Lab Note: The $\Delta^2/12$ formula.

Fact: When a uniform quantizer with level spacing Δ and range $[x_{min}, x_{max}]$ is used to quantize a data sequence $x_1, x_2, ..., x_N$ whose values lie mostly within the quantizer's range and whose standard deviation is much larger than Δ, the mean-squared error can be approximated as

$$\text{MSE} \cong \frac{1}{12}\Delta^2$$

and so the RMS error can be approximated as

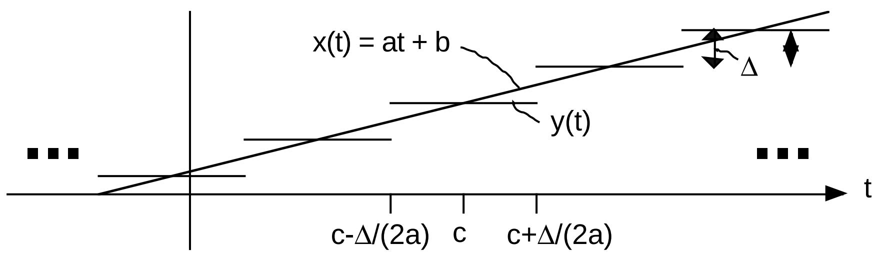
$$\text{RMSE} \cong \sqrt{\frac{1}{12}\Delta^2}$$

Derivation: The basic idea is to derive the formula for the MSE in a certain special case. We then argue that the formula is approximately the same in other cases.

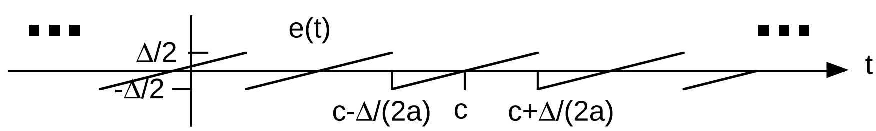
The special case we consider is that of a "straight-line" continuous-time signal, namely,

$$x(t) = a\,t + b$$

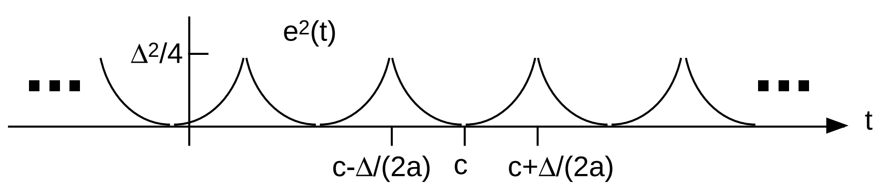
where a and b are known constants. Later we'll address discrete-time signals, i.e. data sequences. The diagram below shows this signal along with the result $y(t)$ of quantizing it with a uniform quantizer with level spacing Δ.



The point marked c in the above will be useful later. The quantization error $e(t) = x(t) - y(t)$ is shown below:



The mean-squared error (MSE) is the mean-squared value of $e(t)$. Equivalently, it is the mean value of $e^2(t)$, which is illustrated below.



Notice that $e^2(t)$ is periodic with period $T_o = \Delta/a$. Because of this, we can compute its mean value by considering only one period. It doesn't matter which one. Specifically, the quantizer's mean-squared error is

$$\text{MSE} = \frac{1}{T_o}\int_{c-T_o/2}^{c+T_o/2} e^2(t)\ dt$$

Now, $e(t) = a(t-c)$ in the time interval $c-T_o/2 \le t \le c+T_o/2$. Therefore,

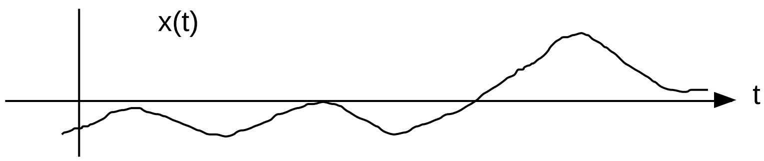
$$\text{MSE} = \frac{1}{T_o} \int_{c-T_o/2}^{c+T_o/2} a^2 (t-c)^2 \, dt = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} a^2 s^2 \, ds , \quad \text{using change of variables: } s=t-c$$

$$= \frac{1}{T_o} a^2 \frac{s^3}{3} \Big|_{-T_o/2}^{T_o/2} = \frac{1}{12} a^2 T_o^2 = \frac{1}{12} \Delta^2 , \quad \text{because } T_o = \Delta/a$$

Thus, we have derived the MSE formula in the special case.

It is important to notice that the MSE formula for a "straightline signal" $x(t) = at+b$ depends neither on the slope of the signal a, nor its offset b.

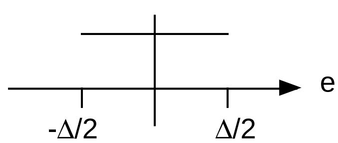
Now consider some arbitrary continuous-time signal $x(t)$, such as that illustrated below.



When Δ is small relative to the standard deviation of the signal, most small time segments of the signal are, approximately, straight lines that cross a number of Δ steps. By the previous analysis the MSE on each of these straightline segments is approximately $\Delta^2/12$. It follows that the overall MSE is approximately $\Delta^2/12$.

Finally, we claim that the same formula applies to the discrete-time case. This is because the discrete-time signal can be thought of as samples of some hypothetical continuous time signal; the MSE in the discrete-time case is just the average of samples of the error function $e(t)$; and the discrete-time average of samples will be approximately the same as the continuous-time average, provided that samples are taken frequently enough. Since we are free to hypothesize a continuous-time signal that is smooth, the continuous and discrete-time averages are approximately the same. Thus, the $\Delta^2/12$ formula holds for discrete-time signals as well.

Alternate derivation: One can also derive this formula by noticing that for continuous-time straightline signal $x(t) = a\,t + b$, the signal value distribution of the error signal $e(t) = x(t) - y(t)$ is approximately constant over the range $-\Delta/2$ to $\Delta/2$, as illustrated below:



In other words, errors of all sizes between $-\Delta/2$ and $\Delta/2$ occur equally often. One can then use this to argue that the averge MSE is just the average of e^2 values, when e ranges from $-\Delta/2$ to $\Delta/2$:

$$\text{MSE} \cong \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} e^2 \, de = \frac{1}{\Delta} \frac{e^3}{3} \Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{12}$$

Then as before one argues that the signal value distribution when quantizing any signal (continuous or discrete-time) with standard deviation much larger than Δ is approximately as shown above. Therefore, the MSE is approximately given by the formula $\Delta^2/12$.