EECS 206F01: EXAM #1
Solutions

What follows are restatements of each exam question, the correct answers, strategies for working each of the problems, and partial credit guidelines. Please look over this material carefully to make sure that you received the credit you deserve for the work you have shown.

Regrading procedure

After you've read through the partial credit guidelines for a given problem, if you believe you should have received more points than given, please write up your petition along with your reasoning and hand the petition *along with your original exam* to either Prof. Neuhoff or Wakefield.

Please remember that the amount of partial credit given for each problem differs according to the problem and the nature of the mistakes. *You may believe that more partial credit should have been given for the work you have shown*, but if we indicate no more than 5 points for a particular body of work, you will receive no more than 5 points, regardless of your petition.

Also, please remember that some answers received NO partial credit, no matter how much work was performed. This reflects the fact that some of the incorrect answers were wrong on fundamental grounds, rather than particular aspects of the problem. A petition requesting that work shown in support of a fundamental error will be rejected automatically.

You have until the end of October to submit your petitions in writing.

1. If x and y are arbitrary complex numbers, then $(x+y^*)(x^*+y)$ is

a. always real and never negative

b. always real, and can be positive or negative

c. always imaginary and never negative

d. always imaginary, and can be positive or negative

e. real, complex or imaginary, depending on the values of x and y

Solution:

Recall that $zz^* = |z|^2$ is always real and nonnegative.

Solution 1: Notice that $x^*+y = (x+y^*)^*$

Therefore, $(x+y^*)(x^*+y) = (x+y^*)(x+y^*)^* = |x+y^*|^2 \geq 0$

Solution 2: Let $x = a + jb$, $y = c + jd$. Then

$$(x+y^*)(x^*+y) = xx^* + x^*y^* + xy + yy^* = |x|^2 + y^*x^* + xy + |y|^2$$

$$= a^2 + b^2 + (c-jd)(a-jb) +(a+jb)(c+jd) + c^2 + d^2$$

$$= a^2 + b^2 + ac - bd - j(ad+bc) +ac -bd +j(bc+ad)+ c^2 + d^2$$

$$= a^2 + b^2 + 2ac - 2bd + c^2 + d^2 \qquad (**)$$

$$= a^2 + 2ac + c^2 +b^2 - 2bd + d^2$$

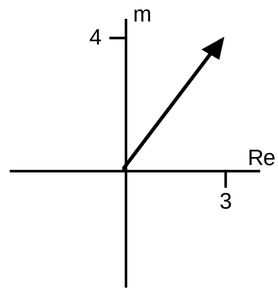
$$= (a+c)^2 + (b-d)^2 \geq 0$$

Partial Credit:

Up to 5 points for showing that $(x+y^*)(x^*+y)$ is real, for example, going as far as (**) above.

Up to 2 points for writing a productive expression for $(x+y^*)(x^*+y)$ and realizing that xx^* and yy^* are real.

2. The diagram



shows the phasor of the signal

a. $x_1(t) = 3.5 \cos(30t+.644)$

b. $x_2(t) = 5 \cos(40t+.644)$

c. $x_3(t) = 3.5 \cos(50t+.927)$

d. $x_4(t) = 5 \cos(60t+.927)$

e. none of the above

Solution:

The diagram shows a nonrotating phasor. The nonrotating phasor of the sinusoid

$$x(t) = A \cos(\omega_o t+\phi)$$

is the complex number $Ae^{j\phi}$. The relationship is

$$x(t) = \text{Re}\{Ae^{j\phi} e^{j\omega_o t}\}$$

Note that the phasor does not depend on ω_o. In other words the same phasor works for any choice of ω_o. The point is that if ω_o is known, $Ae^{j\phi}$ is what distinguishes one sinusoid of frequency ω_o from another.

In this problem, $A = \sqrt{3^2+4^2} = 5$ and $\phi = \tan^{-1}(\frac{4}{3}) = .927$. We cannot tell ω_o from the givens but we don't need to. $x_4(t)$ is the only one of these signals that has the given complex number as its phasor.

Partial Credit:

Up to 5 points for finding A and ϕ in a coherent way.

3. The signal

$$x(t) = 2\sqrt{3}\cos\left(\omega_o t + \frac{\pi}{6}\right) + 2\sin(\omega_o t + \frac{\pi}{6})$$

equals

a. $\boldsymbol{x_1(t) = 4\cos(\omega_o t)}$

b. $x_2(t) = (2+2\sqrt{3})\cos(\omega_o t + \frac{\pi}{6})$

c. $x_3(t) = 4\cos(\omega_o t + \frac{\pi}{3})$

d. $x_4(t) = (2-2\sqrt{3})\cos(\omega_o t + \frac{\pi}{6})$

e. none of the above

Solution:

$$x(t) = 2\sqrt{3}\cos\left(\omega_o t + \frac{\pi}{6}\right) + 2\sin(\omega_o t + \frac{\pi}{6})$$

$$= 2\sqrt{3}\cos\left(\omega_o t + \frac{\pi}{6}\right) + 2\cos(\omega_o t + \frac{\pi}{6} - \frac{\pi}{2})$$

$$= \mathrm{Re}\{2\sqrt{3}\, e^{j\omega_o t + \pi/6} + 2\, e^{j\omega_o t - \pi/3}\}$$

$$= \mathrm{Re}\{\left(2\sqrt{3}\, e^{j\pi/6} + 2\, e^{-j\pi/3}\right) e^{j\omega_o t}\}$$

Now,

$$2\sqrt{3}\, e^{j\pi/6} + 2\, e^{-j\pi/3} = 2\sqrt{3}\,(\cos \pi/6 + j \sin \pi/6) + 2\,(\cos -\pi/3 + j \sin -\pi/3)$$

$$= 2\sqrt{3}\,(\frac{\sqrt{3}}{2} + j\,\frac{1}{2}) + 2(\frac{1}{2} - j\,\frac{\sqrt{3}}{2}) = 3+1 = 4$$

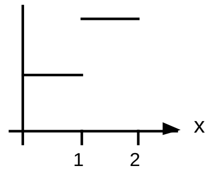
Therefore, $x(t) = \mathrm{Re}\{4\, e^{j\omega_o t}\} = 4\cos\omega_o t$

Partial Credit:

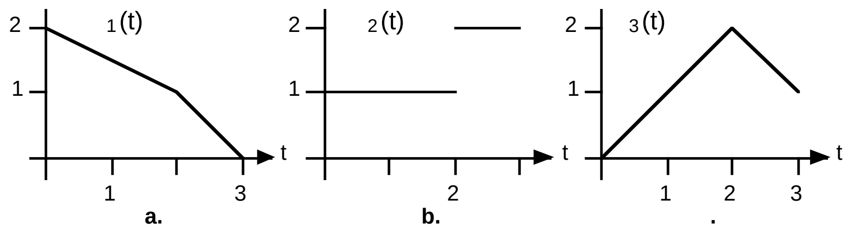
Up to 7 pts if you found c. due to using an approach like that above but with $\sin\theta = \cos\theta + \pi/2$ instead of $\sin\theta = \cos\theta - \pi/2$

Up to 5 points if you used an approach like that above but forget to add or substract $\pi/2$.

4. The function below



is the signal value distribution of



c. more than one of the above, d. none of the above

Solution:

The given signal value distribution shows that values of x between 0 and 1 occur equally often, the values between 1 and 2 occur equally often, and the values between 0 and 1 occur half as often as those between 1 and 2.

Let's now interpret each signal:

$x_1(t)$: The signal values between 1 and 2 are caused by $0 \le t \le 2$. Because the signal is a straight line in this region, all values between 1 and 2 occur equally often. Thus the SVD is constant between 1 and 2. Moreover, the siganl values between 0 and 1 are caused by $2 \le t \le 3$, and all values in this range occur equally often. Thus, the SVD is constant between 0 and 1. Finally, the signal values are between 1 and 2 for twice as long as they are between 0 and 1. Therefore, the SVD between 1 and 2 is twice as large as it is between 0 and 1. This is exactly the given SVD.

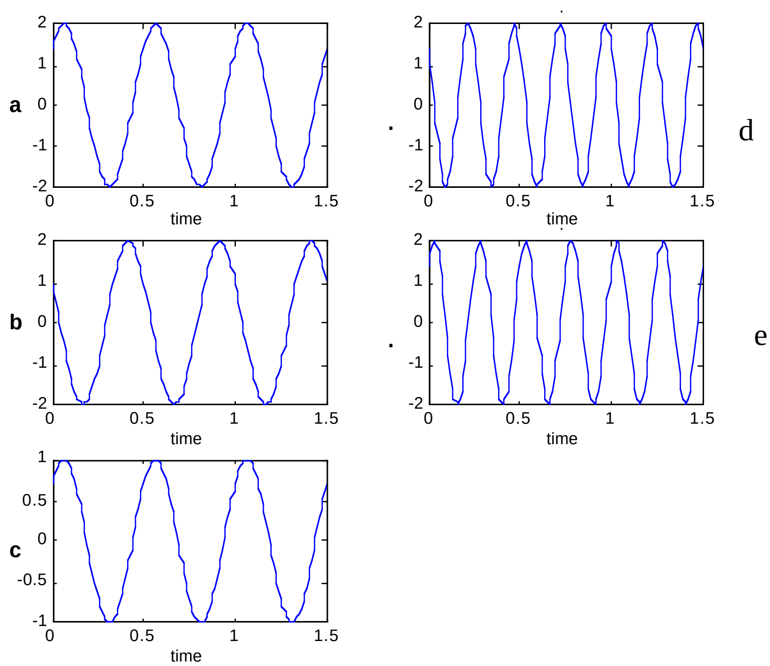
$x_2(t)$: This signal has only two values: 1 and 2. Thus its SVD is concentrated at x=1 and x=2 and is zero elsewhere.

$x_3(t)$: The values between 1 and 2 are caused by $1 \le t \le 3$. Because the signal is made of straight lines in this region, all values between 1 and 3 occur equally often. Thus the SVD is constant between 1 and 2. Moreover, the signal values between 0 and 1 are caused by $0 \le t \le 1$, and all values in this range occur equally often. Thus, the SVD is constant between 0 and 1. Finally, we see that signal value is between 1 and 2 for twice as long as it is between 0 and 1. Therefore, the SVD between 1 and 2 is twice as large as it is between 0 and 1. This is exactly the given SVD.

Partial Credit:

5 points if you answered a. or c. Up to 3 points if you answered b. or e, and explained the meaning of the given SVD or how one gets an SVD for a signal.

5. The signal $x(t) = 2\sin(4\pi t + \pi/4)$ is shown in:



The answer is a.

Solution:

Notice that $x(t) = 2\sin(4\pi t + \pi/4) = 2\cos(4\pi t - \pi/4) = 2\cos(2\pi 2(t-1/16))$

Thus we see that x(t) is a sinusoid with amplitude $A = 2$, frequency $f_o = 2$, fundamental period $T_o = 1/2$, that is time shifted 1/16 sec to the right.

This describes a.

b. has the right amplitude and frequency but the wrong time shift.

c. has the wrong amplitude.

d. and e. have the the wrong frequency. e

Partial Credit:

Up to 5 points if you answered b, c or e and explained why.

6. Which signal is not periodic?

a. $x_1(t) \;=\; \sin\left(\frac{p}{6}\; t\right) + 2\cos\left(\frac{8p}{3}\; t\right)$

b. $x_2(t) \;=\; 5\,(-1)^{\text{floor}(t/3)} \; + 2\cos\left(\frac{2p}{3}\; t\right)$

c. $x_3(t) \;=\; 2\sin\left(\frac{3p}{8}\; t\right) + \cos\left(\frac{3}{8}\; t\right)$

d. $x_4(t) \;=\; e^{-j(11pt+.3)}$

e. more than one of the above

Correct answer: c.
Solution: Go through each of the options to find the common period. c) fails because the ratio is irrational.
Partial Credit Scoring.
Up to 5 points were given for showing c) to be non-periodic when answering e). Otherwise, no points were given.

7. The spectrum of the signal

$$s(t) = 10\sin(2\pi 100t + 0.1\pi)\sin(2\pi 137t)$$

a. consists of four components at frequencies 100, -100, 137 and -137 Hz

b. consists of several components all of which have the same magnitude

c. doesn't exist because the amplitude of the 137 Hz sinusoid is changing over time and the spectrum refers to the amplitudes, frequencies, and phases of a set of *constant-amplitude* sinusoids.

d. more than one of the above is correct

e. none of the above is correct

Correct answer: b

Solution: Let $\alpha_1 = 2\pi 100t + 0.1\pi$ and $\alpha_2 = 2\pi 137t$. We substitute Euler's identities for the sin functions into the expression for $s(t)$ to find the complex exponential expansion. Accordingly,

$$\begin{aligned} s(t) &= 10\sin(2\pi 100t + 0.1\pi)\sin(2\pi 137t) \\ &= 10\sin(\alpha_1 t)\sin(\alpha_2 t) \\ &= 10\left(\frac{e^{j\alpha_1 t} - e^{-j\alpha_1 t}}{2j}\right)\left(\frac{e^{j\alpha_2 t} - e^{-j\alpha_2 t}}{2j}\right) \\ &= \frac{10}{-4}(e^{j(\alpha_1 + \alpha_2)t} + e^{-j(\alpha_1 + \alpha_2)t} - e^{j(\alpha_1 - \alpha_2)t} - e^{-j(\alpha_1 - \alpha_2)t}) \end{aligned}$$

Substituting the expressions for the α 's back into the equation, we have that

$$s(t) = \frac{10}{-4}(e^{j(2\pi 237t + 0.1\pi)} + e^{-j(2\pi 237t + 0.1\pi)} - e^{j(-2\pi 37t + 0.1\pi)} - e^{-j(-2\pi 37t + 0.1\pi)})$$

Therefore,

a) is not correct because the components are at 237, -237, 37, and -37 Hz

b) is correct because the magnitude of each component is $\left|\frac{10}{-4}\right|$

c) is not correct

Partial Credit Scoring.

a) No points were given for this answer because of a fundamental misunderstanding. When we refer to the spectrum of a signal, we are referring to the values of the frequencies, amplitudes, and phases of sinusoids that, when *added together*, make up the signal. The condition in a) corresponds to the *multiplicative* combination of sinusoids, which is not what the spectrum is about.

c) No points were given for this answer because of a fundamental misunderstanding. One of the major points of the lectures (both Neuhoff and Wakefield) and the text is that the type of amplitude modulation in the problem can be represented by an additive combination of constant-amplitude sinusoids.

d) or e) Up to four points were given for going through the calculations that arrived at the expressions shown above. The error occurred most often by failing to interpret *magnitude* correctly - seeing that the phase terms differed, the expressions were interpretted incorrectly as if they reflected differences in magnitude.

8. Suppose $x_1(t)$ and $x_2(t)$ are real-valued signals consisting of the sum of N_1 and N_2 sinusoids, respectively, where both N_1 and N_2 are *even*. Then, the spectrum of the sum of $x_1(t)$ and $x_2(t)$,

$$x(t) = x_1(t) + x_2(t)$$

a. must have exactly $2(N_1 + N_2)$ positive or negative frequency components

b. will not exist unless the N_1 and N_2 are harmonically related

c. if $N_1 = N_2$, could have no positive or negative frequency components

d. cannot have a DC term, since the number of sinusoids in each signal is even

e. none of the above is correct.

Correct answer: c

Solution: The easiest way to solve this problem is to return to the synthesis equation. You are given that

$$x_1(t) = A_{10} + \sum_{k=1}^{N_1 - 1} A_{1k} \cos(2\pi f_{1k} t + \phi_{1k})$$

$$x_2(t) = A_{20} + \sum_{k=1}^{N_2 - 1} A_{2k} \cos(2\pi f_{2k} t + \phi_{2k})$$

so that

$$x(t) = A_{10} + \sum_{k=1}^{N_1 - 1} A_{1k} \cos(2\pi f_{1k} t + \phi_{1k}) + A_{20} + \sum_{k=1}^{N_2 - 1} A_{2k} \cos(2\pi f_{2k} t + \phi_{2k})$$

Given this formulation, a) cannot be correct because of the possible existence of a DC term as well as the possibility that the sets of frequencies $\{f_{1k}\}$ and $\{f_{2k}\}$ may contain elements in common. b) cannot be correct since harmonically related *numbers* makes no sense. Even if the answer was misread as referring to the harmonic relationships among the frequencies, we know that a spectrum is not limited to signals with harmonically related frequencies, only the application of the Fourier series to find the components has such a limitation. d) is also incorrect, by inspection. Finally, c) is correct because it is possible that the sets of frequencies and amplitudes are identical but the phases are 180 degrees of each other, therefore cancelling out all components.

Partial Credit Scoring.

Up to three points were given for answers in support of a) or in rejection of a). No points were given for arguments supporting the fundamentally erroneous statements of b) and d).

9. Suppose $x(t)$ is a periodic square wave, which is defined for one cycle by

$$x(t) = \begin{cases} 10 & 0 \le t < T_0/2 \\ 0 & T_0/2 \le t < T_0 \end{cases}$$

where $T_0 = 10$ msec. Then, the spectrum X_k of $x(t)$

a. has no components at negative frequencies since the amplitude is strictly non-negative

b. $X_k = \dfrac{20}{j\pi k}$, for $k = \pm 1, \pm 3, \pm 5, \ldots$

c. $X_0 = 5$

d. more than one of the above is correct

e. none of the above is correct.

Correct answer: d. Both b) and c) are correct.
Solution: Following the approach for the square wave in the textbook and using the appropriate formulas, we have for the DC term

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}\int_0^{T_0/2} 10dt = 5$$

and for the other frequencies

$$\begin{aligned} X_k &= \frac{2}{T_0}\int_0^{T_0} x(t)e^{(-j2\pi k f_0)t}dt \\ &= \frac{2}{T_0}\int_0^{T_0/2} 10e^{(-j2\pi k f_0)t}dt \\ &= \frac{2}{T_0}\frac{10}{-j2\pi k f_0}e^{(-j2\pi k f_0)t}\Big|_0^{T_0/2} \\ &= -\frac{10}{j\pi k}(e^{-j\pi k} - 1) \end{aligned}$$

The last expression evaluates to 0 when k is even and to $20/(j\pi k)$ when k is odd.
Partial Credit Scoring:
Up to 5 points were given for correct formulation and execution of work establishing either b) or c) as correct. Arithmetic errors were deducted for incorrect calculation of the DC term (3 points) or not interpretting the last expression in the solution above with respect to the special cases of even/odd k (2 points). No credit was given for a), since this answer suggests that positive amplitude is related to positive frequency - a concept clearly not supported by any of our work so far in the text. Answers of e) were given up to 4 points for setting up the problem and carrying out some, but not all, of the work correctly.

10. $Re\left\{e^{j\omega_1 t + j\pi/2}\sin(\omega_2 t)\right\} =$

a. $\cos(\omega_1 t + \pi/2)\sin(\omega_2 t)$

b. $\sin(\omega_2 t)Re\left\{e^{j\omega_1 t + j\pi/2}\right\}$

c. $Re\left\{e^{j\omega_1 t + j\pi/2}e^{j\omega_2 t - j\pi/2}\right\}$

d. more than one of the above is correct

e. none of the above is correct.

Correct answer: d) since a) and b) are correct.
Solution: The argument of the real function consists of a complex number multiplying a real number. In general, we can factor out the real number to obtain b), e.g.,

$$Re\{(a+jb)c\} = Re\{ac+jbc\} = cRe\{a+jb\}$$

so that

$$Re\left\{e^{j\omega_1 + j\pi/2}\sin(\omega_2 t)\right\} = \sin(\omega_2 t)Re\{e^{j\omega_1 + j\pi/2}\}$$

Taking this one step further, we have that

$$\begin{aligned}\sin(\omega_2 t)Re\{e^{j\omega_1 + j\pi/2}\} &= \sin(\omega_2 t)Re\{\cos(\omega_1 + \pi/2) + j\sin(\omega_1 + \pi/2)\} \\ &= \sin(\omega_2 t)\cos(\omega_1 + \pi/2)\end{aligned}$$

Partial Credit Scoring.
Up to 5 points were given for correctly showing either a) or b) to be correct. For setting up and partially solving either problem alone, up to 3 points were given. Attempts to show c) were given up to 2 points. You should go through the work to convince yourself that c) will not work in general.