Nyquist Sampling Theorem

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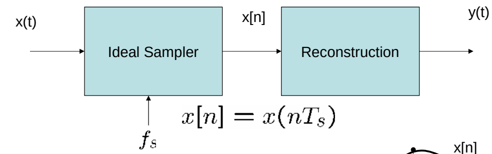
EECS206 F02
Lect 20

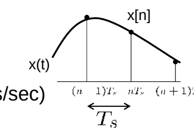
- Special case of sinusoidal signals
- Aliasing (and folding) ambiguities
- Shannon/Nyquist sampling theorem
- Ideal reconstruction of a cts time signal

Sampling and Reconstruction

- Consider time sampling/reconstruction without quantization:



$$x[n] = x(nT_s)$$



- T_s sampling period (secs/sample)
- f_s sampling rate or frequency (samples/sec)

$$f_s = 1/T_s$$

Q. Is there a minimum sampling rate necessary for good reconstruction?

- For simplicity specialize to a sinusoid

$$x(t) = A\cos(2\pi f_o t + \phi)$$

where $f_o \leq f_{max} = 1$ KHz

NOTE:

- frequency f_o is unknown
- frequency f_{max} is known

- After sampling at frequency $f_s = 1/T_s$

$$x[n] = x(nT_s) = A\cos(2\pi f_o T_s n + \phi)$$

Sampling of sinusoid

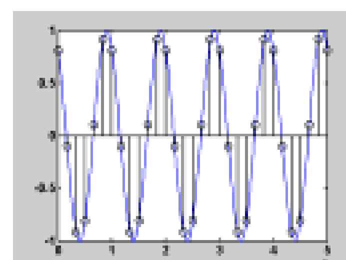
$$x(t) = A\cos(2\pi f_o t + \phi)$$

$\phi = \pi/5$

$f_o = 1000$ Hz

$f_s = 6000$ Hz

6 samples/cycle



Slower sampling of sinusoid

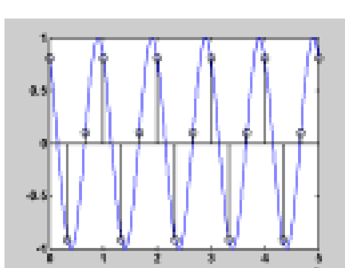
$$x(t) = A\cos(2\pi f_o t + \phi)$$

$\phi = \pi/5$

$f_o = 1000$ Hz

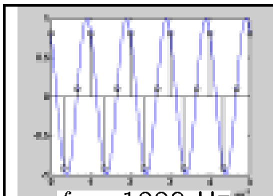
$f_s = 3000$ Hz

3 samples/cycle



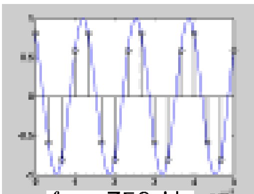
Sampling of Sinusoid: Notes

- There are no other sinusoidal signals with fundamental frequencies less than 1KHz that have exactly the same samples as those in previous two examples.
- Thus, for these sampling rates any fundamental frequency $f_o \leq 1KHz$ can be uniquely identified

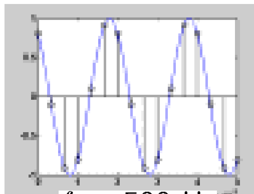


$f_o = 1000$ Hz

$f_s = 3000$ Hz



$f_o = 750$ Hz



$f_o = 500$ Hz

Under sampled sinusoid

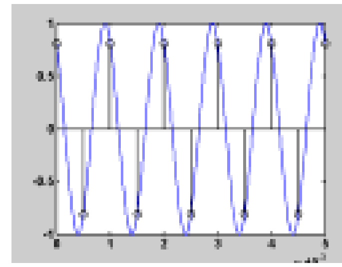
$$x(t) = A\cos(2\pi f_o t + \phi)$$

$\phi = \pi/5$

$f_o = 1000$ Hz

$f_s = 2000$ Hz

2 samples/cycle



Poorly sampled sinusoid

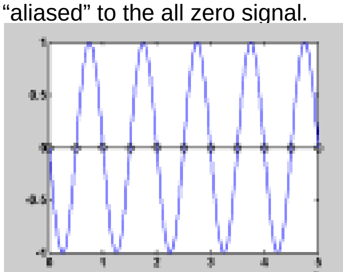
- With only 2 samples/cycle we may confuse x(t) with a sinusoid at lower frequency (0Hz).
- Sampled signal is “aliased” to the all zero signal.

$\phi = \pi/2$

$f_o = 1000$ Hz

$f_s = 2000$ Hz

2 samples/cycle



Poorly sampled sinusoid (ctd)

- Further reduction in sampling rate produces even more ambiguity!

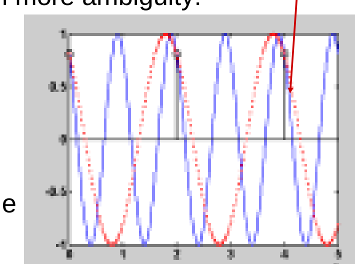
$\phi = \pi/5$

$f_o = 1000$ Hz

$f_s = 500$ Hz

- ½ sample/cycle

Sinusoid at ½ the frequency



What is really going on?

- CT sinusoid at frequency f_o sampled at rate $f_s = 1/T_s$ gives DT sinusoid:

$$x[n] = x(nT_s) \;=\; A\cos(2\pi f_o T_s n + \phi)$$

- Observe: this signal is indistinguishable from a DT sinusoid at any of the frequencies

$$f_o T_s + m, \; m \text{ integer}$$

$$-f_o T_s + m, \; m \text{ integer}$$

Specialize to periodic DT sinusoid

- First we assume relation $f_o/f_s = 1/N$

$$\begin{aligned} x[n] = x(nT_s) &= A\cos(2\pi f_o T_s n + \phi) \\ &= A\cos(2\pi (f_o/f_s) n + \phi) \end{aligned}$$

- DT sinusoid with period N
- Compare to generic periodic sinusoid

$$x[n] = A\cos(2\pi \hat{f}_o n + \phi)$$

- Recognize DT fundamental frequency

$$\hat{f}_o = f_o/f_s = 1/N$$

Recall CT and DT spectra

- CT spectrum (FS): $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk2\pi f_o t}$

$f_o = 1/T_o$

$$\alpha_k = \frac{1}{T_o}\int_0^{T_o} x(t) e^{-jk2\pi f_o t} dt$$

- DT spectrum (DFT): $x[n] = \sum_{k=0}^{N-1} X(k) e^{j2\pi k \hat{f}_o n}$

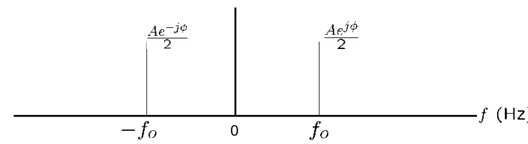
$\hat{f}_o = f_o/f_s = 1/N$

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j2\pi k \hat{f}_o n}$$

Spectrum of CT Sinusoid

- Real sinusoidal signal

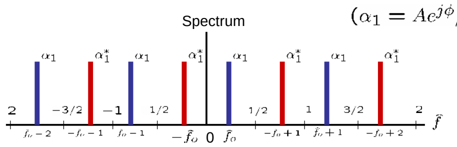
$$A\cos(\omega_o t + \phi) = A\frac{e^{j(\omega_o t+\phi)} + e^{-j(\omega_o t+\phi)}}{2} = \frac{Ae^{j\phi}}{2}e^{j\omega_o t} + \frac{Ae^{-j\phi}}{2}e^{-j\omega_o t}$$

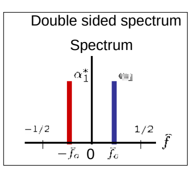


Note: Conjugate symmetry

Spectrum of periodic DT sinusoid with fundamental: $\hat{f}_o < 1/2$

$(\alpha_1 = Ae^{j\phi}/2)$





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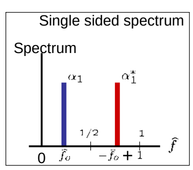


Sampling Spectrum

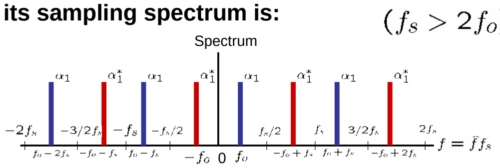
- CT and DT spectra of sinusoid are both line spectra but with lines located at

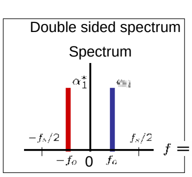
$$f = f_o = 1/T_o \text{ (CT)}$$
$$\hat{f} = \hat{f}_o = f_o/f_s \text{ (DT)}$$

- Define the **sampling spectrum** of $x(nT_s)$ as its DT spectrum with $\hat{f}$ replaced by

$$f = \hat{f} f_s$$

If x[n] is a sampled sinusoid with sampling frequency $f_s = 1/T_s$ **and** $\hat{f}_o = f_o/f_s < 1/2$
Then its sampling spectrum is: $(f_s > 2f_o)$





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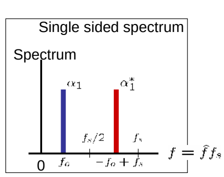
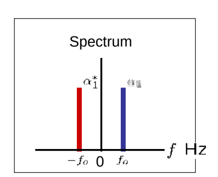


Compare sampling spectrum of CT sinusoid to spectrum of DT sinusoid

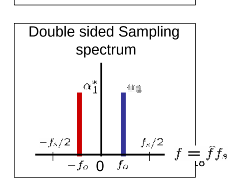
Spectrum of cts time sinusoid

$$x(t) = A\cos(2\pi f_o t + \phi)$$



Sampling spectrum $(f_s > 2f_o)$

$$x(nT_s) = A\cos(2\pi f_o T_s n + \phi)$$

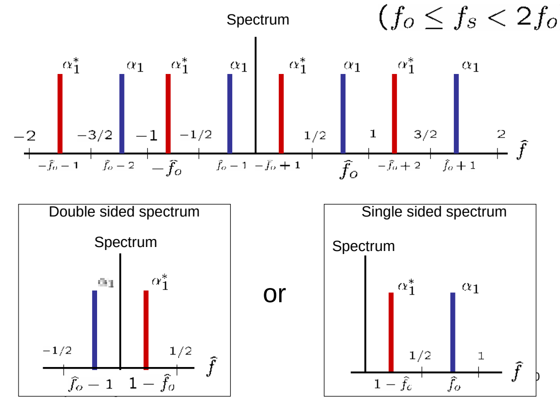


Conclude

- For $\hat{f}_o < 1/2$ (equiv $f_s > 2f_o$)
  - The sampling spectrum of x[n] is **identical** to the spectrum of the input x(t) to digitize.
  - Therefore the fundamental frequency f_o can be recovered (along with sinusoidal amplitude and phase) from the DT spectrum (DFT) of the sampled signal via the formula

$$f_o = \hat{f}_o / T_s = \hat{f}_o f_s$$

Next consider DFT spectrum for case: $1/2 \le \hat{f}_o < 1$ $(f_o \le f_s < 2f_o)$



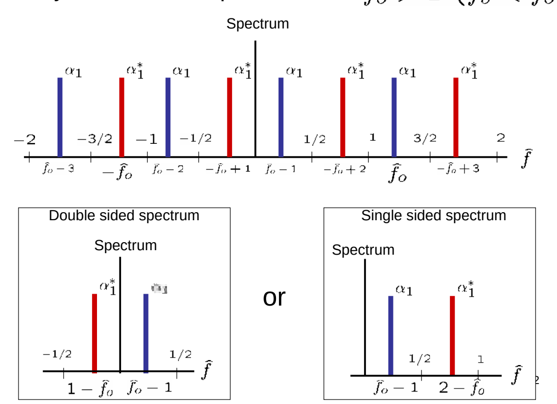
Conclude

- For $1/2 \le \hat{f}_o < 1$ (equiv $f_o \le f_s < 2f_o$)
  - The cts sinusoidal frequency f_o is no longer recoverable from DFT spectrum since cannot tell deifference between

$$f_o = \hat{f}_o f_s \text{ or } f_o = (1 - \hat{f}_o) f_s$$

  - The negative frequency component has been folded into the range $\hat{f} \in [0, 1/2)$
  - The folded amplitude is complex conjugated
  - This phenomenon is called (case II) ALIASING

Finally, consider DT spectrum for: $\hat{f}_o > 1$ $(f_s < f_o)$



Conclude

- For $\hat{f}_o > 1$ (equiv $f_s < f_o$)
  - The cts sinusoidal frequency f_o is no longer recoverable since cannot tell the difference between

$$f_o = \hat{f}_o f_s \text{ or } f_o = (\hat{f}_o - 1) f_s$$

  - A higher positive frequency has been aliased into the frequency range $\hat{f} \in [0, 1/2)$
  - Complex amplitude of the positive frequency component is preserved
  - This phenomenon is called (case I) ALIASING

Summarizing

- When $f_s > 2f_o$ can recover f_o, A, ϕ from DFT

$$f_o = \hat{f}_o / T_s = \hat{f}_o f_s$$

- Otherwise have aliasing and recovery is impossible
  - if $f_o \le f_s < 2f_o$

$$f_o = \hat{f}_o f_s \text{ or } f_o = (1 - \hat{f}_o) f_s \text{ (Case II)}$$

  - if $f_s < f_o$

$$f_o = \hat{f}_o f_s \text{ or } f_o = (\hat{f}_o - 1) f_s \text{ (Case I)}$$

Sampling arbitrary periodic signals

- Let $f_{max} = p f_o$ Hz be the highest frequency component (bandwidth) in FS of a periodic signal x(t) having fundamental frequency $f_o = 1/T_o$

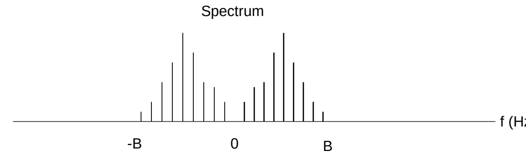
$$x(t) = \sum_{k=-p}^{p} \alpha_k e^{j2\pi k f_o t}$$

- Such a signal is said to be bandlimited to bandwidth f_{max} Hz

Q. What is minimum sampling freq f_s?

Periodic Bandlimited Signal

Spectrum of a periodic signal x(t) with maximum frequency = B Hz



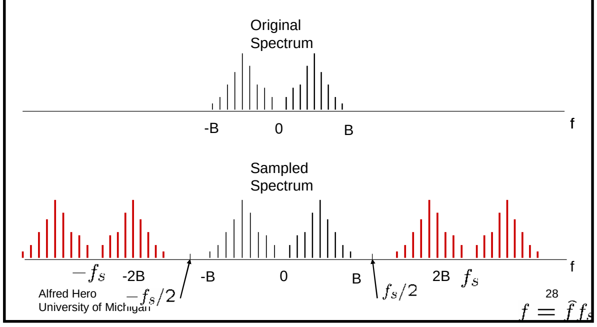
Nyquist Sampling Rate

- Can uniquely recover a periodic signal bandlimited to bandwidth B when f_s is chosen such that

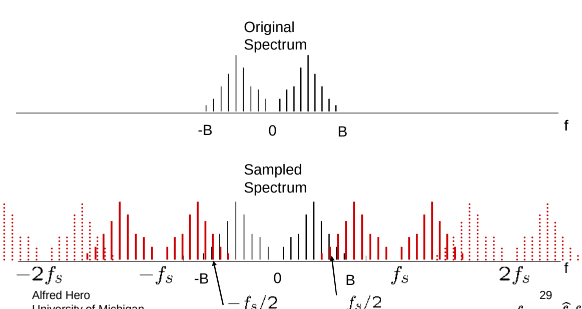
$$f_s > 2f_{max} = 2B$$

- The rate 2B is called the Nyquist sampling rate and it guarantees that no aliasing will occur

No aliasing occurs when exceed Nyquist sampling rate



Aliasing occurs when sample below Nyquist sampling rate

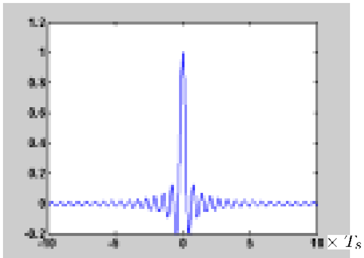


Ideal Reconstruction

- Q. Can we implement a reconstruction algorithm to recover periodic input signal from its Nyquist samples?
- A. Yes. By interpolating between samples with (non-truncated) sinc interpolation function

$$p(t) = \text{sinc}(\pi t/Ts) = \frac{\sin(\pi t T_s)}{\pi t/Ts}$$

Ideal Sinc Interpolation Function



Shannon Sampling Theorem

- If periodic x(t) is bandlimited to bandwidth f_{max} and samples x[n] are obtained from x(t) by sampling at greater than Nyquist rate $f_s > 2f_{max}$ then can exactly reconstruct x(t) from samples using sinc interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\text{sinc}(\pi(t - nT_s)/T_s)$$

- This is also called the cardinal series for x(t)

Q. **Why does sinc interpolation work?**

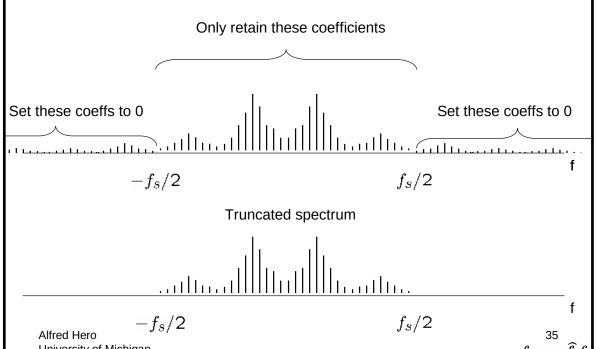
A. The FS of the cardinal series is identical to the FS of x(t)

See EECS306

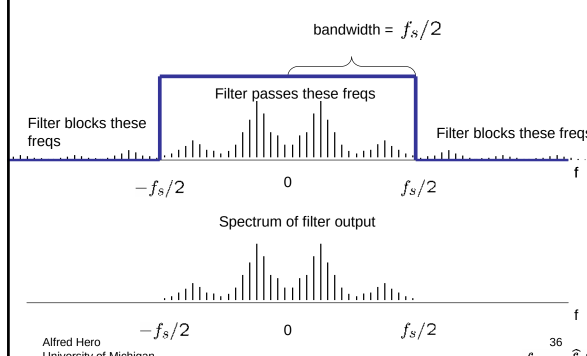
How to avoid aliasing if signal is not bandlimited?

- Must first apply a “Anti-aliasing filter” to eliminate all frequency components above on half the sampling frequency.
- This may destroy some small features of signal but is usually better than aliasing distortion.
- Anti-aliasing filter can be implemented in the frequency domain as a truncation operation which only lets low frequency components pass through.

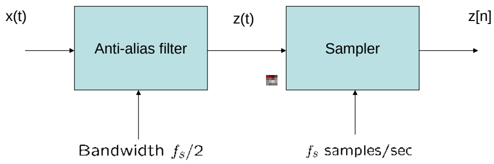
Anti-Alias (Lowpass) Filtering



Anti-Alias (Lowpass) Filtering



Practical Sampling System



In this course we will not further elaborate on CT filters like the anti-alias filter. We will, however, soon treat DT filters in great detail!