EECS 206F01: EXAM #2
Solutions

What follows are restatements of each exam question, the correct answers, strategies for working each of the problems, and partial credit guidelines. Please look over this material carefully to make sure that you received the credit you deserve for the work you have shown.

Regrading procedure

After you've read through the partial credit guidelines for a given problem, if you believe you should have received more points than given, please write up your petition along with your reasoning and hand the petition *along with your original exam* to either Prof. Neuhoff or Wakefield.

Please remember that the amount of partial credit given for each problem differs according to the problem and the nature of the mistakes. *You may believe that more partial credit should have been given for the work you have shown*, but if we indicate no more than 5 points for a particular body of work, you will receive no more than 5 points, regardless of your petition.

Also, please remember that some answers received NO partial credit, no matter how much work was performed. This reflects the fact that some of the incorrect answers were wrong on fundamental grounds, rather than particular aspects of the problem. A petition requesting that work shown in support of a fundamental error will be rejected automatically.

You have until the 11/30/01 to submit your petitions for regrade in writing.

Problem 1. The discrete-time signals for

$$s_1(t) = \cos(2\pi 200t) \qquad s_2(t) = \cos(2\pi 4000t)$$

are

a. identical for a sampling frequency $f_s = 3800$ Hz.

b. identical for at least one sampling frequency f_s such that $1300 < f_s < 1800$

c. identical for a sampling frequency $f_s = 4000$

d. more than one of the above is correct

e. none of the above is correct

Correct answer: d. Both a) and b) are correct.

Solution: Given a sampling frequency of f_s, the sampled forms of $s_1(t)$ and $s_2(t)$ are

$$s_1[n] = \cos\left(\frac{2\pi 200n}{f_s}\right) \qquad s_2[n] = \cos\left(\frac{2\pi 4000n}{f_s}\right)$$

One condition for the two sampled signals to be identical is that their discrete frequencies are the same up to an integer, i.e.,

$$\frac{2\pi 200}{f_s} + 2\pi k = \frac{2\pi 4000}{f_s}$$

where k is integer. Solving for f_s, we have

$$f_s = \frac{3800}{k}$$

Therefore, a) is correct under the condition that $k = 1$. In addition, c) is incorrect, and it would appear that the next two sampling frequencies (1900 and 1266.6...) indicate that b) is also incorrect.

However, a second condition for the two sampled signals (and therefore, the source of the second correct answer) is to recall that cosines are even functions, so that the two sampled signals will also be identical if

$$\frac{2\pi 200}{f_s} + 2\pi k = -\frac{2\pi 4000}{f_s}$$

Solving for f_s, we have

$$f_s = \frac{4200}{k}$$

so that 1400 Hz ($k = 3$) is a valid sampling frequency. Accordingly, b) is also correct.

Partial Credit Scoring.

Up to 5 points were given for *complete* solutions for a) and b), alone. For c) and e), up to three points were given for problem set-up and development.

Problem 2. Given the signal

$$s(t) = (1 + 0.5\cos(2\pi 10t))\cos(2\pi 4000t)$$

Suppose we multiply $s(t)$ by $\cos(2\pi 4000t)$ and filter the continuous-time result with an ideal lowpass filter. Given that the lowpass filter removes all frequencies above 3000 Hz, what is the ***smallest*** sampling frequency for which aliasing will not occur when sampling the ***output*** of the lowpass filter? (Hint: $(\cos\alpha)^2 = 0.5(1 + \cos(2\alpha))$.)

a. 6000 Hz

b. 8020 Hz

c. 16020 Hz

d. 10 Hz

e. 20 Hz

Correct answer: e

Solution: Perform the multiplication as indicated in the problem, using the hint, to obtain

$$\begin{aligned} s(t)\cos(2\pi 4000t) &= 0.5(1 + 0.5\cos(2\pi 10t))(1 + \cos(2\pi 8000t)) \\ &= 0.5(1 + 0.5\cos(2\pi 10t)) + \\ &\quad 0.5(1 + 0.5\cos(2\pi 10t))\cos(2\pi 10t)\cos(2\pi 8000t) \end{aligned}$$

The frequencies of the first two terms on the rhs are 0 and 10 Hz. Based on the amplitude modulation materials found in Section 3.2.3 for the text, the frequencies of the last term of the rhs are 7990, 8000, and 8010 Hz. Therefore, the output of the lowpass filter of this signal has frequencies at 0 and 10 Hz. Twice this frequency (20 Hz) prevents aliasing.

Partial Credit Scoring.

For a), the scoring differed according to the argument presented. (i) Up to 4 points were given for carrying out the multiplication, lowpass filtering, but applying the sampling theorem to the cutoff frequency of the lowpass filter, rather than properly seeing that the lowest frequency in the signal was 10 Hz. (ii). Up to 10 points were given for carrying out the multiplication, lowpass filtering, and identification of the lowest frequency, and then interpretting the sampling theorem as a strict inequality.

For b), up to 2 points were given for realizing that the highest frequency in $s(t)$ was 4010 and applying the sampling theorem.

For c), up to 2 points were given for carrying out the multiplication, realizing the highest frequency was 8010, and applying the sampling theorem.

For d), up to 4 points were given for carrying out the multiplication, properly handling the low-pass filtering operation, but not correctly applying the sampling theorem.

Problem 3. Suppose you are using a D-to-C converter to create a continuous-time signal, $y(t)$, from a discrete-time signal, $y[n]$, according to

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

In order that $y(nT_s) = y[n]$ for all n

a. the D-to-C converter must be ideal

b. it must be the case that $p(t) = 0$ for all $|t| \geq T_s$

c. it must be the case that $p(0) = 1$

d. more than one of the above is correct

e. none of the above is correct

Correct answer: c

Solution:

Options a) and b) can be rejected, and with it, option d, using the following argument:

According to the textbook, sinc interpolation (option a) yields the continuous-time signal $y(t)$ from its samples. Therefore, $y(nT_s) = y[n]$. However, zero-order hold interpolation (option b) also yields $y(nT_s) = y[n]$. Since both of these methods yield the same result, neither is necessary.

Consider option c). Given $y(nT_s) = y[n]$, then $y(nT_s) - y[n] = 0$ so

$$\begin{aligned} y(nT_s) - y[n] &= \sum_{m=-\infty}^{\infty} y[m]p(mT_s - nT_s) - y[n] \\ &= y[n](p(0) - 1) + \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} y[m]p((m-n)T_s) \\ &= 0 \end{aligned}$$

Since this must hold for *any signal* $y(t)$, both expressions on the rhs must be zero. Thus, it must be the case that

$$p(0) = 1$$

In addition, $p(nT_s) = 0$ for $n \neq 0$.

Partial Credit Scoring.

a), b), and d). Up to 5 points of credit were given for identifying the ideal D-to-C converter as a sinc function, providing examples of other D-to-C converters that were discussed in the textbook, or developing the conditions showing c) to be correct. No credit was given for e).

Problem 4.

Correct answer: e

Solution:

a. Not time invariant. See textbook on "time-flip" systems (p. 140).

b. Not linear, since $\cos(a + b) \neq \cos(a) + \cos(b)$

c. Not linear, because of the constant "4".

Partial Credit Scoring.

Up to two points were given for establishing various properties of a) and b). Up to three points were given for establishing the properties of c), including the fact that it was nonlinear.

5. Suppose the response of a certain linear time-invariant system to the unit step sequence $u[n]$ is

$$y[n] = 2\,\delta[n] + \delta[n-1]$$

where $\delta[n]$ denotes the unit impulse and $u[n] = 0,\ n < 0;\ u[n] = 1,\ n \geq 0$.
The impulse response of this system is

a. $h[n] = 2\,\delta[n]$

b. $h[n] = 2\,\delta[n] + 3\,\delta[n-1] + \delta[n-2]$

c. $h[n] = 2\,\delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3] + \delta[n-4] + \ldots$

d. $h[n] = 2\delta[n] - \delta[n-1] - \delta[n-2]$ correct answer

e. None of the above

Solution 1: Notice that $\delta[n] = u[n] - u[n-1]$. Since $u[n] \rightarrow y[n]$, linearity and time invariance implies

$$\delta[n] \rightarrow h[n] = y[n] - y[n-1] = 2\,\delta[n] + 2\,\delta[n-1] - 2\delta[n-1] - \delta[n-2] = 2\delta[n] - \delta[n-2]$$

Solution 2: Work forward from each $h[n]$ to find $y[n]$. Use the fact that

$$\delta[n-k] * x[n] = x[n-k] \text{ which implies } \delta[n-k] * h[n] = h[n-m]$$

a. $h[n] = 2\,\delta[n] \Rightarrow y[n] = 2\,u[n]$ which is not the given $y[n]$

b. $h[n] = 2\,\delta[n] + 3\,\delta[n-1] + \delta[n-2] \Rightarrow y[n] = 2\,u[n] + 3\,\upsilon[n-1] + u[n-2] = \begin{cases} 0, & n<0 \\ 2, & n=0 \\ 4, & n=1 \\ 5, & n\geq 2 \end{cases}$

c. $h[n] = 2\,\delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3] + \delta[n-4] + \ldots$

$$\Rightarrow y[n] = 2\,u[n] - u[n-1] + u[n-2] - u[n-3] + u[n-4] + \ldots = \begin{cases} 0, & n<0 \\ 2, & n=0 \\ 1, & n=1 \\ 2, & n=2 \\ 1, & n=3 \\ 2, & n=4 \\ \text{etc.} \end{cases}$$

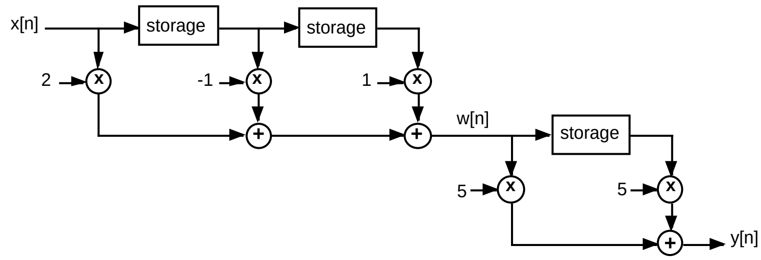
d. $h[n] = 2\delta[n] - \delta[n-1] - \delta[n-2] \Rightarrow y[n] = 2u[n] - u[n-1] - u[n-2] = \begin{cases} 0, & n<0 \\ 2, & n=0 \\ 1, & n=1 \\ 0, & n\geq 2 \end{cases}$

$= 2\delta[n]-\delta[n-1])$

which is the given $y[n]$, so this is $h[n]$

Partial Credit: 2 pts for writing $y[n] = u[n] * h[n]$, 1 pt. for writing $y[n] = \delta[n] * x[n])$, 5 pts for attempting solution 1 or 2 but making serious errors.

6. Consider the discrete-time system shown below.



Its inpulse response is

a. $h[n] = 5\,\delta[n] + 5\,\delta[n-2] + 10\,\delta[n-3]$

b. $h[n] = 10\,\delta[n] + 5\,\delta[n-1] + 5\,\delta[n-3]$ correct answer

c. $h[n] = 10\,\delta[n] - 5\,\delta[n-1] + 5\,\delta[n-2] + 5\,\delta[n-3]$

d. $h[n] = 5\,\delta[n] + 5\,\delta[n-1] + 5\,\delta[n-3]$

e. none of the above

We recognize the above as the cascade of two filters. The first has coefficients {2, -1,1} and the second has coefficients {5,5}.

Time domain solution: The overall system has impulse response

$$h[n] = h_1[n] * h_2[n] = (..., 0, 2, -1, 1, 0, ...) * (..., 0, 5, 5, 0, ...)$$

$$= (..., 0, 10, 5, 5, 0, ...) = 10\,\delta[n] + 5\,\delta[n-1] + 5\,\delta[n-3]$$

Frequency domain solution:

$$H_1(\hat{\omega}) = 2 - e^{-j\hat{\omega}} + e^{-j\hat{\omega}2}, \quad H_2(\hat{\omega}) = 5 + 5\,e^{-j\hat{\omega}}$$

The overall frequency response is

$$H(\hat{\omega}) = H_1(\hat{\omega})\,H_2(\hat{\omega}) = (2 - e^{-j\hat{\omega}} + e^{-j\hat{\omega}2})(5 + 5\,e^{-j\hat{\omega}})$$

$$= 10 + 10\,e^{-j\hat{\omega}} - 5\,e^{-j\hat{\omega}} - 5\,e^{-j\hat{\omega}2} + 5\,e^{-j\hat{\omega}2} + 5\,e^{-j\hat{\omega}3}$$

$$= 10 + 5\,e^{-j\hat{\omega}} + 5\,e^{-j\hat{\omega}3} \Rightarrow h[n] = 10\,\delta[n] + 5\,\delta[n-1] + 5\,\delta[n-3]$$

Partial credit: 3 pts for identifying h_1 and h_2, 7 pts for having right approach but making simple mistake in execution, 4 pts for determining $h[n]$ by following an impulse through the system but neglecting the affect of the last storage element, 6 pts for correct approach but getting h_1 "backwards".

7. The signal

$$x[n] = 1 + 3\cos(\frac{\pi}{3} n + 1)$$

is the input to an FIR filter with coefficients

$$\{b_k\} = \{5, -1\} .$$

The output of the filter is

a. $y[n] = 4 - 12\cos(\frac{\pi}{3} n + 1 + \frac{\pi}{3})$

b. $\mathbf{y[n] = 4 -/+ 3\sqrt{21} \cos(\frac{\pi}{3} n + 1 + \tan^{-1}(\frac{\sqrt{3}}{9}))}$ **correct answer**

c. $y[n] = 5 + 15\cos(\frac{\pi}{3} n + 1)$

d. $y[n] = x[n] H(\hat{\omega})$

e. More than one of the above

This problem was not graded because, unfortunately, there was a typo in b, which was supposed to be the correct answer. The "$4 - 3\sqrt{21}\cos$ ..." should have been "$4 + 3\sqrt{21}\cos$ ...". As the problem was actually written, no answer is correct. In particular, **d.** is incorrect because $y[n] = x[n] H(\hat{\omega})$ **holds only if x[n] is a complex exponential.** Since x[n] in this problem is not a complex exponential, d. is incorrect.

Solution: To compute y[n], we use the general fact that

$$A\cos(\hat{\omega}_o n+\phi) \rightarrow A\,|H(\hat{\omega}_o)|\cos(\hat{\omega}_o n+\phi+\text{angle}(H(\hat{\omega}_o)))$$

We observe that the given x[n] is the sum of two sinusoids with two different frequencies. We apply the above twice.

The frequency response is

$$H(\hat{\omega}) = 5 - e^{-j\hat{\omega}} = 5 - \cos(-\hat{\omega}) - j\sin(-\hat{\omega}) = 5 - \cos(\hat{\omega}) + j\sin(\hat{\omega})$$

$$|H(\hat{\omega})| = \sqrt{25 - 10\cos(\hat{\omega}) + \cos^2(\hat{\omega}) + \sin^2(\hat{\omega})} = \sqrt{26 - 10\cos(\hat{\omega})}$$

$$\text{angle}(H(\hat{\omega})) = \tan^{-1}(\frac{\sin(\hat{\omega})}{5-\cos(\hat{\omega})})$$

1. $\hat{\omega}_o = 0$: $|H(\hat{\omega}_o)| = |H(0)| = 4$, $\text{angle}(H(\hat{\omega}_o)) = 0$. Therefore, $1 \rightarrow 4$

2. $\hat{\omega}_o = \pi/3$: $|H(\hat{\omega}_o)| = 21$, $\text{angle}(H(\hat{\omega}_o)) = \tan^{-1}(\frac{\sqrt{3}}{9})$. Therefore,

$$3\cos(\frac{\pi}{3} n + 1) \rightarrow 3\sqrt{21}\cos(\frac{\pi}{3} n + 1 + \tan^{-1}(\frac{\sqrt{3}}{9}))$$

It follows $x[n] = 1 + 3\cos(\frac{\pi}{3} n + 1) \rightarrow y[n] = 4 + 3\sqrt{21}\cos(\frac{\pi}{3} n + 1 + \tan^{-1}(\frac{\sqrt{3}}{9}))$

8. A periodic signal x[n] with period 4 is the input to a linear time-invariant filter with impulse response

$h[n] = 2\delta[n] + \delta[n-1]$ (this is the filter used in class and Pr. 9 HW 7)

The output y[n] has DFT given by

$Y[0] = 9,\ Y[1] = (2-j)/2,\ Y[2] = 1,\ Y[3] = (2+j)/2$

Then x[n] =

a. $3 + 3\cos(\pi n)$

b. $3 + \cos(\frac{\pi}{4} n) + \cos(\pi n)$

c. $3 + \cos(\frac{\pi}{2} n) + \cos(\pi n)$ correct answer

d. $3 + \cos(\frac{\pi}{2} n)$

e. Not enough information given to determine x[n].

Solution 1: We find the DFT coefficients of the input signal x[n]. The filter has frequency response $H(\omega) = 2 + e^{-j\omega}$. We know the DFT coefficients X[k] of the output are related to the DFT coefficients of the input via $Y[k] = H(\frac{2\pi}{4} k)\, X[k]$. Therefore,

$$X[k] = \frac{Y[k]}{H(\frac{2\pi}{4}k)} \quad k = 0, 1, 2, 3.$$

Applying the above yields

$$X[0] = \frac{Y[0]}{H(0)} = \frac{9}{3} = 3\ ,\ X[1] = \frac{Y[1]}{H(\pi/2)} = \frac{(2-j)/2}{2-j} = \frac{1}{2}\ ,$$

$$X[2] = \frac{Y[2]}{H(\pi)} = \frac{1}{1} = 1,\ X[3] = \frac{Y[3]}{H(3\pi/2)} = \frac{(2+j)/2}{2+j} = \frac{1}{2}$$

Applying the DFT synthesis equation, we find

$$x[n] = X[0] + X[1]\, e^{j\frac{2\pi}{4} n} + X[2]\, e^{j\frac{2\pi}{4} 2 n} + X[3]\, e^{j\frac{2\pi}{4} 3n}$$

$$= 3 + \frac{1}{2} e^{j\frac{\pi}{2} n} + e^{j\pi n} + \frac{1}{2} e^{j\frac{3\pi}{2} n}$$

$$= 3 + \frac{1}{2} e^{j\frac{\pi}{2} n} + (-1)^n + \frac{1}{2} e^{-j\frac{\pi}{2} n} \quad \text{since } e^{j\pi} = -1 \text{ and } e^{j\frac{3\pi}{2}} = e^{-j\frac{\pi}{2}}$$

$$= 3 + \cos\frac{\pi}{2} n + \cos \pi n \quad \text{which is } \mathbf{c}$$

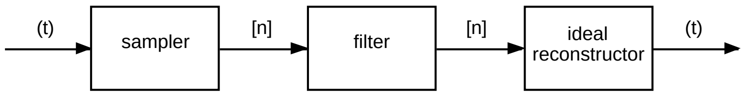
Solution 2: We can see from the DFT of y[n] that y[n] has spectral components at frequencies 0, $\frac{2\pi}{4} = \frac{\pi}{2}$, $\frac{2\pi}{4} 2 = \pi$, and $\frac{2\pi}{4} 3 = \frac{3\pi}{2}$, and also that the $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ really represent a sinsuoid of frequency $\frac{\pi}{2}$. **c** is the only answer that has these three types of components, so it must be the solution.

Partial Credit: up to 5 pts for formulas such as $Y[k] = H(\frac{2\pi}{4} k)\, X[k]$, $H(\hat{\omega}) = 2 + e^{-j\omega}$, and the DFT formula. 7 pts for computing the X[k]'s.

9. The continuous-time signal

$$x(t) = 7\cos(10\pi t)$$

is the input to the system shown below, which consists of a sampler that samples at f_s = 20 samples/sec, a discrete-time filter with coefficients $\{b_k\}$, and an ideal reconstructor.



The output $y(t)$ equals the input $x(t) = 7\cos(10\pi t)$ when the filter coefficients are

a. $\{b_k\} = \{5, 0, 4\}$

b. $\{b_k\} = \{5, 4\}$

c. $\{b_k\} = \{3, 0, 2\}$

d. More than one of the above.

e. None of the above

Solution: Since the sampling frequency is more than twice the highest frequency in x(t), it suffices to choose the filter so that $y[n] = x[n]$. We first an expression for $x[n]$:

$$x[n] = x(nT_s) = 7\cos(10\,\pi\,\tfrac{n}{20}) \;=\; 7\cos(\tfrac{\pi}{2}n) \;=\; 7\cos\hat{\omega}n\,, \quad \text{where } \hat{\omega} = \tfrac{\pi}{2}$$

Since $x[n]$ is a cosine with frequency $\pi/2$, $y[n]$ will be a cosine with frequency $\pi/2$. In particular,

$$y[n] \;=\; 7\,|H(\tfrac{\pi}{2})|\cos(\tfrac{\pi}{2}n + \text{angle}(H(\tfrac{\pi}{2})))$$

To make $y[n] = x[n]$, we must have $|H(\frac{\pi}{2})|=1$ and $\text{angle}(H(\frac{\pi}{2})) = 0$. That is, we need $H(\frac{\pi}{2}) = 1$. Let's find $H(\frac{\pi}{2})$ for each of the given filters. (It's not enough to check only $|H(\hat{\omega})|$.)

a. $H(\hat{\omega}) = 5 + 4e^{-j2\hat{\omega}}$. $H(\frac{\pi}{2}) = 5 + 4e^{-j\pi} = 5 - 4 = 1$. This works.

b. $H(\hat{\omega}) = 5 + 4e^{-j\hat{\omega}}$. $H(\frac{\pi}{2}) = 5 + 4e^{-j\pi/2} = 5 - 4j \neq 1$. This doesn't work.

c. $H(\hat{\omega}) = 3 + 2e^{-j2\hat{\omega}}$. $H(\frac{\pi}{2}) = 3 + 2e^{-j\pi} = 3 - 2 = 1$. This works.

The answer is d. more than one of the above.

Some people worked the problem in the time domain. For example to check a., they computed $y[n] = 5x[n] - 4x[n-2]$ to see if it is $7\cos(\frac{\pi}{2}n)$. Though this approach is possible, it is more difficult, and I don't think anyone did it successfully.

Partial Credit: 3 pts for $y[n] = x[n] = 7\cos(\frac{\pi}{2}n)$ and finding $H(\hat{\omega})$ for systems a.,b.,c.

2 pts for noting that we must have $H(\frac{\pi}{2}) = 1$.

6 pts for showing $H(\frac{\pi}{2}) = 1$ for one but not both of filters a. and c.

2 pts if you answered a. or c. but not both, without adequate explanation.

-1 pt if you wrote $y[n] = x[n]\,H(\hat{\omega})$