More About The Discrete Fourier Transform:

The Discrete Fourier Transform (DFT)

Given a periodic signal $x[n]$ and an integer N that is its fundamental period or a multiple thereof

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}, \quad \text{all } n \qquad \text{synthesis formula}$$

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \quad k = 0,\ldots,N-1 \qquad \text{analysis formula}$$

The DFT synthesis formula shows that $x[n]$ can be interpreted as having N complex exponentional components with frequencies $0, \frac{2\pi}{N}, \frac{4\pi}{N}, \frac{6\pi}{N}, \ldots, \frac{2\pi(N-1)}{N}$ and complex amplitudes $X[0],\ldots,X[N-1]$. Thus, these frequencies and complex amplitudes determine the spectrum of $x[n]$. In addition, the analysis formula shows that $X[k]$, i.e. the spectrum at frequency $\frac{2\pi}{N}k$, is the correlation of $x[n]$ with a complex exponential at this frequency.

Terminology

- The $X[k]$'s are called *DFT coefficients*. Computing them, i.e. applying the analysis formula to compute all the N coefficients, is often called "taking the DFT of $x[n]$".
- $X[k]$ is often written as a shorthand for the entire sequence of N coefficients $X[0], \ldots, X[N-1]$. For example, when we say "$X[k]$ is the N-point DFT of $x[n]$", then we mean "$X[0],\ldots,X[N-1]$" are the DFT coefficients for the entire signal $x[n]$. As another example, it is common to write $X[k] = \text{DFT}\{x[n]\}$ or $X[k] = \mathcal{F}\{x[n]\}$.
- When you hear someone say "X[k] is the N-point DFT of x[n]" you realize that sometimes the word "DFT" refers to the formula for or process of computing the coefficients, and sometimes it refers to the coefficients that result from the formula/process.
- The synthesis formula is often called the *inverse DFT* (IDFT). Applying it is often called "taking the inverse DFT of $X[k]$". It is common to write $x[n] = \text{IDFT}\{X[k]\}$ or $x[n] = \mathcal{F}^{-1}\{X[k]\}$.
- Some people refer to what we call the DFT as the Discrete-Time Fourier Series, because it play the same role for discrete-time periodic signals that the Fourier Series plays for continuous-time signals.
- Some people view the DFT as fundamentally a transform that applies to finite-length signals, whereas we have introduced it as fundamentally applying to infinite-length periodic signals. Either viewpoint is acceptable. If you view it as applying fundamentally to finite-length sequences, then you can apply it to an infinite-length periodic signals by applying it to one period of the signal. Conversely, if you view it as applying fundamentally to infinite-length periodic signals, then you will see from the analysis formula that it only makes use of a finite number of signal values, so it can be applied to finite signals as well.

Examples:

1. exponential (with period that's a divisor of N)

$$x[n] = e^{j2\pi\frac{m}{N}n} \;\Rightarrow\; X[k] = \begin{cases} 1, & k=m \\ 0, & k\neq m \end{cases}$$

2. cosine

$$x[n] = \cos(2\pi\frac{m}{N}n) \;\Rightarrow\; X[k] = \begin{cases} \frac{1}{2}, & k=m,\ N-m \\ 0, & \text{else} \end{cases}$$

3. sine

$$x[n] = \sin(2\pi\frac{m}{N}n) \;\Rightarrow\; X[k] = \begin{cases} \frac{1}{2j}, & k=m \\ -\frac{1}{2j}, & k=N-m \\ 0, & \text{else} \end{cases}$$

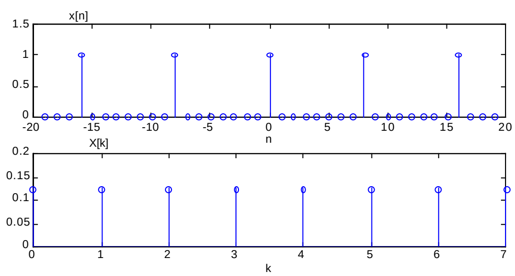
4. exponential with phase shift

$$x[n] = e^{j(2\pi\frac{m}{N}n+\phi)} \;\Rightarrow\; X[k] = \begin{cases} e^{j\phi}, & k=m \\ 0, & k\neq m \end{cases}$$

5. cosine with phase shift

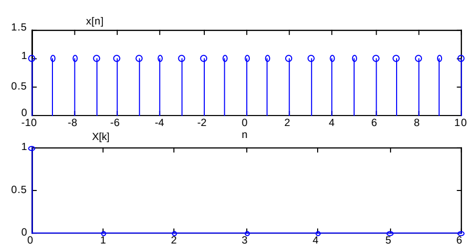
$$x[n] = \cos(2\pi\frac{m}{N}n+\phi) \;\Rightarrow\; X[k] = \begin{cases} \frac{1}{2}e^{j\phi}, & k=m \\ \frac{1}{2}e^{-j\phi}, & k=N-m \\ 0, & \text{else} \end{cases}$$

6. $x[n] = \begin{cases} 1, & n=\text{multiple of } N \\ 0, & \text{else} \end{cases} \;\Rightarrow\; X[k] = \frac{1}{N},\ \ k = 0,\ldots,N-1$



(N=8)

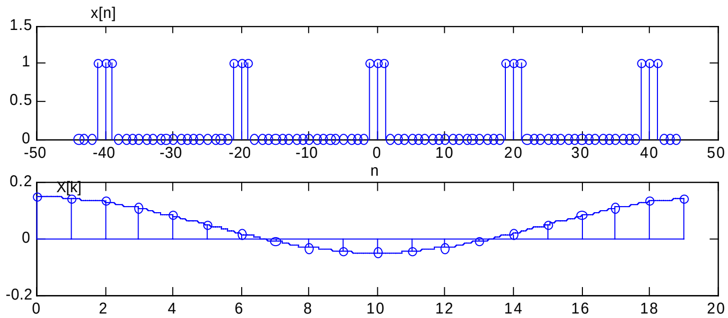
7. $x[n] = 1,\ \text{all } n \;\Rightarrow\; X[k] = \begin{cases} 1, & k=0 \\ 0, & k\neq 0 \end{cases}$



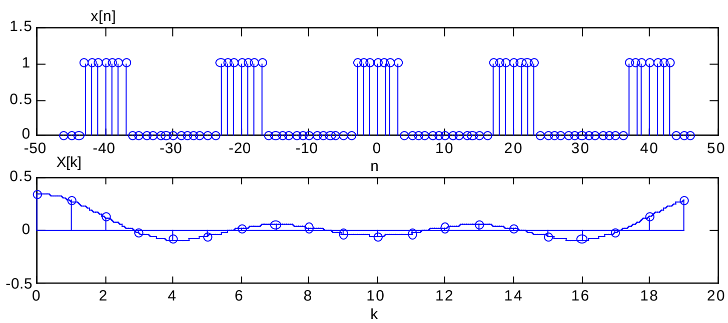
(N=7)

8. $x[n]$ is periodic with fundamental period N and $x[n] = \begin{cases} 0, & -N/2 \le n \le -m-1 \\ 1, & -m \le n \le m \\ 0, & m+1 \le n \le N/2 \end{cases}$

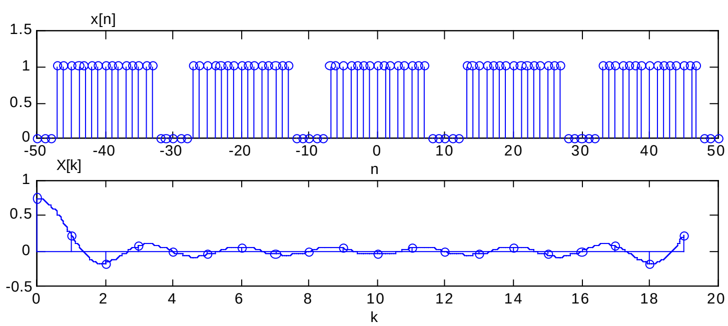
$$\Rightarrow X[k] = \left. \frac{\sin((2m+1)\hat{\omega}/2)}{\sin(\hat{\omega}/2)} \right|_{\hat{\omega}=2\pi k/N}$$



(N=20,m=1)



(N=20,m=3)



(N=20,m=7)

Note that for these examples, $X[k]$ is real valued.

The smooth curve is the "envelope" of the DFT, which depends on N but not m.

It is interesting to notice that as the number of consecutive 1' increases, the spectrum becomes more concentrated at low frequencies. Conversely, as the number of consecutive 1's decreases, the spectrum becomes more spread out in frequency. This is representative of the "rule of the thumb" that generally speaking signals that asmooth

Properties of the DFT:

In the following, assume that x[n] is periodic, N is the fundamental period of x[n] or a multiple thereof, and X[0],...,X[N-1] denotes its N-point DFT.

Most important properties

1. The X[k]'s are complex.

2. $X[k]\, e^{j\frac{2\pi}{N}kn}$ is the component of x[n] at frequency $\frac{2\pi}{N}k$.

3. X[N-k] can be considered to be the frequency component at frequency $-2\pi\frac{k}{N}$.

   Because: $e^{j\frac{2\pi}{N}(N-k)n} = e^{-j\frac{2\pi}{N}kn}$.

4. There is a one-to-one correspondence between periodic time signals with period N and sequences of N complex numbers. That is,

   - X[0],...,X[N-1] is the only N-tuple whose inverse DFT is x[n].
   - x[n] is the only periodic signal with period N whose DFT is X[0],...,X[N-1]

5. Linearity: DFT{ax[n]+by[n]} = a DFT{x[n]} + b DFT{y[n]}

6. Linearity: IDFT{aX[k]+Y[k]} = a IDFT{X[k]} + b IDFT{Y[k]}

7. X[0] is the average or DC value of x[n]; i.e. $X[0] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$

8. If x[n] is real, then $X[N-k] = X^*[k]$,

   and |X[N-k]| = |X[k]|, angle(X[N-k]) = -angle(X[k])

   Because: $$X[N-k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}(N-k)n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{j\frac{2\pi}{N}kn}\, e^{-j\frac{2\pi}{N}Nn}$$

   $$= \left(\frac{1}{N}\sum_{n=0}^{N-1} x^*[n]\, e^{-j\frac{2\pi}{N}kn}\right)^* \times 1 \quad \text{taking conjugates of all terms}$$

   $$= \left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}\right)^* \quad \text{since } x[n] \text{ is real}$$

   $$= X^*[k]$$

9. Suppose a periodic signal x[n] is the input to an LTI system with frequency response $\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}})$ and suppose X[0],...,X[N-1] is the N-point DFT of x[n] where N is the fundamental period of x[n] or a multiple thereof. Then the output y[n] is periodic with fundamental period N or a divisor thereof, and the N-point DFT of y[n] is

$$Y[k] = X[k]\, \mathcal{H}(\frac{2\pi}{N}k)$$

That is,

$$\text{DFT}\{x[n] * h[n]\} = \text{DFT}\{x[n]\}\, \mathcal{H}(\frac{2\pi}{N}k)$$

We see that the DFT turns convolution into multiplication.

Derivation: By the synthesis formula, we know

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$

From the linearity of the system, we know that response to x[n] is the sum of the responses to each of its components X[k] e\s(j \f(2π,N) k n,). Since the response to $X[k]\, e^{j\frac{2\pi}{N}kn}$ is $\mathcal{H}(\frac{2\pi}{N}k)\, X[k]\, e^{j\frac{2\pi}{N}kn}$, the response to x[n] is

$$y[n] = \sum_{k=0}^{N-1} \mathcal{H}(\frac{2\pi}{N}k)\, X[k]\, e^{j\frac{2\pi}{N}kn}$$

This says that $y[n] = \text{IDFT}\{X[k]\, \mathcal{H}(\frac{2\pi}{N}k)\}$, and it follows from Property 4 that $X[k]\, \mathcal{H}(\frac{2\pi}{N}k)$ must in fact be the DFT of y[n], i.e.

$$Y[k] = X[k]\, \mathcal{H}(\frac{2\pi}{N}k)$$

10. Rayleigh's theorem:

$$\frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2 = \sum_{k=0}^{N-1}|X[k]|^2$$

This shows that the power in the signal $x[n]$ equals the energy of its DFT coefficients

Derivation:

$$\sum_{n=0}^{N-1}|x[n]|^2 = \sum_{n=0}^{N-1} x[n]\, x^*[n]$$

$$= \sum_{n=0}^{N-1}\left(\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}\right)\left(\sum_{m=0}^{N-1} X[m]\, e^{j\frac{2\pi}{N}mn}\right)^*, \qquad \text{using the synthesis formula}$$

$$= \sum_{n=0}^{N-1}\left(\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} X[k]\, X^*[m]\, e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}mn}\right), \qquad \text{combining the two inner sums into one double sum}$$

$$= \sum_{k=0}^{N-1}\sum_{m=0}^{N-1} X[k]\, X^*[m] \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}, \qquad \text{interchanging sum orders}$$

$$= \sum_{k=0}^{N-1} X[k]\, X^*[k]\, N, \qquad \text{because } \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n} = \begin{cases} N, & k=m \\ 0, & k\neq m \end{cases}$$

$$= N\sum_{k=0}^{N-1}|X[k]|^2$$

11. When N is a power of 2, e.g. $N = 2^n$, there is fast algorithm for computing the DFT, called the FFT, the Fast Fourier Transform.

Other Properties

12. $x[0]$ is the sum of the $X[k]$'s; i.e. $x[0] = \sum_{k=0}^{N-1} X[k]$.

13. If $x[n]$ is real and N is even, then $X[N/2]$ is real.

Because: Since $N-\frac{N}{2} = \frac{N}{2}$, then by Property 8, $X[\frac{N}{2}] = X[N-\frac{N}{2}] = X^*[\frac{N}{2}]$, which implies that $X[\frac{N}{2}]$ is real.

14. If $x[n]$ is real and $x[N-n] = x[n]$ for all n, then $X[k]$'s are real.

15. Time shifting: $DFT\{x(n-n_o)\} = X[k]\, e^{-j\frac{2\pi}{N}kn_o}$

Because:

$$DFT\{x(n-n_o)\} = \frac{1}{N}\sum_{n=0}^{N-1} x(n-n_o)\, e^{-j\frac{2\pi}{N}kn_o}$$

$$= \frac{1}{N}\sum_{n=-n_o}^{N-1-n_o} x(m)\, e^{-j\frac{2\pi}{N}k(m+n_o)} \quad \text{letting } m = n-n_o$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}km}\, e^{-j\frac{2\pi}{N}kn_o} \quad \text{since summand is periodic, can sum over any interval of length N}$$

$$= X[k]\, e^{-j\frac{2\pi}{N}kn_o}$$

16. Frequency shifting: $DFT\{x[n]\, e^{j\frac{2\pi}{N}k_o n}\} = X[k-k_o]$

Because:

$$DFT\{x[n]\, e^{j\frac{2\pi}{N}k_o n}\} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{j\frac{2\pi}{N}k_o n}\, e^{-j\frac{2\pi}{N}kn_o}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}(k-k_o)n_o} = X[k-k_o]$$

17. The DFT can be used to compute the inverse DFT:

$$IDFT\{X[k]\} = N\left(DFT\{X^*[k]\}\right)^*$$

Because

$$N\left(DFT\{X^*[k]\}\right)^* = \left(N\,\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, e^{-j\frac{2\pi}{N}kn}\right)^*$$

$$= \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn} = x[n]$$

18. The DFT can be used on aperiodic signals, by applying it to segments of the signal, as discussed in Lab 6.

The Discrete-Time Fourier Transform (DTFT)

If you continue the study of discrete-time signals and systems, you will soon discover that there is another Fourier transform for discrete-time signals, called the *discrete-time Fourier transform* (DTFT). Indeed, it is a bit unfortunate that its name is so similar to the transform we have already studied. This transform is primarily intended for aperiodic signals that have finite energy (unlike periodic signals) or that are finitely summable in the sense that

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \ .$$

Analysis formula: The discrete-time Fourier transform of a signal $x[n]$ is the function $\tilde{X}(\hat{\omega})$ defined by

$$\tilde{X}(\hat{\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\hat{\omega}n} \ , \quad -\infty < \hat{\omega} < \infty.$$

Synthesis formula:

$$x[n] = \int_{-\pi}^{\pi} \tilde{X}(\hat{\omega})\, e^{j\hat{\omega}n} d\hat{\omega}$$

The DTFT can be thought of as decomposing $x[n]$ into a continuum of complex exponential components -- one for each value of $\hat{\omega}$.

The DTFT bears a close relationship to the DFT. Indeed the DFT can be thought of as being obtained by taking samples of the DTFT. Specifically, if $x[n]$ is periodic with period N, then

$$X[k] = \tilde{X}(\frac{2\pi}{N} k)$$

where $\tilde{X}(\hat{\omega})$ denotes the DTFT of just one period of $x[n]$.

We won't discuss the DTFT further, except to emphasize that the main purpose of the DTFT is to analyze aperiodic signals that have finite energy or are absolutely summable, whereas the main purpose of the DFT is to analyze periodic signals with infinite energy. However, each one can be adapted to the job of the other. It turns out that in practical situations where the DTFT is needed, the DFT is often used to approximate the DTFT.