**Correlation is a linear function**

1. If $\alpha$ is a scalar constant and x and y are signals

$$c(ax,y) = a\ c(x,y)$$

2. If x, y and z are signals

$$c(x+y, z) = c(x,z) + c(y,z)$$

3. Combine 1 and 2:

$$c(\alpha x+\beta y,\gamma z) = \alpha\gamma\ c(x,z) + \beta\gamma\ c(y,z)$$

**The performance of Correlating Detector in a Binary Communication System**

Consider a discrete-time version of the binary communication system described earlier. The transmitter sends the discrete-time signal

$$s[n] = b[0]\, p[n] + b[1]\, p[n-M] + b[2]\, p[n-2M] + \ ....$$

where the b[n]'s are ±1 message bits that we desire to send to the receiver, and p[n] is a pulse of duration M time units, i.e. p[n]=0, n < 0 and n>M . This system sends one message bit for every M transmitted samples.

During transmission, the transmitted signal is attenuated and noise is added, with the result that receiver is given the signal

$$r[n] \ = \ a\, s[n] + N[n]$$

where $a<<1$ is the attenuation factor, and N[n] is the noise sequence. We will assume that the noise sequence has mean value zero and mean squared value, i.e. power, denoted $P_N$.

To detect the b[i]'s, we use a correlating detector. In particular, to detect b[i] we correlate r[n] with p[n-iM] producing

$$C[i] \ = \ c(r[n], p[n-iM]) \ .$$

From C[i] we make a decision denoted $\hat{b}[i]$ about b[i]. Specifically,

$$\hat{b}[i] \ = \ \begin{cases} +1, \text{ if } C[i]\geq 0 \\ -1, \text{ if } C[i]<0 \end{cases} .$$

This makes sense because if there is no noise, then when detecting b[i], the correlator produces a positive C[i] if b[i]=+1, and a negative C[i] if b[i]=-1. Specifically, when there is no noise, the correlator produces C[i] = a b[i] E(p), as we will now show. Notice that in the interval where p[n-iM] is not zero, r[n] = a b[i] s[n] = a b[i] p[n-iM]. Therefore

$$C[i] \ = \ c(r[n], p[n-iM]) \ = \ c(a\, b[i]\, p[n-iM], p[n-iM])$$

$$= \ a\, b[i]\, c(p[n-iM], p[n-iM]) \qquad \text{by the linearity of correlation}$$

$$= \ a\, b[i]\, E(p) \qquad \text{because correlating a signal with itself gives its energy}$$

When there is noise, the correlator produces

$$C[i] \ = \ c(r[n], p[n-iM]) \ = \ c(a\, s[n]+N[n], p[n-iM])$$

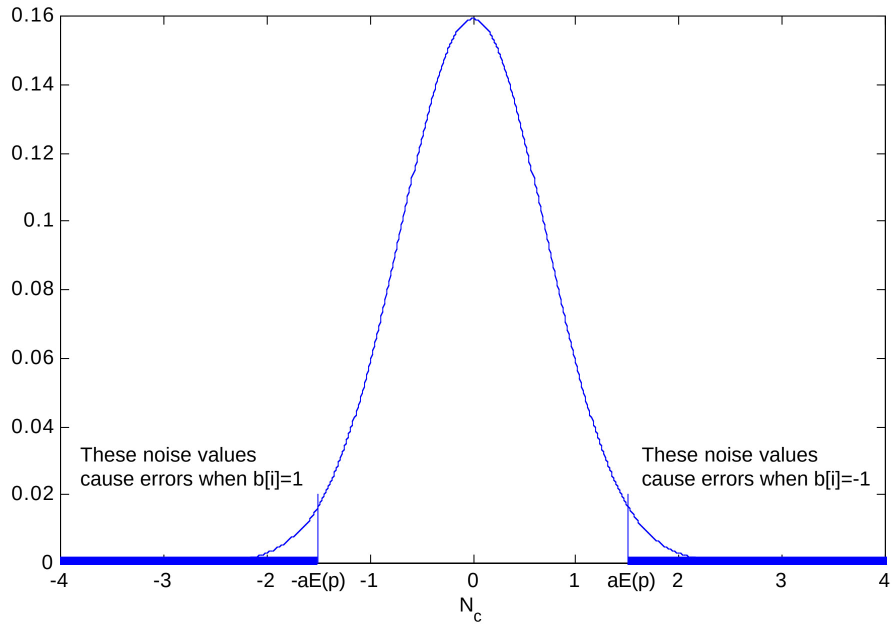
$$= \ a\, c(s[n], p[n-iM]) + c(N[n],p[n-iM]) \qquad \text{by linearity of correlation}$$

$$= \ a\, b[i]\, E(p) + N_c[i]$$

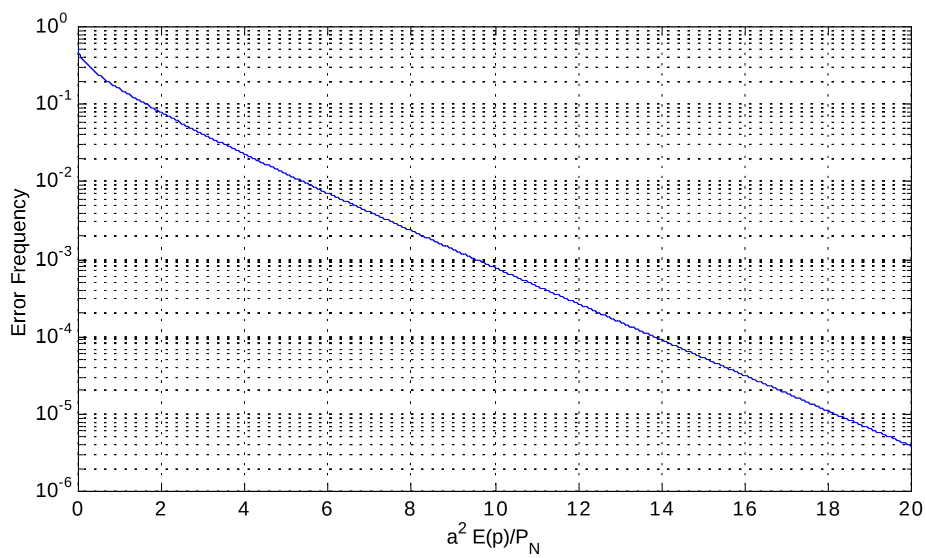
where the first term comes from the previous derivation and the second term, $N_c[i]$, is simply shorthand notation for c(N[n],p[n-iM]) .

We can now see what would cause the detector to make an error: If b[i] = -1, an error occurs when and only when $C[i] \geq 0$, which happens when and only when $N_c[i] \geq aE(p)$. Thus for those bits that are -1, the frequency of detection errors is the frequency with which $N_c[i] \geq a\,E(p)$. Similarly, for those bits that are +1, an error occurs when and only when $C[i] < 0$, which happens when and only when $N_c[i] < aE(p)$. Thus for those bits that are +1, the rate of detection errors is the frequency with which $N_c[i] > a\,E(p)$. It follows that the rate of errors is determined by the signal value distribution of the $N_c[i]$'s.

In many situations, good assumptions about the $N_c[i]$'s are that they have zero mean, power[4] (i.e. mean squared value) $P_N E(p)$, and signal value distribution that is a classic bell-shaped "Gaussian" (a.k.a. "normal") curve[5] illustrated below. The thick lines at the bottom and the short vertical lines are intended to delineate the values of $N_c[i]$ that cause errors.




Though we won't derive it here, the formula for the Gaussian curve above can be used to compute the rate of errors (It's the frequency of noise values larger than $aE(p)$.). The result is that the rate of errors depends on $a^2E(p)/P_N$ as shown in the graph below. For example, if $a^2E(p)/P_N = 10$, then the error rate is a little less than 1 in 10,000.




We conclude from all of this discussion that our ability to detect bits transmitted in noise depends on what is commonly called the "signal-to-noise ratio" $a^2E(p)/P_N$. (The numerator is

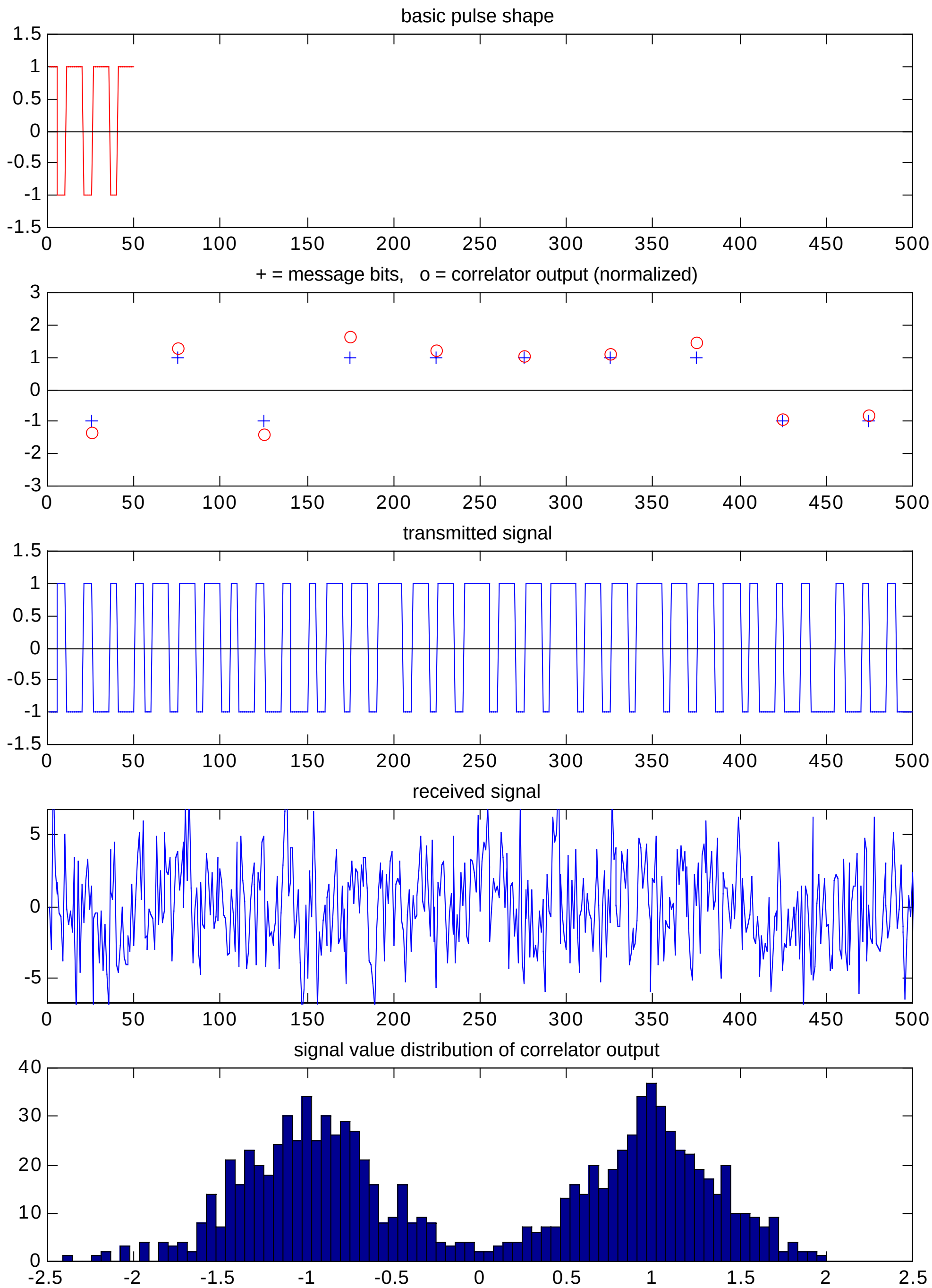
[4]It should be intuitive that since $N_c[i]$ is produced by multiplying the noise by $p[n-iM]$ and summing, its mean squared value will be proportional to the energy of $p$.
[5]The numerical labels are not intended to be accurate.

the energy of one received pulse $a\,p[n]$; the denominator is the power in the noise $N[n]$.) To get small error frequency we need this ratio to be large. Note that to make this ratio large, it is not necessary that the pulses have larger magnitude than the noise, i.e. it is not necessary that $a\,p[n]$ be large relative to $N[n]$. Rather it is only necessary that $a^2 E(p)$ be large relative to $P_N$. And for the latter to happen, it suffices for $p[n]$ to be a long pulse. For example, if $p[n] = \pm 1$, then $E(p) = M$. So no matter how much attenuation occurs or how much noise power there is, simply by making the basic pulse length $M$ large enough, reliable detection can be obtained.

The plot on the next page shows a typical example. The basic pulse shape $p[n]$ is shown in the top panel. (It is typical of a spread-spectrum type pulse.) A sequence of 1000 message bits was transmitted using this pulse. The first 10 such message bits are shown as +'s in the second panel. The first 10 transmitted pulse are shown in the third panel. The portion of the received signal corresponding to the first 10 bits is shown in the fourth panel. It is assumed that the signal attenuation is $a=1$ and the added noise has power $P_N = 8$, meaning that the noise has 8 times as much power as the signal[6]. The correlator output for each pulse (suitably normalized) is shown with o's in the second panel. Notice how they are generally close to the corresponding message bit. In this example, the detector made 9 errors, for an error frequency of 0.009. For this situation, $E(p) = 50$ and $a^2 E(p)/P_N = 50/8 = 6.25$. The plot shown above predicts that the error frequency should be 0.006, which is tolerably close to the actual value. The prediction (0.006) will become more accurate as the number of message bits increases.

Example: underwater round the world sonar transmission to measure water temperature.

[6]To keep things simple we used a large noise power rather than a small attenuation factor.


basic pulse shape
+ = message bits, o = correlator output (normalized)
transmitted signal
received signal
signal value distribution of correlator output