Correlation, Energy and Components for Periodic Signals with Infinite Duration.

In the discussion of the Discrete Fourier Transform to follow, we will be interested in determining the component of a periodic signal $x[n]$ that is like the complex exponential signal $e^{j\hat{\omega}n}$. The discussion of the previous pages, suggests that the component of $x[n]$ that is like the $e^{j\hat{\omega}n}$ is

$$\frac{c(x, e^{j\hat{\omega}n})}{E\left(e^{j\hat{\omega}n}\right)} e^{j\hat{\omega}n}$$

However, the cautious reader will note that the denominator, which is the energy of $e^{j\hat{\omega}n}$, is infinite and that because the numerator involves a sum of infinitely many terms, it is questionable whether it is well defined.

To rectify the situation, we go back and modify slightly our original approach to the question of how to find the component of a signal x that is like another signal p. When both signals are periodic, let us find the value of α that minimizes the *time-normalized* energy, i.e. the *power*, of the difference between x and p. Specifically, the power of a periodic signal is

$$P(x) = \frac{1}{N} \sum_{n=1}^{N} |x[n]|^2$$

where N is the fundamental period of x or any integer multiple thereof. Repeating the previous derivation with power replacing energy, we find that the value of α that minimzes $P(x-\alpha p)$ is

$$\alpha = \frac{\frac{1}{N} \sum_{n=1}^{N} x[n]\, p^*[n]}{\frac{1}{N} \sum_{n=1}^{N} |p[n]|^2} = \frac{\sum_{n=1}^{N} x[n]\, p^*[n]}{\sum_{n=1}^{N} |p[n]|^2} = \frac{c_N(x,p)}{E_N(p)}$$

where N is any multiple of the fundamental periods of both x and p, and where $c_N(x,y)$ and $E_N(y)$ denote the correlation and energy, respectively, over the time interval[7] $[1,N]$. Thus the component of $x[n]$ that is like $p[n]$ is

$$\frac{c_N(x,p)}{E_N(p)}\, p[n] \,.$$

In effect, we find the component of a periodic signal $x[n]$ by finding the component of the signal ... 0 0 x[1] x[2] ... x[N] 0 0 ... that is like ... 0 0 p[1] p[2] ... p[N] 0 0 ... .

Summary:

If $x[n]$ and $p[n]$ are periodic discrete-time signals, then the component of $x[n]$ that is like $y[n]$ is

$$\frac{c_N(x,p)}{E_N(p)}\, p[n]$$

where N is any multiple of the fundamental periods of both x and p.

Similarly, if $x(t)$ and $p(t)$ are periodic continuous-time signals, a similar derviation shows that the component of $x(t)$ that is like $p(t)$ is

$$\frac{c_T(x,p)}{E_T(p)}\, p(t)$$

where T is any multiple of the fundamental periods of both x and p, and $c_T(x,p)$ and $E_T(p)$ denote correlation and energy over the time interval $[0,T]$.

[7]Or any other interval of n_o consecutive times.

II. Correlation As the Basis of Fourier/Spectral Theory

II.1 The Discrete Fourier Transform

Let us focus on discrete-time signals. We will use what we have learned about components of signals to develop the discrete Fourier transform. We begin with the following questions.

> Given a discrete-time periodic signal $x[n]$ and a frequency $\hat{\omega}$, what is the component of $x[n]$ that is like the complex exponential $e^{j\hat{\omega}n}$? How much $e^{j\hat{\omega}n}$ is in $x[n]$?

In view of the previous discussion, we are ready to answer this question, at least in the case that $e^{j\hat{\omega}n}$ is periodic. However, recall that in discrete-time, not all complex exponentials are periodic. Instead, the complex exponential signal $e^{j\hat{\omega}n}$ is periodic when and only when its frequency has the form $\hat{\omega} = 2\pi\, u/v$ where u and v are integers, i.e. the radian frequency must be a rational multiple of 2π.

Now, assuming $\hat{\omega}$ is of this form, the component of a periodic signal $x[n]$ that is like $e^{j\hat{\omega}n}$ is

$$\frac{c_N(x, e^{j\hat{\omega}n})}{E_N\left(e^{j\hat{\omega}n}\right)}\, e^{j\hat{\omega}n} ,$$

where N is any multiple of the fundamental periods of x and $e^{j\hat{\omega}n}$. The numerator is

$$c_N(x, e^{j\hat{\omega}n}) = \sum_{n=1}^{N} x[n] \left(e^{j\hat{\omega}n}\right)^* = \sum_{n=1}^{N} x[n]\, e^{-j\hat{\omega}n} .$$

The denominator is

$$E_N\left(e^{j\hat{\omega}n}\right) = \sum_{n=1}^{N} \left|e^{j\hat{\omega}n}\right|^2 = \sum_{n=1}^{N} 1 = N .$$

> Therefore, the the component of the periodic signal $x[n]$ that is like the periodic exponential $e^{j\hat{\omega}n}$ is
>
> $$\left(\frac{1}{N} \sum_{n'=1}^{N} x[n']\, e^{-j\hat{\omega}n}\right) e^{j\hat{\omega}n} .$$
>
> where N is any multiple of the fundamental periods of $x[n]$ and $e^{j\hat{\omega}n}$. (Note the use of a dummy summation variable n' that is different than n.)

Instead of saying the "component of $x[n]$ that is like $e^{j\hat{\omega}n}$", it is more common to say "the component of $x[n]$ at frequency $\hat{\omega}$". And instead of saying the "amount of $e^{j\hat{\omega}n}$ in $x[n]$", it is more common to say "the complex amplitude or spectrum of $x[n]$ at frequency $\hat{\omega}$".

The attentive reader may recognize that what multiplies $e^{j\hat{\omega}n}$ in the above expression is a DFT coefficient when $\hat{\omega} = \frac{2\pi}{N} k$ for some k. But let us not leap to the conclusion that we have already rediscovered the DFT. There's more that we need to learn.

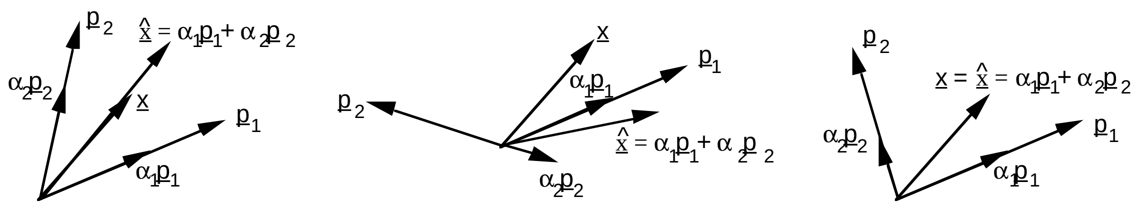
Exercise: Find an expression for the component of a periodic signal $x[n]$ that is like the periodic sinusoid $p[n] = \cos(\hat{\omega}n)$.

We have learned how to find the component of a periodic signal at any frequency $\hat{\omega}$ that is a rational multiple of 2π. There are infinitely many such frequencies. If we are interested in determining the spectrum of $x[n]$, at which frequencies should we compute the components? For example, should we compute the components at all rational frequencies?

One way to narrow the possible choices of frequencies is to take into account the desirable property that the components of a signal should sum to give the original signal. That is, we would like to find a collection of frequencies $\hat{\omega}_1, \hat{\omega}_2, \ldots$ such that the components of $x[n]$ at these frequencies sum to x[n]. If this happens, then the set of frequencies and the set of complex amplitudes at these frequencies forms a complete representation of the signal, in the sense that they completely determine the signal.

Vector Geometry

Let us return to the case of vectors for guidance. Each plot below shows a vector $\underline{x} = (x_1,x_2)$, a pair of vectors $\underline{p}_1, \underline{p}_2$, the components of $\underline{x}$ that are like $\underline{p}_1$ and $\underline{p}_2$, and the sum $\hat{\underline{x}}$ of these components.



In this example, we see that the only case where the components of $\underline{x}$ sum to equal $\underline{x}$ is the case on the far right where $\underline{p}_1$ and $\underline{p}_2$ are orthogonal. This happens to be the key idea that generalizes. Specifically,

> When $\underline{p}_1,...,\underline{p}_N$ are mutually orthogonal N-dimensional vectors, i.e. $\underline{p}_k \circ \underline{p}_j = 0,\ k \neq j$, then every N-dimensional vector $\underline{x}$ is the sum of its components that are like $\underline{p}_1,...,\underline{p}_N$, respectively. That is,
>
> $$\underline{x} = \sum_{k=1}^{N} \frac{(\underline{x} \circ \underline{p}_k)}{\|\underline{p}_k\|^2}\, \underline{p}_k .$$

To demonstrate this, let us assume that $\underline{p}_1,...,\underline{p}_N$ are mutually orthogonal, let $\underline{x}$ be an arbitrary N-dimensional vector, and let us try to find $\alpha_1,...,\alpha_N$ such that

$$\underline{x} = \sum_{k=1}^{N} \alpha_k\, \underline{p}_k .$$

i.e. let us try to show that $\underline{x}$ is a linear combination of the $\underline{p}_k$'s. In the process of doing this, we will find that α_k must equal $(\underline{x} \circ \underline{p}_k)/\|\underline{p}_k\|^2$ for every k. Let us view $\underline{x}$, $\underline{\alpha}$ and the $\underline{p}_k$'s as column vectors, and let us rewrite the above as a matrix equation

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{bmatrix} = \alpha_1 \begin{bmatrix} p_{1,1} \\ p_{1,2} \\ \cdots \\ p_{1,N} \end{bmatrix} + \alpha_2 \begin{bmatrix} p_{2,1} \\ p_{2,2} \\ \cdots \\ p_{2,N} \end{bmatrix} + ... + \alpha_N \begin{bmatrix} p_{N,1} \\ p_{N,2} \\ \cdots \\ p_{N,N} \end{bmatrix}$$

Equivalently,

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{bmatrix} = \begin{bmatrix} p_{1,1} & p_{1,1} & \cdots & p_{1,N} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,N} \\ p_{3,1} & p_{3,2} & \cdots & p_{3,N} \\ & & & \\ p_{N,1} & p_{N,2} & \cdots & p_{N,N} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_N \end{bmatrix}$$

or $\underline{x} = P\,\underline{\alpha}$, where P is the $N \times N$ matrix

$$P = \begin{bmatrix} | & | & \dots & | \\ \underline{p}_1 & \underline{p}_2 & \dots & \underline{p}_N \\ | & | & \dots & | \end{bmatrix} = \begin{bmatrix} p_{1,1} & p_{1,1} & \dots & p_{1,N} \\ p_{2,1} & p_{2,2} & \dots & p_{2,N} \\ p_{3,1} & p_{3,2} & \dots & p_{3,N} \\ & & & \\ p_{N,1} & p_{N,2} & \dots & p_{N,N} \end{bmatrix}$$

A few facts about matrices are reviewed in subsequent pages.

Our question now becomes: Does there exist a solution α to the equation $\underline{x} = P\,\underline{\alpha}$? From matrix theory, the answer is: Yes, if and only if the matrix P is *nonsingular*, or equivalently, if its *determinant* is not zero, or equivalently, if it has an *inverse*[8]. In fact, P does have an inverse, namely,

$$Q = \begin{bmatrix} \text{---}\underline{p}_1/\|\underline{p}_1\|^2\text{---} \\ \text{---}\underline{p}_2/\|\underline{p}_2\|^2\text{---} \\ \text{---}\underline{p}_3/\|\underline{p}_3\|^2\text{---} \\ \\ \text{---}\underline{p}_N/\|\underline{p}_N\|^2\text{---} \end{bmatrix}$$

(the dashes in the above indicate that each $\underline{p}_i$ has been converted to a row vector) as one may verify by showing that Q times P is the identity matrix I:

$$QP = \begin{bmatrix} \text{---}\underline{p}_1/\|\underline{p}_1\|^2\text{---} \\ \text{---}\underline{p}_2/\|\underline{p}_2\|^2\text{---} \\ \text{---}\underline{p}_3/\|\underline{p}_3\|^2\text{---} \\ \\ \text{---}\underline{p}_N/\|\underline{p}_N\|^2\text{---} \end{bmatrix} \begin{bmatrix} | & | & \dots & | \\ \underline{p}_1 & \underline{p}_2 & \dots & \underline{p}_N \\ | & | & \dots & | \end{bmatrix}$$

$$= \begin{bmatrix} \underline{p}_1\circ\underline{p}_1/\|\underline{p}_1\|^2 & \underline{p}_1\circ\underline{p}_2/\|\underline{p}_1\|^2 & \dots & \underline{p}_1\circ\underline{p}_N/\|\underline{p}_1\|^2 \\ \underline{p}_2\circ\underline{p}_1/\|\underline{p}_2\|^2 & \underline{p}_2\circ\underline{p}_2/\|\underline{p}_2\|^2 & \dots & \underline{p}_2\circ\underline{p}_N/\|\underline{p}_2\|^2 \\ \underline{p}_3\circ\underline{p}_1/\|\underline{p}_3\|^2 & \underline{p}_3\circ\underline{p}_2/\|\underline{p}_3\|^2 & \dots & \underline{p}_3\circ\underline{p}_N/\|\underline{p}_3\|^2 \\ & & & \\ \underline{p}_N\circ\underline{p}_1/\|\underline{p}_N\|^2 & \underline{p}_N\circ\underline{p}_2/\|\underline{p}_N\|^2 & \dots & \underline{p}_N\circ\underline{p}_N/\|\underline{p}_N\|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & & & \\ 0 & 0 & \dots & 1 \end{bmatrix} = I$$

where the next to last equality derives from the facts that $\underline{p}_k\circ\underline{p}_k = \|\underline{p}_k\|^2$ for every k and $\underline{p}_k\circ\underline{p}_j = 0$ for $k \neq j$. We conclude that since P has an inverse, there must indeed be a choice of the α_k's such that

$$\underline{x} = \sum_{k=1}^{N} \alpha_k\,\underline{p}_k\ .$$

Multiplying both sides of the equation $\underline{x} = P\underline{\alpha}$ by Q shows what the α_k's are:

$$Q\,\underline{x} = Q\,P\,\underline{\alpha} = \underline{\alpha} \quad \text{since } Q\,P \text{ is the identity matrix..}$$

Therefore,

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_N \end{bmatrix} = \underline{\alpha} = Q\,\underline{x} = \begin{bmatrix} \text{---}\underline{p}_1/\|\underline{p}_1\|^2\text{---} \\ \text{---}\underline{p}_2/\|\underline{p}_2\|^2\text{---} \\ \text{---}\underline{p}_3/\|\underline{p}_3\|^2\text{---} \\ \\ \text{---}\underline{p}_N/\|\underline{p}_N\|^2\text{---} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_N \end{bmatrix} = \begin{bmatrix} \underline{x}\circ\underline{p}_1/\|\underline{p}_1\|^2 \\ \underline{x}\circ\underline{p}_2/\|\underline{p}_2\|^2 \\ \underline{x}\circ\underline{p}_3/\|\underline{p}_3\|^2 \\ \\ \underline{x}\circ\underline{p}_N/\|\underline{p}_N\|^2 \end{bmatrix},$$

[8]The definition of an inverse is given on a subsequent page.

or equivalently,

$$\alpha_k = \frac{(\underline{x} \circ \underline{p}_k)}{||\underline{p}_k||^2} .$$

Thus we see that the α_k's that make the $\alpha_k\, \underline{p}_k$'s sum to equal $\underline{x}$ are precisely those that make $\alpha_k\, \underline{p}_k$ the component of $\underline{x}$ that is equal to $\underline{p}_k$. In other words, when there are N orthogonal $\underline{p}_k$'s, then the N components with respect to these $\underline{p}_k$'s sum to give $\underline{x}$.

Review of vector and matrix multiplication.

Let $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_N \end{bmatrix}$ and an N-dimensional vector in column form, i.e. a column vector, and let $\underline{y} = [y_1,y_2,...,y_N]$ be a vector in row form. Then

$$\underline{x}\, \underline{y} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_N \end{bmatrix} [y_1\ y_2\ ...\ y_N] = \sum_{i=1}^{N} x_i\, y_i = \underline{x} \circ \underline{y} .$$

(The dot product notation ignores the row and column natures of $\underline{x}$ and $\underline{y}$.)

Let $A = [A_{i,j}]$ be an N×N matrix, i.e

$$A = \begin{bmatrix} A_{1,1}\ A_{1,1} \dots A_{1,N} \\ A_{2,1}\ A_{2,2} \dots A_{2,N} \\ A_{3,1}\ A_{3,2} \dots A_{3,N} \\ \\ A_{N,1} A_{N,2} \dots A_{N,N} \end{bmatrix} = \begin{bmatrix} ---\underline{A}_1--- \\ ---\underline{A}_2--- \\ ---\underline{A}_3--- \\ \\ ---\underline{A}_N--- \end{bmatrix},$$

with rows $\underline{A}_1,...,\underline{A}_N$. Multiplying A by a column vector $\underline{x}$ on the right yields the N-dimensional vector

$$A\, \underline{x} = \begin{bmatrix} \underline{A}_1\, \underline{x} \\ \underline{A}_2\, \underline{x} \\ \underline{A}_3\, \underline{x} \\ \\ \underline{A}_N\, \underline{x} \end{bmatrix}.$$

Now consider another N×N matrix

$$B = \begin{bmatrix} B_{1,1} & B_{1,1} & \dots & B_{1,N} \\ B_{2,1} & B_{2,2} & \dots & B_{2,N} \\ B_{3,1} & B_{3,2} & \dots & B_{3,N} \\ \\ B_{N,1} & B_{N,2} & \dots & B_{N,N} \end{bmatrix} = \begin{bmatrix} | & | & \dots & | \\ \underline{B}_1 & \underline{B}_2 & \dots & \underline{B}_N \\ | & | & \dots & | \end{bmatrix},$$

whose columns are $\underline{B}_1,...,\underline{B}_N$. Multiplying A by B on the right yields the N×N matrix

$$AB = \begin{bmatrix} ---\underline{A}_1--- \\ ---\underline{A}_2--- \\ ---\underline{A}_3--- \\ \\ ---\underline{A}_N--- \end{bmatrix} \begin{bmatrix} | & | & \dots & | \\ \underline{B}_1 & \underline{B}_2 & \dots & \underline{B}_N \\ | & | & \dots & | \end{bmatrix}$$

$$= \begin{bmatrix} \underline{A}_1\underline{B}_1 & \underline{A}_1\underline{B}_2 & \dots & \underline{A}_1\underline{B}_N \\ \underline{A}_2\underline{B}_1 & \underline{A}_2\underline{B}_2 & \dots & \underline{A}_2\underline{B}_N \\ \underline{A}_3\underline{B}_1 & \underline{A}_3\underline{B}_2 & \dots & \underline{A}_3\underline{B}_N \\ \\ \underline{A}_N\underline{B}_1 & \underline{A}_N\underline{B}_2 & \dots & \underline{A}_N\underline{B}_N \end{bmatrix} = \begin{bmatrix} \underline{A}_1\circ\underline{B}_1 & \underline{A}_1\circ\underline{B}_2 & \dots & \underline{A}_1\circ\underline{B}_N \\ \underline{A}_2\circ\underline{B}_1 & \underline{A}_2\circ\underline{B}_2 & \dots & \underline{A}_2\circ\underline{B}_N \\ \underline{A}_3\circ\underline{B}_1 & \underline{A}_3\circ\underline{B}_2 & \dots & \underline{A}_3\circ\underline{B}_N \\ \\ \underline{A}_N\circ\underline{B}_1 & \underline{A}_N\circ\underline{B}_2 & \dots & \underline{A}_N\circ\underline{B}_N \end{bmatrix}$$

The inverse of an $N\times N$ matrix A is any matrix such B such that $BA = I$, where I denotes the $N\times N$ identity matrix

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

I is called the identity matrix because $I\underline{x} = \underline{x}$ for every $\underline{x}$.

Back to Signals

Let $x[n]$ be a periodic discrete-time signal. Recall that our goal is to choose a collection of frequencies $\hat{\omega}_1, \hat{\omega}_2, \ldots$ such that the components of $x[n]$ at these frequencies sum to $x[n]$. A development just like that for vectors shows that if N is the fundamental period of $x[n]$, or a multiple thereof, then it suffices to choose N frequencies such that the resulting complex exponentials are orthogonal. What N frequencies cause the complex exponentials to be orthogonal? The answer is

$$0, \frac{2\pi}{N}, \frac{4\pi}{N}, \frac{6\pi}{N}, \ldots, \frac{2\pi(N-1)}{N},$$

and the resulting complex exponentials are

$$1,\ e^{j\frac{2\pi}{N}n},\ e^{j\frac{2\pi}{N}2n},\ e^{j\frac{2\pi}{N}3n},\ \ldots,\ e^{j\frac{2\pi}{N}(N-1)n}.$$

The following verifies the orthogonality of $e^{j\frac{2\pi}{N}kn}$ and $e^{j\frac{2\pi}{N}mn}$ for $k\neq m$:

$$c_N(e^{j\frac{2\pi}{N}kn}, e^{j\frac{2\pi}{N}mn}) = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}mn}\left(e^{j\frac{2\pi}{N}kn}\right)^*$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}$$

$$= \frac{\sin(\frac{\pi}{N}(k-m)N)}{\sin(\frac{\pi}{N}(k-m))}\, e^{j\frac{\pi}{N}(k-m)\frac{1}{2}(N-1)} \qquad \text{by formula on p. 179, 331}$$

$$= 0 \qquad \text{since the numerator is the sine of a multople of } \pi.$$

We now collect what we have learned. Let $X[k]\, e^{j\frac{2\pi}{N}kn}$ denote the component of $x[n]$ at frequency $\frac{2\pi}{N}k$. Then by our earlier discussion of what constitutes a component, we have

$$X[k] \;=\; \frac{c_N(x, e^{j\frac{2\pi}{N}kn})}{\|e^{j\frac{2\pi}{N}kn}\|^2} \;=\; \frac{\sum_{n=0}^{N-1} x[n] \left(e^{j\frac{2\pi}{N}kn}\right)^*}{\sum_{n=0}^{N-1} |e^{j\frac{2\pi}{N}kn}|^2}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

And since now the components sum to give $x[n]$, we have

$$x[n] \;=\; \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$

These two relationships constitute the Discrete-Fourier Transform (DFT). They are summarized below.

The Discrete Fourier Transform (DFT)

Given a periodic signal $x[n]$ and an integer N that is its fundamental period or a multiple thereof

$$x[n] \;=\; \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn} \qquad \text{synthesis formula}$$

$$X[k] \;=\; \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \qquad \text{analysis formula}$$

The DFT synthesis formula shows that $x[n]$ can be interpreted as having N complex exponentional components with frequencies $0, \frac{2\pi}{N}, \frac{4\pi}{N}, \frac{6\pi}{N}, ..., \frac{2\pi(N-1)}{N}$ and complex amplitudes $X[0],...,X[N-1]$. Thus, these frequencies and complex amplitudes determine the spectrum of $x[n]$. In addition, the analysis formula shows that $X[k]$, i.e. the spectrum at frequency $\frac{2\pi}{N}k$, is the correlation of $x[n]$ with a complex exponential at this frequency.