II. Signal Components

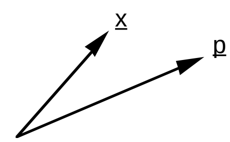
What does it mean for one signal to be a component of another?

Questions: Suppose we are given signals $x(t)$ and $p(t)$ (or $x[n]$ and $p[n]$).

- Is there a component of $x(t)$ that is like $p(t)$? (or of $x[n]$ is like $p[n]$)
- If so, how much $p(t)$ is in $x(t)$? (or $p[n]$ in $x[n]$)
- How to define "how much of ___ is in ___ "?

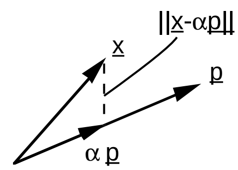
Example: Is there a component of $x(t)$ that is a sinusoid at frequency 2 hz? If so, what?

Vector Geometry: These questions are like familiar questions in vector geometry.



Question: Suppose $\underline{x} = (x_1,...,x_N)$ and $\underline{p} = (p_1,...,p_N)$ are vectors. Is there a component of $\underline{x}$ that is like $\underline{p}$? How much of $\underline{p}$ is in $\underline{x}$?

Approach: Find the value α such that $\alpha\,\underline{p}$ is as close to $\underline{x}$ as possible, i.e. such that $||\underline{x}-\alpha\underline{p}||$ is as small as possible, where $||\underline{u}-\underline{v}||$ is the Euclidean distance between $\underline{u}$ and $\underline{v}$ as defined by



$$||\underline{u}-\underline{v}|| = \sqrt{\sum_{i=1}^{N} (u_i-v_i)^2}$$

Actually, it's a bit easier to find the value of α that minimizes $||\underline{x}-\alpha\underline{y}||^2$ because this avoids the square root. To find α, let's equate to zero the derivative of $||\underline{x}-\alpha\underline{y}||^2$ with respect to α. First let's rewrite $||\underline{x}-\alpha\underline{y}||^2$:

$$||\underline{x}-\alpha p||^2 = \sum_{i=1}^{N} (x_i-\alpha p_i)^2 = \sum_{i=1}^{N} x_i^2 - 2\alpha \sum_{i=1}^{N} x_i\,p_i + \alpha^2 \sum_{i=1}^{N} p_i^2$$

$$= ||\underline{x}||^2 - 2\,\alpha\,(\underline{x} \circ \underline{p}) + \alpha^2\,||\underline{p}||^2$$

where $||\underline{x}||$ and $||\underline{p}||$ denote the lengths of $\underline{x}$ and $\underline{p}$, respectively, and $(\underline{x} \circ \underline{p})$ is the dot product defined by

$$(\underline{x} \circ \underline{p}) = \sum_{i=1}^{N} x_i\,p_i$$

Now differentiating and equating to zero gives

$$0 = \frac{d}{d\alpha} ||\underline{x}-\alpha p||^2 = \frac{d}{d\alpha}\Big(||\underline{x}||^2 - 2\,\alpha\,(\underline{x} \circ \underline{p}) + \alpha^2\,||\underline{p}||^2 \Big)$$

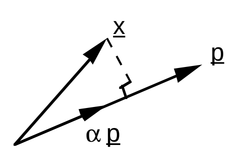
$$= -2\,(\underline{x} \circ \underline{p}) + 2\alpha||\underline{p}||^2 ,$$

which yields

$$\alpha = \frac{(\underline{x} \circ \underline{p})}{||\underline{p}||^2}$$

Answer: The component of $\underline{x}$ that is like $\underline{p}$ is $\dfrac{(\underline{x} \circ \underline{p})}{||\underline{p}||^2}\,\underline{p}$.

Fact: $\alpha = \dfrac{(\underline{x} \circ \underline{p})}{||\underline{p}||^2}$ is the unique value of α that makes the residual vector $(\underline{x}-\alpha\underline{p})$ and $\underline{p}$ orthogonal, where $\underline{u}$ and $\underline{v}$ are said to be orthogonal if $\underline{u} \circ \underline{v} = 0$.



Proof: The dot product of $(\underline{x}-\alpha\underline{p})$ and $\underline{p}$ is

$$(\underline{x}-\alpha\underline{p}) \circ \underline{p} = (\underline{x} \circ \underline{p}) - \alpha(\underline{p} \circ \underline{p}) \quad \text{by the linearity of the dot product}$$

$$= (\underline{x} \circ \underline{p}) - \alpha \, ||\underline{p}||^2$$

which is zero when and only when $\alpha = \frac{(\underline{x} \circ \underline{p})}{||\underline{p}||^2}$, i.e. when and only when $(\underline{x}-\alpha\underline{p})$ and $\underline{p}$ are orthogonal.

Back to Signals: Let us now return to the original questions for signals:

Questions: Suppose we are given signals x(t) and p(t).

- Is there a component of x(t) that is like p(t)? If so, what is it?
- How much of p(t) is in x(t)?
- How to define "how much of ___ is in ___ " and "component of ___ like ___"?

Approach: Find the value α such that the difference energy $E(x(t) - \alpha\, p(t))$ is as small as possible. We will then say that "α p(t) is the component of x(t) that is like p(t)" and "α is the amount of p(t) that is in x(t)".

The idea is that the question we are asking is just like the question for vectors, and we can use the same approach. The only difference is that instead of Euclidean distance as a measure of similarity we use difference energy. Indeed, for discrete-time signals the question is *exactly the same* because difference energy is Euclidean distance squared. Thus in the discrete-time case, we can simply use the answers to the vector question. In doing so, we recognize that what is called "dot product" in the "vector domain", is just what we have called "correlation". Morever, it is easy to check that with correlation replacing dot product and energy replacing length squared, the answer we found to the vector question applies to continuous-time signals as well as to discrete-time signals. Therefore, we immediately obtain the following:

Answers:

- The value of α that minimizes the difference energy $E(x(t) - \alpha\, p(t))$ is $\alpha = \frac{c(x,p)}{E(p)}$.
- The amount p(t) that is in x(t) is $\frac{c(x,p)}{E(p)}$.
- The component of x(t) that is like p(t) is $\frac{c(x,p)}{E(p)}\, p(t)$. (++)
- $\alpha = \frac{c(x,p)}{E(p)}$ is the unique value that makes the residual signal $(x(t)-\alpha p(t))$ and p(t) orthogonal.
- These answers apply to discrete-time signals as well with p[n] replacing p(t).
- These answers apply to complex-valued signals, in discrete or continuous time.

Comments: Engineers have long recognized the connnections between signals and vectors. As a result, basic ideas from geometry, and more generally from linear algebra, are commonly used in signals and systems analysis. One of the most beneficial transferences is the idea that we can draw geometric pictures that represent signals and their relationships, such as those on the previous page. For example, orthogonal signals are drawn at right angles to one another. It often happens that a geometric picture will help one to understand some complex signal situation. It is also true that studying linear algebra will lead to increased understanding of signals and systems. For example, you might wish to learn as much as possible about linear algebra in Math 216 and to take Math 419 as an elective.