2\pi f_0 t + \phi) \]

\[ \phi = \pi/5 \]
\[ f_0 = 1000 \text{ Hz} \]
\[ f_s = 6000 \text{ Hz} \]
6 samples/cycle

Slower sampling of sinusoid

\[ x(t) = A \cos(2\pi f_0 t + \phi) \]

\[ \phi = \pi/5 \]
\[ f_0 = 1000 \text{ Hz} \]
\[ f_s = 3000 \text{ Hz} \]
3 samples/cycle

Sampling of Sinusoid: Notes

• There are no other sinusoidal signals with fundamental frequencies less than 1KHz that have exactly the same samples as those in previous two examples.

• Thus, for these sampling rates any fundamental frequency \( f_0 \leq 1 \text{ KHz} \) can be uniquely identified
Under sampled sinusoid

$$ x(t) = A \cos(2\pi f_0 t + \phi) $$

- $\phi = \pi/5$
- $f_0 = 1000$ Hz
- $f_s = 2000$ Hz
- 2 samples/cycle

Poorly sampled sinusoid

- With only 2 samples/cycle we may confuse $x(t)$ with a sinusoid at lower frequency (0Hz).
- Sampled signal is “aliased” to the all zero signal.

- $\phi = \pi/2$
- $f_0 = 1000$ Hz
- $f_s = 2000$ Hz
- 2 samples/cycle

Poorly sampled sinusoid (ctd)

- Further reduction in sampling rate produces even more ambiguity!

- $\phi = \pi/5$
- $f_0 = 1000$ Hz
- $f_s = 500$ Hz
- $1/2$ sample/cycle

What is really going on?

- CT sinusoid at frequency $f_0$, sampled at rate $f_s = 1/T_s$ gives DT sinusoid:

  $$ x[n] = x(nT_s) = A \cos(2\pi f_0 T_s n + \phi) $$

- Observe: this signal is indistinguishable from a DT sinusoid at any of the frequencies

  $f_0 T_s + m, m$ integer

  $-f_0 T_s + m, m$ integer

Specialize to periodic DT sinusoid

- First we assume relation $f_0/f_s = 1/N$

  $$ x[n] = x(nT_s) = A \cos(2\pi f_0 T_s n + \phi) $$

- DT sinusoid with period $N$
- Compare to generic periodic sinusoid

  $$ x[n] = A \cos(2\pi f_0 n + \phi) $$

- Recognize DT fundamental frequency

  $$ f_0 = f_0/f_s = 1/N $$
Recall CT and DT spectra

- CT spectrum (FS): \[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_o t} \]
  \[ a_k = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x(t) e^{-j2\pi f_o t} dt \]

- DT spectrum (DFT): \[ x[n] = \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N} \]
  \[ X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \]

Spectrum of CT Sinusoid

- Real sinusoidal signal
  \[ A \cos(\omega_o t + \phi) = A \frac{e^{j(\omega_o t + \phi)} + e^{-j(\omega_o t + \phi)}}{2} = \frac{Ae^{j\phi}}{2} e^{j\omega_o t} + \frac{Ae^{-j\phi}}{2} e^{-j\omega_o t} \]

Sampling Spectrum

- CT and DT spectra of sinusoid are both line spectra but with lines located at
  \[ f = f_o = 1/T_o \text{ (CT)} \]
  \[ \tilde{f} = \tilde{f}_o = f_o/f_s \text{ (DT)} \]

- Define the sampling spectrum of \( x(nT_s) \) as its DT spectrum with \( \tilde{f} \) replaced by
  \[ f = \tilde{f} f_s \]

If \( x[n] \) is a sampled sinusoid with sampling frequency \( f_s = 1/T_s \) and \( \tilde{f}_o = f_o/f_s < 1/2 \)

Then its sampling spectrum is: \( f_s > 2f_o \)

Compare sampling spectrum of CT sinusoid to spectrum of DT sinusoid

Spectrum of cts time sinusoid
\[ x(t) = A \cos(2\pi f_o t + \phi) \]

Sampling spectrum: \( f_s > 2f_o \)
\[ x(nT_s) = A \cos(2\pi f_o T_s n + \phi) \]

Conclude

- For $f_o < 1/2$ (equiv $f_s > 2f_o$)
  - The sampling spectrum of $x[n]$ is identical to the spectrum of the input $x(t)$ to digitizer.
  - Therefore the fundamental frequency $f_o$ can be recovered (along with sinusoidal amplitude and phase) from the DT spectrum (DFT) of the sampled signal via the formula
    \[ f_o = \frac{f_o}{T_s} = f_o f_s \]

Next consider DFT spectrum for case: $1/2 \leq f_o < 1$ ($f_o \leq f_s < 2f_o$)

Conclude

- For $1/2 \leq f_o < 1$ (equiv $f_o \leq f_s < 2f_o$)
  - The cts sinusoidal frequency $f_o$ is no longer recoverable from DFT spectrum since cannot tell difference between
    \[ f_o = f_o f_s \text{ or } f_o = (1 - f_o) f_s \]
  - The negative frequency component has been folded into the range $f \in [0, 1/2]
  - The folded amplitude is complex conjugated
  - This phenomenon is called (case II) ALIASING

Finally, consider DT spectrum for: $f_o > 1$ ($f_s < f_o$)

Conclude

- For $f_o > 1$ (equiv $f_s < f_o$)
  - The cts sinusoidal frequency $f_o$ is no longer recoverable since cannot tell the difference between
    \[ f_o = f_o f_s \text{ or } f_o = (f_o - 1) f_s \]
  - A higher positive frequency has been aliased into the frequency range $f \in [0, 1/2]
  - Complex amplitude of the positive frequency component is preserved
  - This phenomenon is called (case I) ALIASING

Summarizing

- When $f_s > 2f_o$ can recover $f_o, A, \phi$ from DFT
  \[ f_o = \frac{f_o}{T_s} = \frac{f_o}{f_s} \]
- Otherwise have aliasing and recovery is impossible
  - if $f_o \leq f_s < 2f_c$ \[ f_o = f_o f_s \text{ or } f_o = (1 - f_o) f_s \text{ (Case II)} \]
  - if $f_s < f_c$ \[ f_o = f_o f_s \text{ or } f_o = (f_o - 1) f_s \text{ (Case I)} \]
Sampling arbitrary periodic signals

- Let \( f_{\text{max}} = p f_0 \) Hz be the highest frequency component (bandwidth) in FS of a periodic signal \( x(t) \) having fundamental frequency \( f_0 = 1/T_c \)

\[
x(t) = \sum_{k=-p}^{p} a_k e^{j2\pi k f_0 t}
\]

- Such a signal is said to be bandlimited to bandwidth \( f_{\text{max}} \) Hz

- Q. What is minimum sampling freq \( f_s \)?

**Periodic Bandlimited Signal**

Spectrum of a periodic signal \( x(t) \) with maximum frequency = \( B \) Hz

Nyquist Sampling Rate

- Can uniquely recover a periodic signal bandlimited to bandwidth \( B \) when \( f_s \) is chosen such that

\[
f_s > 2f_{\text{max}} = 2B
\]

- The rate \( 2B \) is called the Nyquist sampling rate and it guarantees that no aliasing will occur

No aliasing occurs when exceed Nyquist sampling rate

Aliasing occurs when sample below Nyquist sampling rate

Ideal Reconstruction

- Q. Can we implement a reconstruction algorithm to recover periodic input signal from its Nyquist samples?

- A. Yes. By interpolating between samples with (non-truncated) sinc interpolation function

\[
p(t) = \text{sinc}(\pi t/T_s) = \frac{\sin(\pi tT_s)}{\pi tT_s}
\]
Q. Why does sinc interpolation work?

A. The FS of the cardinal series is identical to the FS of x(t)

See EECS306

Shannon Sampling Theorem

- If periodic x(t) is bandlimited to bandwidth $f_{\text{max}}$ and samples $x[n]$ are obtained from x(t) by sampling at greater than Nyquist rate $f_s > 2f_{\text{max}}$, then can exactly reconstruct x(t) from samples using sinc interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \text{sinc}(\pi(t - nT_s)/T_s)$$

- This is also called the cardinal series for x(t)

How to avoid aliasing if signal is not bandlimited?

- Must first apply a “Anti-aliasing filter” to eliminate all frequency components above on half the sampling frequency.
- This may destroy some small features of signal but is usually better than aliasing distortion.
- Anti-aliasing filter can be implemented in the frequency domain as a truncation operation which only lets low frequency components pass through.
Practical Sampling System

In this course we will not further elaborate on CT filters like the anti-alias filter. We will, however, soon treat DT filters in great detail!