Lecture on Continuous-Time Signals and Systems

Why study continuous-time given that discrete-time signal processing is taking over?

Though the solutions are increasing digital, the problems addressed by signals and systems are, essentially, are most about continuous-time signals and systems. To solve a problem well, it is importatn to understand it in its original domain.

The sampling theorem is derived in the continuous-time domain. Moreover, when sampling we know we must sample at rate at least equal to twice the highest frequency. How do we determine the highest frequency? We need to understand continuous-time spectra. Additionally, most signals do not have a largest frequency. We need to choose an approximate highest frequency, and we need to apply a continuous-time filter to block all signal components with frequency higher than half the sampling rate (in the continuous time domain) because these will alias into the frequency range 0 to $f_s/2$.

What to study?

Fourier series:

Spectral analysis of periodic continuous-time signals

Continuous-time systems, e.g. filters:

Properties:

Linearity, time-invariance, causality, stability

Described by	(corresponding discrete-time property)
A. differential equation	(difference equation)
B. block diagram	(block diagram)
C. impulse response	(impulse response)
D. frequency response	(frequency response)
E. transfer function	(system function)
F. poles and zero's	(poles and zeros)

Continuous-time Signals: more spectral analysis techniques

Fourier Transforms

Laplace Transforms

Other topics

Sampling Theorem

Filter Design

Random signals

Applications:

- Communication signals
- Feedback control systems
- Model actual physical systems

Fourier Series (See also the Quick Primer by Prof. Wakefield)

Let $x(t)$ be a periodic continuous-time signal. Let T_o be its fundamental period or a multiple thereof. Let $f_o = 1/T_o$ be called the *fundamental frequency*.

Exponential Fourier Series

Synthesis formula: $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi k f_o t}$

Analysis formula: $\alpha_k = \frac{1}{T_o} \int_0^{T_o} x(t)\, e^{-j2\pi k f_o t}\, dt$

Sinusoidal Fourier Series (the Fourier series of the book)

Synthesis formula: $x(t) = A_o + \sum_{k=1}^{\infty} A_k \cos(2\pi f_o k t + \phi_k)$

Analysis formulas: $A_o = \alpha_o$, $A_k = 2|\alpha_k|$, $\phi_k = \text{angle}(A_k)$

We will discuss the derivation of the exponential Fourier series, the relationships between the exponential and sinusoidal Fourier series, some properties and examples.

Derivation of Exponential Fourier Series

Assume $x(t)$ is periodic continuous-time signal. Let T_o be its fundamental period or a multiple thereof. Let $f_o = 1/T_o$ be called the *fundamental frequency*.

The component of $x(t)$ that is like $e^{j2\pi k f_o t}$ is

$$\alpha_k e^{j2\pi k f_o t}$$

where

$$\alpha_k = \frac{c_{T_o}(x, e^{j2\pi k f_o t})}{E_{T_o}(e^{j2\pi k f_o t})}$$

Why? Because as shown earlier, this minimizes difference energy

$$E_{T_o}\left(x(t) - \alpha_k e^{j2\pi k f_o t}\right) = \int_0^{T_o} \left|x(t) - \alpha_k e^{-j2\pi k f_o t}\right|^2 dt$$

Let us now find a formula for α_k. By the definitions of correlation and energy

$$c_{T_o}\left(x, e^{j2\pi k f_o t}\right) = \int_0^{T_o} x(t)\, e^{-j2\pi k f_o t}\, dt$$

$$E_{T_o}\left(e^{j2\pi k f_o t}\right) = \int_0^{T_o} \left|e^{-j2\pi k f_o t}\right|^2 dt = T_o$$

Therefore,

$$\alpha_k = \frac{1}{T_o} \int_0^{T_o} x(t)\, e^{-j2\pi k f_o t}\, dt$$

Are we done? No, we need to see that these α_k's make sense in the Fourier sum.

Note: The $e^{j2\pi k f_o t}$'s are mutually orthogonal, i.e.

$$c_{T_o}\left(e^{j2\pi k f_o t}, e^{j2\pi m f_o t}\right) = 0, \text{ when } k \neq m$$

We will now show that for any N, the above choices of $\alpha_1,...,\alpha_N$ minimize the difference energy

$$E_{T_o}\left(x(t) - \sum_{k=-N}^{N} \alpha_k \, e^{j2\pi k f_o t}\right)$$

As shorthand, let $p_k(t) = e^{j2\pi k f_o t}$. Then, omitting some of the messy detail:

$$E_{T_o}\left(x(t) - \sum_{k=-N}^{N} \alpha_k \, e^{j2\pi k f_o t}\right) = \int_0^{T_o} \left| x(t) - \sum_{k=-N}^{N} \alpha_k \, p_k(t) \right|^2 dt$$

$$= E_{T_o}(x) - 2\, c_{T_o}\left(x, \sum_{k=-N}^{N} \alpha_k \, p_k(t)\right) + E_{T_o}\left(\sum_{k=-N}^{N} \alpha_k \, p_k(t)\right)$$

$$= \sum_{k=-N}^{N} E_{T_o}(x) - 2 \sum_{k=-N}^{N} \alpha_k \, c_{T_o}(x,p_k) + \sum_{k=-N}^{N} \alpha_k^2 \, E_{T_o}(p_k) - (N-1)E_{T_o}(x)$$

$$= \sum_{k=-N}^{N} \left(E_{T_o}(x) - 2\alpha_k \, c_{T_o}(x,p_k) + \alpha_k^2 \, E_{T_o}(p_k) \right) - (N-1)E_{T_o}(x)$$

$$= \sum_{k=-N}^{N} E(x - \alpha_k p_k) - (N-1)E_{T_o}(x)$$

From this we see that the choice of the α_k's that minimizes the energy in the difference between x and the sum of the $\alpha_k \, p_k$'s is precisely the α_k's such that $\alpha_k p_k$ is closest to x, i.e. such that $\alpha_k p_k$ is the component of x like p_k. This justifies the formula for the α_k's.

Question: What happens as $N \to \infty$?

Definition:

$$\sum_{k=-\infty}^{\infty} \alpha_k \, e^{j2\pi k f_o t} = \lim_{N\to\infty} \sum_{k=-N}^{N} \alpha_k \, e^{j2\pi k f_o t}.$$

Fact:

$$E_{T_o}\left(x(t) - \sum_{k=-N}^{N} \alpha_k \, e^{j2\pi k f_o t}\right) \to 0 \text{ as } N \to \infty$$

Derivation**:** It's beyond the scope of all courses in EECS except EECS 600.

The above fact shows the sense in which $x(t)$ equals $\sum_{k=-\infty}^{\infty} \alpha_k \, e^{j2\pi k f_o t}$. We don't know for certain that $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \, e^{j2\pi k f_o t}$ for every t. However, the energy in the difference is zero, so they can only differ at isolated points.

The following theorem gives conditions for equality and tells us what happens when equality does not hold.

Dirichlet theorem

(a) $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi k f_o t}$ at all times t such that x(t) is continuous at t, i.e. $x(t+\varepsilon) \to x(t)$ as $\varepsilon \to 0$

(b) $\sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi k f_o t} = \frac{1}{2}(x(t_+)+x(t_-))$ if x has a jump at time t, $x(t_+)$ = limit of x from the right and $x(t_-)$ = limit of x from the left

if one of the following three conditions holds:

1. Absolute integrability:

$$\int_0^{T_o} |x(t)|\, dt < \infty$$

2. bounded variation: In any finite interval time x(t) has at most a finite numbe of maxima and minima.

3. Finite number of discontinuities: In any finite interval of time, x(t) is continuous or has finite number of discontinuities, and these discontinuities are finite in magnitude.

Gibbs Phenomenon

See p. 65 of *DSP First*

Terminology

- $\sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi k f_o t}$ is called a "Fourier series", because it it is an infinite sum, i.e. a series, discovered/invented by Fourier.
- The α_k's are called the coefficients. Computing them is often called "taking the Fourier series".

Relationship Between Exponential and Sinusoidal Fourier Series

- Follows from Euler's formula.
- The α_k's are what the book has been asking us to plot all along.
- See the discussion in the Quick Primer by Prof. Wakefield.

Properties of Fourier Series

1. α_0 = average value of signal

2. When x(t) is real,

$$\alpha_{-k} = \alpha_k^* \quad \text{(see derivation in the Quick Primer)}$$

3. Parseval's theorem (see derivation in Quick Primer)

   The power in x(t) equals the energy of the coefficients, i.e.

$$\frac{1}{T_o}\int_0^{T_o} |x|^2(t)\,dt = \sum_{k=-\infty}^{\infty} |\alpha_k|^2$$

4. Linearity: the FS coeff's of x+y are the sum of the FS coef's of each

   Linearity of inverse: ditto

5. Consider a periodic signal x(t) with period T_o and Fourier coefficients $\{\alpha_k\}$. When x(t) is the input to a continuous time linear time-invariant system with frequency response $H(\omega)$, then the output y(t) is periodic with period T_o and with Fourier series coefficients $\{\beta_k\}$ where

$$\beta_k = \alpha_k\, H(2\pi \tfrac{1}{T_o} k)$$

Continuous-time systems

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Properties

Linearity: $a_1 x_1(t) + a_2 x_2(t) \to a_1 y_1(t) + b_1 y_2(t)$, for any $x_1(t)$, $x_2(t)$, a_1, a_2

Time-invariance: $x(t-t_o) \to y(t-t_o)$, for any x(t) and t_o

Causality: The output at time t depends only on the inputs up through time t

Stability: If the input is bounded, the output is bounded

Descriptions of linear time-invariant systems

A. Input-output relationship described by differential equations

let $x(t)$ denote the input, $y(t)$ the output, most filters are described by coefficients $a_0,\ldots,a_N$, $b_0,\ldots,b_M$ and the differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Make filters out of multipliers, adders, differentiators

Differentiators are implemented with

capacitors: $i = C \frac{dv}{dt}$

inductors: $v = L \frac{di}{dt}$

these plus operational amplifiers

Integrators can be used in addition to or instead of differentiators:

$v = \frac{1}{C} \int i \, dt$, $i = \frac{1}{L} \int v \, dt$, these plus op amps

Model actual physical systems as discrete-time filters

B. Describe with block diagrams

C. Impulse response

$\delta(t)$ = idealized impulse (very narrow, very tall, area 1)

$h(t)$ = impulse response = $y(t)$ when $x(t) = \delta(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

this is called "convolution"

D. Frequency response

When $x(t) = e^{j\omega t}$, then $y(t) = H(\omega)\, e^{j\omega t}$,

where

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt = \text{frequency response}$$

$$= \text{correlation of } h \text{ with } e^{j\omega t} = \text{Fourier transform of } h(t)$$

Inverse relationship

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, e^{j\omega t}\, d\omega = \text{inverse Fourier transform of } H(\omega)$$

Input-output relationship

$$Y(\omega) = H(\omega)\, X(\omega)$$

where

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \text{Fourier transform of } x(t)$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt \quad = \text{ Fourier transform of } y(t)$$

Inverse Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

E. Transfer function

When $x(t) = e^{st}$, for some $s = \sigma + j\omega$,

then $y(t) = H_L(\omega)\, e^{st}$

where

$$H_L(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt \quad \text{where } s \text{ ranges over all complex numbers}$$

$$Y_L(s) = H_L(s)\, X_L(s)$$

where

$$X_L(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \quad = \text{ Laplace transform of } x(t)$$

$$Y_L(s) = \int_{-\infty}^{\infty} y(t)\, e^{-st}\, dt \quad = \text{ Laplace transform of } y(t)$$

$$H_L(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \quad = \text{ Laplace transform of impulse response}$$

Frequency response = transfer function on the imaginary axis

$$H(\omega) = H_L(j\omega)$$

F. Poles and zeros

$$H_L(s) = \frac{\prod_{k=1}^{M} (s-z_k)}{\prod_{k=1}^{M} (s-p_k)}$$

Poles and zeros give an indication of the frequency response.